THE POINCARÉ CONJECTURE IS TRUE
IN THE PRODUCT OF ANY GRAPH WITH A DISK

DAVID GILLMAN

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Abstract. We prove that the only compact 3-manifold-with-boundary which has trivial rational homology, and which embeds in the product of a graph with a disk, is the 3-ball. This implies that no punctured lens space embeds in the product of a graph with a disk. It also implies our title.

The proof relies on a general position argument which enables us to perform surgery.

I. Introduction and summary

We work with finite polyhedra and piecewise linear maps throughout. The term 3-manifold $M$ denotes a compact, but not necessarily connected, 3-manifold-with-boundary. We denote by $b_i(M)$ the $i$th betti number of $M$; that is, the rank of the $i$th homology group of $M$ with rational coefficients; thus, $b_0(M) = 1$ if and only if $M$ is connected.

Definition 1. Let $n$ be a nonnegative integer. The $n$-punctured ball is the topologically unique 3-manifold obtained by removing $n$ disjoint open neighborhoods of $n$ interior points from the 3-dimensional ball.

Theorem. Let $G$ denote a graph (that is, a 1-complex), and $I$ denote an interval. Let $M$ be a 3-manifold embedded in $G \times I \times I$. If $b_0(M) = 1$ and $b_1(M) = 0$, then $M$ is homeomorphic to the $n$-punctured ball with $n = b_2(M)$.

In other words, any 3-manifold $M$ in $G \times I \times I$ with $b_0(M) = 1$ and $b_1(M) = 0$ is completely classified by its three betti numbers $b_0$, $b_1$, and $b_2$.

Corollary 1. Let $M$ denote any connected 3-manifold with trivial rational homology. If $M$ embeds in $G \times I \times I$, then $M$ is the 3-ball.

Corollary 2. The punctured lens spaces do not embed in $G \times I \times I$. The product $P^2 \times I$ of a projective plane and an interval does not embed in $G \times I \times I$. Furthermore, these 3-manifolds, punctured any finite number of times, fail to embed in $G \times I \times I$.

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This fact about the punctured lens spaces does not follow the odd–even pattern of known results about embeddings in Euclidean $4$-space. The odd punctured lens spaces embed in $4$-space [5]. The even ones do not (differentiably) embed in $4$-space [1].

**Corollary 3.** The Poincaré Conjecture holds in $G \times I \times I$; that is, the only contractible $3$-manifold in $G \times I \times I$ is the $3$-ball.

**Question.** Does the Theorem hold in the product $K^2 \times I$ of any contractible $2$-complex $K^2$ and an interval?

An affirmative answer to this question implies the Poincaré Conjecture, because: Let $K_{\text{std}}$ denote any standard spine of a contractible $3$-manifold $M$. It is shown in [2] that $M$ embeds in $K_{\text{std}} \times I$.

Indeed, we do not know the answer to this question for the special case, where $K^2$ is the cone over a graph.

II. **Alternative statements of the theorem**

Let us call the statement given above Version I of the Theorem. To begin, we verify that Version I is valid in $3$-space.

**Lemma I.** Any $3$-manifold $M$ in $I \times I \times I$ with $b_0(M) = 1$ and $b_1(M) = 0$ is an $n$-punctured ball with $n = b_2(M)$.

**Proof.** Because $M$ lies in $I \times I \times I$, $M$ is orientable. Since $b_1(M) = 0$ and $M$ is orientable, $\partial M$ is a finite collection of $2$-spheres [4, p. 231]. An induction argument is now used on the number of $2$-spheres. The $3$-dimensional Schoenflies theorem [3, Ch. 17] is applied on the innermost $2$-sphere to complete the proof of Lemma 1.

If one removes the restriction that $b_0(M) = 1$, a counterexample to classification of manifolds in $3$-space by betti numbers is easily constructed: Let $b_0(M) = 2$, $b_1(M) = 0$, and $b_2(M) = 2$. Two distinct submanifolds of $3$-space have these betti numbers. Both manifolds consist of two disjoint $3$-balls with two punctures. One may place both punctures in the same $3$-ball, or one in each $3$-ball.

