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A NOTE ON SUBDIAGONALITY FOR TRIANGULAR AF ALGEBRAS

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ABSTRACT. We consider the analogue, for triangular AF algebras, of the notion of subdiagonality for subalgebras of von Neumann algebras. We show that a subalgebra $\mathscr{T}$ of the AF algebra $\mathscr{A}$ is subdiagonal if and only if it is strongly maximal.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathscr{A}$ be an AF algebra, and let $\mathscr{D}$ be a maximal abelian self-adjoint subalgebra (masa) of $\mathscr{A}$, obtained through the Stratila–Voiculescu diagonalization process (cf. [SV, Theorem I.1.10]). A norm closed subalgebra $\mathscr{T}$ of $\mathscr{A}$ is said to be triangular AF if $\mathscr{T} \cap \mathscr{T}^* = \mathscr{D}$. J. Peters, Y. Poon, and B. Wagner have introduced two different notions of maximality for triangular AF algebras (cf. [PPW, Proposition 2.24. and Example 3.25.]). One of them is strong maximality, which we now recall. $\mathscr{T}$, a triangular subalgebra of $\mathscr{A}$, is said to be strongly maximal if there is a sequence $\{\mathscr{A}_n\}_{n=1}^{\infty}$ of finite-dimensional subalgebras of $\mathscr{A}$ such that $\mathscr{A} = \overline{(\bigcup_{n=1}^{\infty} \mathscr{A}_n)}$ and $\mathscr{T} \cap \mathscr{A}_n$ is a maximal triangular algebra in $\mathscr{A}_n$.

In this note we consider an analogue, for triangular AF algebras, of the notion of subdiagonal subalgebra of a von Neumann algebra, introduced by W. Arveson in [A]. Our definition follows.

Definition 1.1. Let Φ be the conditional expectation of $\mathscr{A}$ with respect to $\mathscr{D}$. A norm closed subalgebra $\mathscr{S}$ of $\mathscr{A}$ containing $\mathscr{D}$ is said to be subdiagonal if

1. $\mathscr{S} + \mathscr{S}^*$ is norm dense in $\mathscr{A}$.
2. $\Phi(TS) = \Phi(T)\Phi(S)$ for all S, $T \in \mathscr{S}$.

We say that $\mathscr{S}$ is maximal subdiagonal if $\mathscr{S}$ is not contained in any larger subalgebra of $\mathscr{A}$ containing $\mathscr{D}$, on which Φ is multiplicative.

Subdiagonal subalgebras of von Neumann algebras play a very important role

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in the theory of triangular algebras developed by P. Muhly, K-S Saito, and B. Solel in [MSS].

Our motivation for introducing Definition 1.1 is to consider maximal subdiagonality as an alternative for the two notions of maximality that already exist. We found out that this is not the case. In fact, the main result of this note asserts that a subalgebra $\mathscr{T}$ of an AF algebra $\mathscr{A}$ is strongly maximal triangular if and only if it is maximal subdiagonal.

After this paper had been submitted for publication the author learned that P. Muhly and B. Solel, in [MS], have obtained a characterization of the maximal C^*-subdiagonal subalgebras of the C^*-algebras associated with amenable, r-discrete, principal groupoids (cf. [MS, Theorem 4.2]) that leads to an alternate proof of our Theorem 2.1.

For the remainder of this paper, $\mathscr{A}$ will denote an AF algebra, $\{\mathscr{A}_n\}_{n=1}^{\infty}$ will be a sequence of finite-dimensional subalgebras of $\mathscr{S}$ such that $\mathscr{A} = \overline{(\bigcup_{n=1}^{\infty} \mathscr{A}_n)}$ and $\mathscr{D} = \overline{(\bigcup_{n=1}^{\infty} \mathscr{A}_n \cap \mathscr{D})}$. Φ will denote the conditional expectation of $\mathscr{A}$ with respect to $\mathscr{D}$ and $\mathscr{A}_k^{(n)}$, $k = 1, \dots, N(n)$, will be the central factors of $\mathscr{A}_n$, that is $\mathscr{A}_n = \oplus_{k=1}^{N(n)} \mathscr{A}_k^{(n)}$, and $\mathscr{A}_k^{(n)}$ is $*$-isomorphic to $M_{N(n,k)}$, the $N(n,k) \times N(n,k)$ matrices.

