PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 110, Number 3, November 1990

A CLOSURE THEOREM FOR σ-COMPACT SUBGROUPS OF LOCALLY COMPACT TOPOLOGICAL GROUPS

TA-SUN WU

(Communicated by Jonathan M. Rosenberg)

ABSTRACT. We describe the closure of certain subgroups of a locally compact group.

D. Ž. Djoković proved the following theorem.

Theorem ([1]). *Let G be a real Lie group, A a closed subgroup of G, and B an analytic subgroup of G. We assume that B normalizes A and that AB is closed in G. Then we have*

$$B^{-} = (A \cap B)^{-} \cdot B.$$

In particular, B is closed in G if and only if $A \cap B$ is closed in G.

For many interesting applications of the above theorem, we refer to [1] and [2]. In this note, we generalize it into the following theorem.

Theorem. *Let G be a locally compact (Hausdorff) topological group. Let B be a σ-compact subgroup of G. Suppose there exists a closed subgroup A of G such that B normalizes A and BA is closed. Then the closure B^{-} of B is the group $(A \cap B)^{-} \cdot B$.*

Since an analytic subgroup is σ-compact, Djoković's result is an immediate consequence of the above theorem.

Our proof is simple, using a known categorical argument for topological groups which we state as a lemma (cf. Theorem 5.29 of [4] for a similar result).

Lemma. *Let F be a σ-compact topological group. If there exists a continuous isomorphism f from F onto a locally compact topological group H, then F is locally compact and f is a topological isomorphism, i.e. f is an open map.*

Proof. First, we show that F is locally compact. Since F is σ-compact, there exists a sequence of compact subsets $\{D_i : i = 1, 2, \dots\}$ of F such that $F = \bigcup_{i=1}^{\infty} D_i$. Since $H = f(F) = \bigcup_{i=1}^{\infty} f(D_i)$, $f(D_i)$ has nonvoid interior $f(D_i)^0$

Received by the editors July 24, 1989 and, in revised form, October 19, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 22A05.

©1990 American Mathematical Society
0002-9939/90 \$1.00 + \$.25 per page

for some D_i by the Baire category theorem. Then $f^{-1}(f(D_i)^0)$ is an open subset of F with compact closure. Therefore F is a locally compact group.

Now, we show that f is an open map. Let U be any compact neighborhood of identity 1_F of F. Let V be a compact neighborhood of 1_F of F such that $V = V^{-1} \subset V^2 \subset U$. Since F is σ-compact, $F = \bigcup_{i=1}^{\infty} x_i V$ where $\{x_i : i = 1, 2, \ldots\}$ is a sequence of elements in F. Again, by the Baire category theorem, $f(x_i V)$ has nonvoid interior for some $f(x_i V)$. Since $f(V) = f(x_i)^{-1} \cdot f(x_i V)$, $f(V)$ has a nonvoid interior. Now let h be an interior point of $f(V)$. Let $x = f^{-1}(h)$. Then $f(1_F) = f(x^{-1})f(x) = h^{-1}h \in h^{-1}f(V)^0 \subset f(U)$. Hence $f(1_F) = 1_H \in f(U)^0$, and f is an open map at the identity. Both F and H are homogeneous spaces. It follows that f is an open map. The proof of the lemma is now complete.

Proof of the theorem. Without loss of generality, we assume that $G = BA$. A is a closed normal subgroup of G. Since B normalizes $A \cap B$, B normalizes $(A \cap B)^-$. Therefore $B \cdot (A \cap B)^-$ is a subgroup of G. It is straightforward to check that $[B \cdot (A \cap B)^-] \cap A = (A \cap B)^-$. Let ϕ be the inclusion map from $B \cdot (A \cap B)^-$ into G. Then we have the continuous isomorphism ϕ' induced by ϕ from $B \cdot (A \cap B)^- / (A \cap B)^-$ onto G/A. Since $B \cdot (B \cap A)^- / (B \cap A)^-$ is the homomorphic image of B, it is σ-compact. Since G/A is locally compact, ϕ' is an open map and $B \cdot (A \cap B)^- / (A \cap B)^-$ is locally compact by the above lemma. Since $(A \cap B)^-$ is locally compact, therefore $B \cdot (A \cap B)^-$ is locally compact (cf. [3], Theorem 5.25 of [4], or Theorem 2.2 of [5]). Hence $B \cdot (A \cap B)^-$ is closed. Since $B \subset B \cdot (A \cap B)^- \subset B^-$, so $B^- = B \cdot (A \cap B)^-$. The proof is now complete.

References

1. D. Ž. Djoković, *A closure theorem for analytic subgroups of real Lie groups*, Canad. Math. Bull. **19** (1976), 435–439.
2. ——, *Irreducible connected Lie subgroups of* $\mathrm{GL}_n(R)$ *are closed*, Israel J. Math. **28** (1977), 175–176.
3. A. Gleason, *Groups without small subgroups*, Ann. of Math. **56** (1952), 193–212.
4. E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, Springer-Verlag, New York, 1963.
5. D. Montgomery and L. Zippin, *Topological transformation groups*, Interscience, 2nd printing, New York, 1964.

Department of Mathematics, Case Western Reserve University, Cleveland, Ohio 44106