If one instead removes the restriction that $b_1(M) = 0$ from Lemma 1, then a solid torus and a cube-with-a-knotted-hole provide the counterexample to classification by betti numbers.

**Version II of the theorem.** Any $3$-manifold $M$ in $G \times I \times I$ with $b_1(M) = 0$ is embeddable in $I \times I \times I$.

Without the hypothesis that $b_1(M) = 0$, Version II is false. The $3$-manifold Pune $S^2 \times S^1$, obtained by puncturing $S^2 \times S^1$ exactly once, is a counterexample. Specifically, Pune $S^2 \times S^1$ embeds in $T \times I \times I$, where $T$ denotes a triod. To see this, first place a solid torus in $3$-space. Observe that a meridional annulus on its boundary is planar in $3$-space. Onto this planar annulus, we now attach a $2$-handle which lies in the “third page” of $T \times I \times I$, forming Pune $S^2 \times S^1$. 

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Version II implies Version I, by Lemma I. To prove that Version I implies Version II, we introduce a third version.

**Definition 2.** A **punctured ball sum** is the disjoint union of a finite number of punctured balls. Each punctured ball may be punctured any finite number of times.

**Version III of the theorem.** Let $M$ be a 3-manifold embedded in $G \times I \times I$. If $b_1(M) = 0$, then $M$ is a punctured ball sum.


In our fourth and final version, we make a slight change in Version III; namely, in place of a graph $G$, we substitute a tree with only one vertex of high order.

**Definition 3.** For any positive integer $n$, an **$n$-odd $Y$** is a tree with one vertex $v$ of order $n$, and $n$ other vertices of order 1.

**Version IV of the theorem.** Let $M$ be a 3-manifold which embeds in $Y \times I \times I$, for some $n$-odd $Y$. If $b_1(M) = 0$, then $M$ is a punctured ball sum.

Version III immediately implies Version IV, because every $n$-odd is a graph. Version IV implies Version III because any $G \times I \times I$ embeds in $Y \times I \times I$ for some $n$-odd $Y$. To establish this, we prove a simpler and stronger statement:

**Lemma 2.** For any graph $G$, there exists an $n$-odd $Y$ such that $G \times I$ embeds in $Y \times I$.

**Proof.** Let $N$ denote a regular neighborhood in $G$ of the vertices of $G$. We embed $N$ in $Y \times I$ so that each page of $Y \times I$ contains exactly one edge of $N$. The embedding of $N$ extends to an embedding of $G$ in $Y \times I$. This extension may be easily constructed on each arc of $G - N$ individually. Lastly, one extends this embedding of $G$ in $Y \times I$ to an embedding of $G \times I$ in $Y \times I$. This completes the proof of Lemma 2.

We thank the referee for suggesting Lemma 2.

The rest of this paper is devoted to a proof of Version IV of the Theorem.

**III. Transverse position of $M$ in $Y \times I \times I$**

Regarding $Y \times I \times I$ as a "book with $n$ pages", we call the disk $B = v \times I \times I$ the **binding**; the $n$ components $P_1, P_2, \ldots, P_n$ of $Y \times I \times I - B$ are called the **pages**.

**Definition 4.** If $W$ is a subset of a 3-manifold $M$, we say that $W$ is **neighborhood-bicollared** in $M$ if a regular neighborhood of $W$ in $M$ is of the form $W \times [-1, 1]$, with $W$ identified with the level $W \times 0$.

Observe that if $W$ is neighborhood-bicollared in $M$, then $W$ must be a 2-manifold. To see this, let $p$ be a point of $W$. The link of $p$ in $W$ must
be connected, and has no vertex of order greater than 2, because $W \times [-1, 1]$ is a 3-manifold at the point $p \times 0$. Thus, this link is either an arc or a simple closed curve, depending if $p$ is a boundary point or interior point of $W$.

**Definition 5.** A 3-manifold $M$ in $Y \times I \times I$ is transverse to the binding $B$ if each component $W$ of $M \cdot B$ is neighborhood-bicollared in $M$ by the set $W \times [-1, 1]$, and the two one-sided collars $W \times [-1, 0)$ and $W \times (0, 1]$ lie in two distinct pages $P_i$ and $P_j$ of $Y \times I \times I$.