$$\left\{E_{ij}^{(n,k)} : 1 \le i, j \le N(n,k), \ 1 \le k \le N(n), \ n = 1, 2, \dots\right\}$$

will be a system of matrix units consistent with the diagonalization of $\mathscr{A}$ by $\mathscr{D}$. This means that $\{E_{ij}^{(n,k)} : 1 \le i, j \le N(n,k)\}$ is a system of matrix units for $\mathscr{A}_k^{(n)}$, and that any $E_{ij}^{(n,k)}$ is the sum of some matrix units $E_{pq}^{(n+1,r)}$. That this choice is always possible, follows from [SV, Proposition I.1.8].

2. Subdiagonality and strong maximality

Theorem 2.1. *Let $\mathscr{T}$ be a subalgebra of $\mathscr{A}$. Then $\mathscr{T}$ is strongly maximal triangular iff $\mathscr{T}$ is maximal subdiagonal.*

Proof. Let us first assume that $\mathscr{T}$ is strongly maximal triangular subalgebra of $\mathscr{A}$, and that $\mathscr{A}_n$, $n = 1, 2, \dots$, satisfies that $\mathscr{T} \cap \mathscr{A}_n$ is maximal triangular in $\mathscr{A}_n$. Then $(\mathscr{T} \cap \mathscr{A}_n) + (\mathscr{T}^* \cap \mathscr{A}_n) = \mathscr{A}_n$. Therefore $\mathscr{T} + \mathscr{T}^*$ is norm dense in $\mathscr{A}$. We finish the proof of this implication in the next two lemmas.

Lemma 2.2. *If $\mathscr{T}$ is strongly maximal triangular, then Φ is multiplicative on $\mathscr{T}$.*

Proof. If $i \neq j$, then

$$\Phi_n\left(E_{ij}^{(n,k)}\right) = \sum_{r=1}^{N(n)} \sum_{p=1}^{N(n,r)} E_{pp}^{(n,r)} E_{ij}^{(n,k)} E_{pp}^{(n,r)} = 0$$

and since $\Phi_{n+1}|_{\mathscr{A}_n} = \Phi_n|_{\mathscr{A}_n}$ [SV, I.1.3], we obtain that $\Phi(E_{ij}^{(n,k)}) = 0$.

Let $E_{ij}^{(n,k)}$ and $E_{jl}^{(n,k)} \in \mathscr{T} \cap \mathscr{A}_n$. If $i = j$, then $E_{jj}^{(n,k)} E_{jl}^{(n,k)} = E_{jl}^{(n,k)}$, and

$$\Phi\left(E_{jj}^{(n,k)}\right)\Phi\left(E_{jl}^{(n,k)}\right) = E_{jj}^{(n,k)}\Phi\left(E_{jl}^{(n,k)}\right) = \begin{cases} 0 & \text{if } j \neq l, \\ E_{jj}^{(n,k)} & \text{if } j = l. \end{cases}$$

Thus $\Phi(E_{jj}^{(n,k)}E_{jl}^{(n,k)}) = \Phi(E_{jj}^{(n,k)})\Phi(E_{jl}^{(n,k)})$. If $i \neq j$, then $l \neq i$, otherwise $E_{ij}^{(n,k)}$ would belong to $\mathscr{T} \cap \mathscr{T}^* = \mathscr{D}$. Thus $\Phi(E_{il}^{(n,k)}) = 0$, and $\Phi(E_{ij}^{(n,k)}) \times\Phi(E_{lj}^{(n,k)}) = 0$, since $\Phi(E_{ij}^{(n,k)}) = 0$. It follows that Φ is multiplicative on the matrix units of $\mathscr{A}$ contained in $\mathscr{T}$. Thus Φ is multiplicative on $\bigcup_{n=1}^{\infty} \mathscr{A}_n \cap \mathscr{T}$. Therefore, by continuity of Φ and the fact that $\mathscr{T}$ is a $\mathscr{D}$-bimodule, Φ is multiplicative on $\mathscr{T}$ (cf. [P, Lemma 1.3]). □

