THE CESÀRO OPERATOR IS BOUNDED ON $H^1$

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Abstract. The purpose of this note is to give a direct proof that the Cesàro operator is bounded on the Hardy space $H^1$.

1. INTRODUCTION

On the Hardy spaces $H^p$ of the unit disc, the Cesàro operator is

$$
\mathcal{C}(f)(z) = \sum_{n=0}^{\infty} \left( \frac{1}{n+1} \sum_{k=0}^{n} a_k \right) z^n,
$$

where $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is in $H^p$. It is known that $\mathcal{C}$ is bounded on $H^p$ for $1 \leq p < \infty$. For $1 < p < \infty$ this can be obtained by using the Hardy result [1] concerning trigonometric series and M. Riesz's theorem, but this proof does not cover the case $p = 1$. The boundedness of $\mathcal{C}$ on $H^1$ was obtained as a byproduct in [3], where, for the purpose of finding the norm of $\mathcal{C}$ on $H^p$, a related strongly continuous semigroup of weighted composition operators was studied. The purpose of this note is to give a direct proof of the boundedness of $\mathcal{C}$ on $H^1$, avoiding the semigroups.

We use the usual notation $M_p(r, f)$ for the integral means on $|z| = r$ of an analytic $f$, and will make use of the following Hardy–Littlewood result which we state as a lemma.

Lemma [2, p. 412]. If $f \in H^1$ and $q > 1$ then

$$
\int_0^1 M_q(r, f)(1-r)^{-1/q} dr \leq k \|f\|_1,
$$

and the constant $k$ depends only on $q$.

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2. The proof

A computation shows that (1) can be written

\[ \mathcal{E}(f)(z) = \frac{1}{z} \int_{0}^{\pi} f(\zeta) \frac{1}{1 - \zeta} \, d\zeta = \int_{0}^{1} f(tz) \frac{1}{1 - tz} \, dt. \]

For \(0 < r < 1\) and \(f \in H^1\), we have

\[
M_1(r, \mathcal{E}(f)) = \frac{1}{2\pi} \int_{0}^{2\pi} \left( \int_{0}^{1} f(rte^{i\theta}) \frac{1}{1 - rte^{i\theta}} \, dt \right) \, d\theta 
\leq \frac{1}{2\pi} \int_{0}^{2\pi} |f(rte^{i\theta})| \frac{1}{|1 - rte^{i\theta}|} \, d\theta \, dt 
\leq \frac{1}{2\pi} \int_{0}^{2\pi} \left( \frac{1}{2\pi} \int_{0}^{2\pi} |f(rte^{i\theta})|^2 \, d\theta \right)^{1/2} \left( \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1}{|1 - rte^{i\theta}|^2} \, d\theta \right)^{1/2} \, dt 
= \int_{0}^{1} M_2(rt, f) \left( \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1}{1 - r^2t^2} P(rt, \theta) \, d\theta \right)^{1/2} \, dt,
\]

where H"older's inequality was used in the third step and \(P(r, \theta)\) is the Poisson kernel. Since \((1/2\pi) \int_{0}^{2\pi} P(r, \theta) \, d\theta = 1\), the last integral is dominated by

\[
\int_{0}^{1} M_2(rt, f)(1 - r^2t^2)^{-1/2} \, dt \leq \int_{0}^{1} M_2(t, f)(1 - t)^{-1/2} \, dt \leq k \|f\|_1,
\]

by applying the Lemma with \(q = 2\) in the last step. This finishes the proof.

It is interesting to see, upon close examination of the argument, that the proof cannot be adapted for other values \(0 < p < 1\) or \(1 < p < \infty\).

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References