If a 3-manifold $M$ is transverse to the binding in $Y \times I \times I$, then each component $W$ of $M \cdot B$ is an $n$-punctured disk in $B$, since $W$ is a connected 2-manifold subset of $B$. This fact is important, in that it provides us with an innermost curve of $bdM \cdot B$ on which we will later perform surgery.

**Lemma 3.** Any 3-manifold $M$ in $Y \times I \times I$ may be placed transverse to the binding. That is, if $M$ is embedded in $Y \times I \times I$, then there exists another embedding of $M$ in $Y \times I \times I$ which is transverse to the binding.

**Proof.** In one lower dimension, an easy example shows the difficulty and how to avoid it. Consider the subdisk $D$ of $T \times I$, for $T$ a triod, consisting of all of page 1, the top half of page 2, and the bottom half of page 3. Then $D$ is a 2-manifold in $T \times I$ which is not transverse to the binding. The difficulty is that the midpoint $m$ of the binding is a "bad" point, in that any neighborhood of $m$ in $D$ is "3-paged." Note that $m$ is a boundary point on the disk $D$.

On the other hand, at any interior point, the disk $D$ is "2-paged." Our plan to avoid the difficulty caused by 3-paged points is to push $D$ into its own interior, thus making it locally "2-paged." The new disk may be placed in transverse position by resorting to the usual notion of general position in a 2-paged (Euclidean space) setting. Fortunately, this idea is valid in the higher dimensional situation as well:

**Definition 6.** A subset $X$ of $B \cdot M$ is called 2-paged if there are two integers $i$ and $j$ such that an open neighborhood of $X$ in $M$ is contained in $P_i + P_j + B$.

Every point $p$ of $B \cdot \text{Int}M$ is 2-paged. To see this, suppose every open neighborhood of $p$ intersects the $i$th page $P_i$. There exists an open neighborhood $N_i$ of $p$ in the "closed $i$th page" $P_i + B$ such that $N_i$ lies entirely in $\text{Int}M$. The existence of three distinct such neighborhoods $N_i$, $N_j$, and $N_k$ would imply that $M$ must contain the set $T \times I \times I$, for $T$ a triod. Of course, no 3-manifold can contain such a set. In fact, this argument shows that the two integers $i$ and $j$ that we associate with the point $p$ are distinct; furthermore, $i$ and $j$ do not change if we move $p$ in the binding to any nearby point $q$. Thus, every component $C$ of $B \cdot \text{Int}M$ is 2-paged in $Y \times I \times I$.

We now push $M$ into its own interior. That is, select any homeomorphic copy $M^*$ of $M$ such that $M^*$ lies in $\text{Int}M$. Let $C^*$ denote a component of $B \cdot M^*$. Since $C^*$ is a subset of a component $C$ of $B \cdot \text{Int}M$, there exists
a regular neighborhood $R^*$ of $C^*$ in $M^*$ such that $R^*$ lies in two pages of $Y \times I \times I$. That is, there exist two distinct integers $i$ and $j$ such that $R^*$ lies in $P_i + P_j + B$.

The 3-manifold $R^*$ is now moved into general position with respect to $B$ in the set $P_i + P_j + B$. Here, we regard $P_i + P_j + B$ as a cube in Euclidean 3-space, and we use the term “general position” in its usual sense in 3-space (no three points colinear, no four points coplanar). In general position, $R^*$ intersects $B$ transversally. Observe that this move of $R^*$ into general position with respect to $B$ may be entirely performed in a small neighborhood of $B$, so that it extends to an embedding of the 3-manifold $M^*$ in $Y \times I \times I$. The set $C^*$ has been replaced by transverse intersection of $M^*$ with $B$.

This procedure is now repeated for every component of $B \cdot M^*$.

This completes the proof of Lemma 3.