Lemma 2.3. *Let $\mathscr{T}$ be a strongly maximal triangular subalgebra of $\mathscr{A}$. If $\mathscr{S}$ is a subdiagonal subalgebra of $\mathscr{A}$ containing $\mathscr{T}$, then $\mathscr{S} = \mathscr{T}$.*

Proof. $\mathscr{S}$ is a $\mathscr{D}$-bimodule, so [P, Lemma 1.3] implies that

$$\mathscr{S} = \left(\overline{\bigcup_{n=1}^{\infty} \mathscr{S} \cap \mathscr{A}_n}\right).$$

Thus, if $\mathscr{T} \subseteq \mathscr{S}$ and $\mathscr{T} \neq \mathscr{S}$, there exists some n such that $\mathscr{T} \cap \mathscr{A}_n$ is properly contained in $\mathscr{S} \cap \mathscr{A}_n$. Since $\mathscr{T} \cap \mathscr{A}_n$ is maximal triangular in $\mathscr{A}_n$, there is a matrix unit $E_{ij}^{(n,k)}$ with $i \neq j$ such that $E_{ij}^{(n,k)}$, $E_{ji}^{(n,k)} \in \mathscr{S} \cap \mathscr{A}_n$. But then

$$\Phi\left(E_{ij}^{(n,k)}E_{ji}^{(n,k)}\right) = E_{ii}^{(n,k)} \quad \text{and} \quad \Phi\left(E_{ij}^{(n,k)}\right)\Phi\left(E_{ji}^{(n,k)}\right) = 0,$$

contradicting that $\mathscr{S}$ is subdiagonal. □

Now we prove the converse. Let $\mathscr{S}$ be a maximal subdiagonal subalgebra of $\mathscr{A}$, and let $\{\mathscr{A}_n\}_{n=1}^{\infty}$ be a sequence of finite-dimensional subalgebra of $\mathscr{A}$ and let $C^*(\mathscr{A}_n, \mathscr{D}_m)$ denote the C^*-algebra generated by $\mathscr{A}_n$ and $\mathscr{D}_m = \mathscr{A}_m \cap \mathscr{D}$.

Lemma 2.4. *Given n there exists $m(n)$ such that $C^*(\mathscr{A}_n, \mathscr{D}_{m(n)}) \cap \mathscr{S}$ is triangular and $\mathscr{A}_n \subseteq (C^*(\mathscr{A}_n, \mathscr{D}_{m(n)}) \cap \mathscr{S}) + (C^*(\mathscr{A}_n, \mathscr{D}_{m(n)}) \cap \mathscr{S}^*)$.*

Proof. Let $E_{ij}^{(n,k)}$ be a matrix unit in $\mathscr{A}_n$. We pointed out in the proof of Lemma 2.3, that $\mathscr{S} = (\overline{\bigcup_{n=1}^{\infty} \mathscr{S} \cap \mathscr{A}_n})$. Thus, given ε such that $0 < \varepsilon < 1$, there is m and A, $B \in \mathscr{S} \cap \mathscr{A}_m$ such that

$$\left\| E_{ij}^{(n,k)} - \left(A + B^*\right) \right\| < \varepsilon.$$

Let $E_{ij}^{(n,k)} = \sum_{p=1}^{N} F_p$, where the F_p's belong to the set $\{E_{q,r}^{(m,s)} : 1 \leq q, r \leq N(m,s), 1 \leq s \leq N(m)\}$. Recall that these matrix units are chosen as in [SV, I.1.8], so $F_p \in C^*(\mathscr{A}_n, \mathscr{D}_m)$. Since $\mathscr{S}$ is a $\mathscr{D}$-bimodule, we have that $(F_pF_p^*)A(F_p^*F_p)$ and $(F_p^*F_p)B(F_pF_p^*)$ belong to $\mathscr{S}$.

