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$L(X, C(K))$ AS A DUAL SPACE

T.S.S.R.K. RAO

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ABSTRACT. We exhibit a class of Banach spaces X, with X^* having nontrivial centralizer for which the space of operators $L(X, C(K))$ is a dual space implies that K is hyperstonian.

INTRODUCTION

It is well known that the space $C(K)$ of continuous functions on a compact Hausdorff space K is isometrically isomorphic to the dual of a Banach space iff K is hyperstonian (see [5, §11]. So for a hyperstonian space K, if Y denotes the predual of $C(K)$ then for any Banach space X, the space of bounded linear operators $L(X, C(K))$ has the projective tensor product space $X \hat{\otimes} Y$ as a predual (see [4]). In the converse direction, Behrends proved recently, [2] that for a Banach space X with centralizer of X^* trivial, if $L(X, C(K))$ is a dual space then K is hyperstonian. In this short note we exhibit several classes of Banach spaces with X^* having nontrivial centralizer for which $L(X, C(K))$ is a dual implies that K is hyperstonian.

The notation, terminology and results of M-structure theory that we will be using here can be found in [1].

Main result. Our results are based on the following interesting observation of Lima [7, Proposition 6.3]).

If P is an L-projection in the Banach space X, then the map $T \to T \circ P$ is an M-projection in $L(X, Y)$.

Theorem 1. *Let X be a Banach space having a one dimensional L-ideal. For any Banach space Y, if $L(X, Y)$ is a dual space then Y is a dual space.*

Proof. Let $x_0 \in X$, $\|x_0\| = 1$, and $\operatorname{span}\{x_0\}$ an L-ideal in X. We can choose a $f_0 \in X^*$ such that $f_0(x_0) = 1 = \|f_0\|$ and such that the projection $P: x \to f_0(x)x_0$ is the L-projection from X onto $\operatorname{span}\{x_0\}$. Now using the observation of Lima, we see that $Q: T \to T \circ P$ is an M-projection in $L(X, Y)$. For any $y \in Y$, the operator $y \otimes f_0$ is in the range of Q and for any $T \in \operatorname{range} Q$, $T = y_0 \otimes f_0$, where $y_0 = Tx_0$. Hence Y is isometric to the range of Q.

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Now if $L(X, Y)$ is a dual space then since the range of an M-projection in a dual space is an ω^*-closed subspace, we see that Y is isometric to a dual space.

Remark. Clearly for the X considered above, the centralizer of X^* is nontrivial. An infinite dimensional L-space with an extreme point in the unit ball (say $X = l^1$), is an example of such an X, with X^* having an infinite dimensional centralizer.

Corollary. *For a Banach space X as in the above theorem, $L(X, C(K))$ is a dual space iff K is hyperstonian.*

Theorem 2. *Let X be a Banach space, not containing an isomorphic copy of $L^1[0, 1]$. If $L(X, C(K))$ is a dual then K is hyperstonian.*

Proof. It follows from the observations made by Li [6, Proposition 8] that there exists a family of L-ideals $\{X_a\}_{a\in A}$ in X such that no X_a has a nontrivial L-ideal and X is the l^1-direct sum of the spaces $\{X_a\}_{a\in A}$. Since X_a has no nontrivial L-ideals, the centralizer of X_a^* is trivial (see [1]). Fix $a \in A$ and let P_a denote the L-projection from X onto X_a. As in the proof of Theorem 1, we can see that $T \to T \circ P_a$ is an M-projection whose range is isometric to $L(X_a, C(K))$. Consequently, if $L(X, C(K))$ is a dual space, we can conclude that $L(X_a, C(K))$ is a dual space for each a and since X_a^* has trivial centralizer an appeal to the result of Behrends mentioned before, gives us that K is hyperstonian.

Remark 1. A brief list of spaces satisfying the hypothesis of the above theorem includes spaces with unconditional basis, spaces with the R.N.P. or w.R.N.P. and spaces with the dual having the R.N.P. or w.R.N.P.

Remark 2. It follows from Proposition 5.3 of [1] that if X^* is such that the centralizer is finite dimensional, then X has a finite decomposition of the kind, considered above. If X is such that X^* has no isomorphic copy of c_0, then the centralizer of X^* is finite dimensional [1, Corollary 4.24] and hence for such an X, $L(X, C(K))$ is a dual iff K is hyperstonian.

Theorem 3. *Let X be a Banach space with an extreme point in the unit ball. If $L(X, C(K))$ is a dual space, then K is hyperstonian.*

Proof. We may assume that X has nontrivial L-ideals. Let e be an extreme point in the unit ball of X and let N denote the smallest L-ideal containing e. Then either $N = \operatorname{span}\{e\}$ or N has no nontrivial L-ideals. In either case the desired conclusion can be obtained by using arguments given during the proofs of Theorems 1 and 2.

Remark. The identification of $L(X, C(K))$ as the space of X^*-valued, w^*-continuous functions on K, equipped with the supremum norm, shows the connection between this work and the recent work of Cambern and Greim [3], where they settle the uniqueness question for the preduals of $L(X, C(K))$, when X is reflexive. It can also be deduced from the remarks made in [3, §3]

that when X is either an $L^1(\mu)$ (μ, nonatomic) or contains an $L^1(\mu)$ as an L-ideal, then $L(X, C(K))$ is a dual space implies that K is hyperstonian.

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INDIAN STATISTICAL INSTITUTE, R. V. COLLEGE POST OFFICE, BANGALORE–560 059, INDIA