IV. Classification of surgeries as “reducing” and “enlarging”

Our general plan is to reduce the number of simple closed curves in which $bdM$ intersects the binding $B$. This is accomplished by a sequence of “reducing surgeries”. This reduction procedure finally yields a manifold $M'$ which is disjoint from the binding. Each reducing surgery causes either $b_0(M)$ or $b_2(M)$ to increase by 1, but $b_1(M)$ remains zero throughout. Thus, the fully reduced manifold $M_f$, which embeds in 3-space, must be a punctured ball sum. The inverse of reducing surgeries is called “enlarging surgeries”. We show that each enlarging surgery preserves punctured ball sum, so we may retrace our steps from $M_f$ back to $M$, proving that $M$ is a punctured ball sum. Note that the manifold itself is not being reduced in any sense by this procedure. The complexity of placement of $M$ in $Y \times I \times I$, that is, the intersection of $bdM$ with $B$ in $Y \times I \times I$, is being reduced.

Definition 7. We define two types of reducing surgery.

2-handle addition: Let $D \times I$ be the product of a disk and an interval. Suppose that $Int D \times I$ is disjoint from $M$, and $bdD \times I$ lies in $bdM$. The reduced manifold $M'$ is defined by $M' = M + Int D \times I$.

1-handle subtraction: Let $D \times I$ be the product of a disk and an interval. Suppose that $Int D \times I$ lies in $Int M$, and $bdD \times I$ lies in $bdM$. The reduced manifold $M'$ is defined by $M' = M - Int D \times I$.

Lemma 4. Both types of reducing surgery preserve the property that the first betti number is zero. That is, if $b_1(M) = 0$ before reducing surgery, then $b_1(M') = 0$ after reducing surgery.

Proof. Adding a 2-handle to $M$ may be viewed homologically as attaching a disk $D$ to $M$ along $bdD$. This disk does not add nontrivial elements to the first homology group.

Since $b_1(M) = 0$, any 1-handle in $M$ separates $M$, so $b_1$ is also preserved under 1-handle subtraction.
**Definition 8.** We define two types of *enlarging surgery*. They are the two inverse operations of the two reducing surgeries. In the notation of Definition 7, they turn $M'$ back into $M$.

2-handle subtraction: Let $D \times I$ be the product of a disk and an interval. Suppose that $D \times \text{Int } I$ lies in $\text{Int } M'$, and $D \times \text{bd } I$ lies in $\text{bd } M'$. The enlarged manifold $M$ is defined by $M = M' - D \times \text{Int } I$.

1-handle addition: Let $D \times I$ be the product of a disk and an interval. Suppose that $D \times \text{Int } I$ is disjoint from $M'$, and $D \times \text{bd } I$ lies in $\text{bd } M'$. The enlarged manifold $M$ is defined by $M = M' + \text{Int } D \times I$.

**Lemma 5.** Let $M'$ be a punctured ball sum. Let $M'$ be altered by enlarging surgery, forming $M$. If $b_1(M) = 0$, then $M$ is a punctured ball sum.

**Proof.** If $M$ is formed from $M'$ by 2-handle subtraction, then the 2-handle lies in a component of $M'$, a punctured ball $P$. Furthermore, this 2-handle connects two different components of $\text{bd } P$, because $b_1(M) = 0$. Surgery changes $P$ to a punctured ball with one less puncture.

If $M$ is formed by 1-handle addition, then the 1-handle must connect two different components of $M'$ as a boundary connected sum, because $b_1(M) = 0$. The boundary connected sum of two punctured balls is a punctured ball. This completes the proof of Lemma 5.

**V. Conclusion**

We now assemble the various Lemmas into a proof.

**Proof of Version IV of the theorem.** Let $M$ be a 3-manifold which embeds in $Y \times I \times I$, for some $n$-odd $Y$. By Lemma 3, we place $M$ transverse to the binding in $Y \times I \times I$. Starting with the innermost simple closed curve of $\text{bd } M \cdot B$, we perform reducing surgery on $M$, which terminates with $M_r$, with $M_r$ disjoint from the binding. Since $b_1(M) = 0$, Lemma 4 asserts that $b_1(M_r) = 0$. Since $M_r$ is disjoint from the binding in $Y \times I \times I$, any component of $M_r$ is embedded in $I \times I \times I$. Furthermore, each component of $M_r$ has $b_1 = 0$. Lemma 1 tells us that each component is a punctured ball, so $M_r$ is a punctured ball sum. Lemma 5 guarantees that $M$ is a punctured ball sum. This completes the proof of the Theorem.

**References**


**Department of Mathematics, University of California, Los Angeles, California 90024-1766**