From

$$\left\|F_p - \left(F_pF_p^*\right)\left(A+B^*\right)\left(F_p^*F_p\right)\right\| \\ = \left\|\left(F_pF_p^*\right)\left(E_{ij}^{(n,k)} - \left(A+B^*\right)\right)\left(F_p^*F_p\right)\right\|$$

and $\|(F_pF_p^*)(E_{ij}^{(n,k)} - (A+B^*))(F_p^*F_p)\| < \varepsilon$ and since $E_{ij}^{(n,k)}$ is the restriction of a permutation to a subset of the minimal projections of $\mathscr{D}_m$, we obtain that

$$\left\|E_{ij}^{(n,k)} - \sum_{p=1}^{N}\left(F_pF_p^*\right)\left(A+B^*\right)\left(F_p^*F_p\right)\right\| < \varepsilon .$$

It follows that we can assume that $A = \sum_{p=1}^{N}\lambda_pF_p$ and $B = \sum_{p=1}^{N}\mu_pF_p$. In this case, the fact that $E_{ij}^{(n,k)}$ is a partial permutation of the minimal projections of $\mathscr{D}_m$, yields that $|1-(\lambda_p+\mu_p)| < \varepsilon$. Hence, either $\lambda_p \neq 0$ or $\mu_p \neq 0$. If $\lambda_p \neq 0$ and $\mu_p \neq 0$, we would have that $F_p \in \mathscr{S} \cap \mathscr{S}^* \cap \mathscr{A}_m$. But in that case, the argument given in the proof of Lemma 2.3 will show that Φ is not multiplicative on $\mathscr{S}$, a contradiction. We conclude that $\lambda_p \neq 0$ (resp. $\mu_p \neq 0$) implies that $\mu_p = 0$ (resp. $\lambda_p = 0$).

Therefore, given $p \in \{1, \dots, N\}$, either $F_p \in \mathscr{S}$ or $F_p \in \mathscr{S}^*$, but we cannot have both. It follows that $E_{ij}^{(n,k)} \in (\mathscr{S} \cap C^*(\mathscr{A}_n, \mathscr{D}_m)) + (\mathscr{S}^* \cap C^*(\mathscr{A}_n, \mathscr{D}_m))$.

Since there are only finitely many matrix units in $\mathscr{A}_n$, we can find $m(n)$ such that

$$\mathscr{A}_n \subseteq \left(\mathscr{S} \cap C^*\left(\mathscr{A}_n, \mathscr{D}_{m(n)}\right)\right) + \left(\mathscr{S}^* \cap C^*\left(\mathscr{A}_n, \mathscr{D}_{m(n)}\right)\right) .$$

In this case, every matrix unit in $\{E_{q,r}^{(m(n),s)} : 1 \le q, r \le N(m(n), s),\ 1 \le s \le N(m(n))\}$ that is in $C^*(\mathscr{A}_n, \mathscr{D}_{m(n)})$ appears with coefficient one in the expansion of some $E_{ij}^{(n,k)}$. Therefore $\mathscr{S} \cap C^*(\mathscr{A}_n, \mathscr{D}_{m(n)})$ is triangular. □

Let $\mathscr{B}_n = C^*(\mathscr{A}_n, \mathscr{D}_{m(n)})$. Lemma 2.4 implies that $\mathscr{B}_n \cap \mathscr{S}$ is triangular and that

$$\mathscr{B}_n = (\mathscr{S} \cap \mathscr{B}_n) + \left(\mathscr{S}^* \cap \mathscr{B}_n\right) .$$

It follows that $\mathscr{S} \cap \mathscr{B}_n$ is maximal triangular in $\mathscr{B}_n$. We have that $\bigcup_{n=1}^{\infty}\mathscr{A}_n \subseteq \bigcup_{n=1}^{\infty}\mathscr{B}_n$. Therefore $\mathscr{S}$ is strongly maximal. □

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