THE CONSTRUCTION OF GLOBAL ATTRACTORS

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Abstract. The purpose of this note is to show that every inverse limit space of an interval mapping can be realized as a global attractor for a homeomorphism of the plane.

The purpose of this note is to describe a simple method for the construction of an abundance of global attractors. In order to facilitate understanding, we describe this method in the plane.

Suppose that $I$ is an interval and that $f: I \to I$ is continuous. Let $(I, f)$ be the inverse limit space $\{(x_0, x_1, \ldots) | x_i \in I$ and $f(x_{i+1}) = x_i\}$, with metric $d((x_0, x_1, \ldots), (y_0, y_1, \ldots)) = \sum_{i=0}^{\infty} |x_i - y_i|/2^i$. We will show that $(I, f)$ can be topologically realized as a global attractor in the plane. These inverse limit spaces are examples of what Bing has called "snakelike continua", see [Bi] or [Wa]. Furthermore, the dynamics on $(I, f)$ can be understood in terms of the dynamics of $f$, see [B-M1; B-M2; B-M3]. In [Wi], Williams discusses inverse limits of branched 1-manifolds as "generalized solenoids", which have attracting neighborhoods.

The idea is a simple one. Imagine that $D$ is a disk and that $I \subset \text{int } D$. We construct a map $H: D \to D$ such that (1) $H(I) = I$, and $H|I = f$; (2) $H|\text{Bdry}(D) = \text{id}$; (3) if $x \in \text{int } D$ there is a positive integer $n$, such that $H^n(x) \in I$; and (4) $H$ is uniformly approximated by homeomorphisms. Then let $X$ be the inverse limit of $D$ with bonding map $H$. Using (4), it follows from a theorem of M. Brown [Br] that $X$ is a topological disk. Using conditions (1) and (3) we see that $(I, f)$ is embedded in $X = (D, H)$, and if $x \in \text{int } X$, then $d((\hat{H})^n(x), (I, f)) \to 0$. Here $\hat{H}$ is the homeomorphism on $X$ induced by $H$. Furthermore, the homeomorphism $\hat{H}|(I, f)$ is just the homeomorphism $\hat{f}: (I, f) \to (I, f)$, induced by $f$.

Definitions. If $Z$ is a compact metric space, and $g: Z \to Z$ is continuous, the inverse limit space $(Z, g)$ is $\{(z_0, z_1, \ldots) | z_i \in Z$ and $g(z_{i+1}) = z_i\}$ with the metric $\rho((z_0, z_1, \ldots), (y_0, y_1, \ldots)) = \sum_{i=0}^{\infty} d(z_i, y_i)/2^i$. The in-
duced homeomorphism \( \hat{g} : (Z, g) \to (Z, g) \) is given by \( \hat{g}((z_0, z_1, \ldots)) = (g(z_0), z_0, z_1, \ldots) \).

If \( A \) is a subset of the plane \( E^2 \), the statement that \( A \) is a global attractor means that there is a homeomorphism \( h : E^2 \to E^2 \) such that (1) \( h(A) = A \); (2) if \( x \in E^2 \) then \( d(h^n(x), A) \to 0 \) as \( n \to \infty \) and; (3) if \( U \) is open and \( A \subset U \), then there is an open set \( V \) and a positive integer \( N \) such that \( A \subset V \subset U \) and if \( n > N \), then \( h^n(V) \subset U \).

**Construction of the examples.** For \( i = 1, 2, 3 \), let \( B_i \subset E^2 \) be \( \{(x, y)| -t < x < i \text{ and } -i < y < i\} \). Let \( I \) be the interval \( \{(t, 0)| -1 < t < 1\} \) and suppose that \( f : I \to I \) is continuous.

Now let \( h : B_3 \to B_3 \) be a homeomorphism such that (1) \( h|B_3 - B_2 = \text{id} \); (2) if \( (t, 0) \in I \) then \( h((t, 0)) = (f(t), t) \). This last condition insures that \( h \), followed by vertical projection onto \( I \), is \( f \). See the diagram.

![Diagram](image.png)

We now construct a continuous function \( G : B_3 \times [0, 1] \to B_3 \). Denoting \( G|B_3 \times \{t\} \) by \( G_t \), we will have the following properties:

1. \( G_0 = \text{id} \);
2. \( G_t \) is a homeomorphism if \( 0 \leq t < 1 \);
3. for each \( t \), \( G_t|\text{Bdry}(B_3) = \text{id} \);
4. if \( (t, 0) \in I \), then \( \{(t, s)| -1 \leq s \leq 1\} \subset G_t^{-1}((t, 0)) \);
5. \( G_1(B_2) = B_1 \);
6. if \( x \in \text{int} B_3 \), there is an integer \( n \) such that \( G_1^n(x) \in I \).

Roughly speaking, \( B_2 \) is gradually squeezed down to \( B_1 \) while the vertical
intervals in $B_1$ are shrunk down to points in $I$.

Now, let $H = G_1 \circ h$ and let $X = (B_3, H)$ be the inverse limit space of $B_3$ with bonding map $H$. From Condition (2), it follows that $H$ is uniformly approximated by homeomorphisms. Using [Br, Theorem 4], it follows that $X$ is a topological disk. Let $A = \{(x_0, x_1, \ldots) | (x_0, x_1, \ldots) \in X \text{ and } x_i \in I\}$. Then $A \subset X$, and it follows from (5), (6), and the fact that $h = \text{id}$ on $B_3 - B_2$, that $A$ is a global attractor for $\text{int} \ X$ under $H$.

Now suppose that $(t, 0) \in I$. Then $H(t, 0) = G_1((f(t), t)) = (f(t), 0)$, by (4). From this it follows that $A$ is homeomorphic with $(I, f)$. Notice that if $(x_0, x_1, \ldots) \in A$, then $\tilde{H}((x_0, x_1, \ldots)) = (H(x_0), x_0, x_1, \ldots) = (f(x_0), x_0, x_1, \ldots) = f((x_0, x_1, \ldots))$. so the homeomorphism induced on $A$ by $\tilde{H}$ is just $f$.

Remarks. Notice that if $\rho \in \text{int} \ X$, then there is a point $q \in A$ such that $d((\tilde{H})^n(\rho), (\tilde{H})^n(q)) \to 0$. Also it is clear that this construction, and elaborations of it, can be carried out in much greater generality. We will discuss these results elsewhere.

References


