THE CESÁRO OPERATOR IS BOUNDED ON \(H^1\)

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Abstract. The purpose of this note is to give a direct proof that the Cesàro operator is bounded on the Hardy space \(H^1\).

1. Introduction

On the Hardy spaces \(H^p\) of the unit disc, the Cesàro operator is

\[
\theta(f)(z) = \sum_{n=0}^{\infty} \left( \frac{1}{n+1} \sum_{k=0}^{n} a_k \right) z^n,
\]

where \(f(z) = \sum_{k=0}^{\infty} a_k z^k\) is in \(H^p\). It is known that \(\theta\) is bounded on \(H^p\) for \(1 \leq p < \infty\). For \(1 < p < \infty\) this can be obtained by using the Hardy result [1] concerning trigonometric series and M. Riesz's theorem, but this proof does not cover the case \(p = 1\). The boundedness of \(\theta\) on \(H^1\) was obtained as a byproduct in [3], where, for the purpose of finding the norm of \(\theta\) on \(H^p\), a related strongly continuous semigroup of weighted composition operators was studied. The purpose of this note is to give a direct proof of the boundedness of \(\theta\) on \(H^1\), avoiding the semigroups.

We use the usual notation \(M_p(r, f)\) for the integral means on \(|z| = r\) of an analytic \(f\), and will make use of the following Hardy–Littlewood result which we state as a lemma.

Lemma [2, p. 412]. If \(f \in H^1\) and \(q > 1\) then

\[
\int_0^1 M_q(r, f)(1-r)^{-1/q} \, dr \leq k\|f\|_1,
\]

and the constant \(k\) depends only on \(q\).
2. The proof

A computation shows that (1) can be written

\[ \mathcal{C}(f)(z) = \frac{1}{z} \int_0^z f(\zeta) \frac{1}{1 - \zeta} \, d\zeta \]

\[ = \int_0^1 f(tz) \frac{1}{1 - tz} \, dt. \]

For \( 0 < r < 1 \) and \( f \in H^1 \), we have

\[ M_1(r, \mathcal{C}(f)) = \frac{1}{2\pi} \int_0^{2\pi} \left| \int_0^1 f(rte^{i\theta}) \frac{1}{1 - rte^{i\theta}} \, dt \right| \, d\theta \]

\[ \leq \int_0^1 \frac{1}{2\pi} \int_0^{2\pi} \left| f(rte^{i\theta}) \right| \frac{1}{|1 - rte^{i\theta}|} \, d\theta \, dt \]

\[ \leq \int_0^1 \left( \frac{1}{2\pi} \int_0^{2\pi} \left| f(rte^{i\theta}) \right|^2 \, d\theta \right)^{1/2} \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1 - rte^{i\theta}|^2} \, d\theta \right)^{1/2} \, dt \]

\[ = \int_0^1 M_2(rt, f) \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1 - r^2 t^2|^2} P(rt, \theta) \, d\theta \right)^{1/2} \, dt, \]

where Hölder's inequality was used in the third step and \( P(r, \theta) \) is the Poisson kernel. Since \((1/2\pi) \int_0^{2\pi} P(r, \theta) \, d\theta = 1\), the last integral is dominated by

\[ \int_0^1 M_2(rt, f) (1 - r^2 t^2)^{-1/2} \, dt \leq \int_0^1 M_2(t, f) (1 - t)^{-1/2} \, dt \leq k \|f\|_1, \]

by applying the Lemma with \( q = 2 \) in the last step. This finishes the proof.

It is interesting to see, upon close examination of the argument, that the proof cannot be adapted for other values \( 0 < p < 1 \) or \( 1 < p < \infty \).

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References

1. G. H. Hardy, _Notes on some points in the integral calculus_ LXVI, Messenger of Math. 58 (1929), 50–52.