A LUECKING-TYPE SUBSPACE OF $\mathcal{L}_a^1$ AND ITS DUAL

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Abstract. The purpose of this investigation is to determine the extent to which Luecking's decomposition of Bergman spaces $\mathcal{L}_a^p$, $(1 < p < \infty)$ [4] can be extended to $\mathcal{L}_a^1$. The set of functions for which an atomic decomposition (using the reproducing kernel of the Bergman space) is possible turns out to be only a small part of $\mathcal{L}_a^1$. In this note we equip each of such functions with a new norm and study the resulting Banach space. We describe its dual and predual.

1. Notation

The Bergman space $\mathcal{L}_a^p$ $(1 \leq p < \infty)$ consists of analytic functions for the open unit disk $\mathbb{D}$ whose $p$th powers are integrable with respect to $dA$, the normalized area measure of $\mathbb{D}$. Of particular interest to us is the space $\mathcal{L}_a^1 = \{f : f : \partial D \to \mathbb{C}, \int_0 |f(z)|^2 \, dA(z) < \infty\}$ and its reproducing kernel $k_\alpha(z) = 1/(1 - \alpha z)^2$, where $\alpha \in \mathbb{D}$. For more details see [2, p. 2].

If $f(z) = \sum_{n=1}^{\infty} a_n k_{\lambda_n}(z)$, where $\{\lambda_n\} \subseteq \mathbb{D}$ and $\sum |a_n| k_{\lambda_n} < \infty$, then clearly $f \in \mathcal{L}_a^1$ and $\|f\|_1 \leq \sum_{n=1}^{\infty} |a_n| k_{\lambda_n}$. If we write

$$\|f\|_* = \inf \sum_{n=1}^{\infty} |a_n| k_{\lambda_n},$$

where the infimum is taken over the set of all such decompositions of $f$, then $\|f\|_* \geq \|f\|_1$, $\|k_{\lambda_n}\|_1 \leq \|k_{\lambda_n}\|_1$ for all $\lambda$ and hence $Y$ is the completion of the set of such functions in the $\| \cdot \|_*$ norm is a subspace of $\mathcal{L}_a^1$. We call $X = \{f, f : \mathbb{D} \to \mathbb{C} \text{ analytic}, \sup_{|\alpha| < 1} |f(\alpha)| k_{\alpha}^{-1} < \infty\}$. Note that $X$ is complete in the norm $\|f\|_* = \sum_{\alpha \in \mathbb{D}} |f(\alpha)| k_{\alpha}^{-1}$ and $X_0 = \{f \in X, \lim_{|\alpha| \to 1} |f(\alpha)| k_{\alpha}^{-1} = 0\}$ is a closed subspace of $X$. (In both cases, the proof is perfectly standard, and hence it is omitted.) The Bloch space $B$...
consists of analytic functions \( \{ g, g : D \to \mathbb{C}, \sup_{z \in D} (1 - |z|^2)|g'(z)| < \infty \} \), and the little Bloch space \( \mathcal{B}_0 \) consists of those functions in \( \mathcal{B} \) for which \( \lim_{|z| \to 1} (1 - |z|^2)|g'(z)| = 0 \). We conclude this section with the remark that as a simple consequence of the well-known fact that \( (\mathcal{L}_a^1)^* = \mathcal{B} \), it follows that whenever \( g \in \mathcal{B} \) we have \( g(\alpha) = \langle g, k_\alpha \rangle \) and hence \( |g(\alpha)| \|k_\alpha\|^{-1} \leq \|g\|_a \). In other words, \( \mathcal{B} \subseteq X \) and \( \|g\|_* \leq \|g\|_a \) whenever \( g \in \mathcal{B} \). It is also true that \( \mathcal{B}_0 \subseteq X_0 \), but the proof requires a little more detail and we defer it to the next section.

2.

In this section we start out by giving an intrinsic description of \( Y \). The description turns out not to be particularly surprising, but it enables us to show that \( Y \) is the dual of \( X_0 \) and predual of \( X \). This in turn allows us to conclude that \( X_0 \) is indeed the closure of \( \mathcal{B}_0 \) (and hence of polynomials) in the \( \| \|_* \) norm on analytic functions on the disk.

**Proposition 1.** \( Y = \{ f(z) = \int_D k_\alpha(z) d\mu(\alpha), \text{ where } \mu \text{ is a Borel measure on } D \text{ with } \|\mu\|_1(\partial D) = 0 \text{ and } \int_D \|k_\alpha\|_1 d\mu(\alpha) < \infty \} \text{ with } \|f\|_* = \inf_{\mu} \{ \int_D \|k_\alpha\|_1 \times d\mu(\alpha) \}, \) where the infimum is taken over all such decompositions of \( f \).

The only nontrivial implication that any function \( f(z) \) of the form specified in the proposition can be approximated in the \( \| \|_{*} \) norm by discrete sums follows from standard measure-theoretic arguments. We sketch the proof for completeness.

The topological argument in our proof is based on the Whitney covering lemma [7, I, Theorem 3, p. 16, VI, Theorem 1, p. 167].

**Lemma 0.** Let \( F \) be a nonempty closed set in \( \mathbb{R}^n \). Then its complement \( \Omega \) is the union of a sequence of rectangles \( Q_k \) whose sides are paralleling to the axes, whose interiors are mutually disjoint, and whose diameters are approximately proportional to their distances from \( F \).

**Corollary 1.** Given \( \epsilon > 0 \) there exists a family of disks \( \{ D_k \}_{k=1}^\infty \), \( D_k = D_k(\lambda_k, r_k) \), and a disk centred at \( \lambda_k \) and having radius \( r_k \), satisfying the following conditions:

1. \( D_k \subseteq D, \quad |D \setminus \bigcup_{k=1}^\infty D_k| = 0, \) where \( | \| \) denotes area measure.
2. \( D_k \cap D_\ell = \emptyset \) if \( k \neq \ell \).
3. \( \text{diam} D_k = 2r_k \leq \min\{ \epsilon \text{ dist}(D_k, \partial D) \}. \)

**Proof.** We apply the covering lemma recursively, starting with \( F = \mathbb{R}_2 \setminus D \). Subdividing the rectangles further if necessary, we may choose a family of rectangles \( \{ R_1, R_2, \ldots \} \) with disjoint interiors, so that they fill up \( D \) and \( \text{diam} R_k \leq \min\{ \epsilon, \text{ dist}(R_k, \partial D) \} \). We now subdivide each rectangle \( R_k \) by means of disks in the following simple way; we have a family of disks \( \{ D_k^{(1)} \}_{k=1}^\infty \subseteq R_k \), \( | \bigcup_k D_k^{(1)} | > \frac{1}{2}|R_k| \), as shown in Figure 1.
By successive applications of the lemma to \( \mathbb{R}^2 \setminus \bigcup_{j=1}^{n} \bigcup_{k} D^{(j)} \), we obtain a countable family of open disks \( \{ D^{(j)} \}_{j \geq 1} \) such that \( \left| \mathbb{R}^2 \setminus \bigcup_{j=1}^{n} \bigcup_{k} D^{(j)} \right| < \left( \frac{3}{4} \right)^{n} \pi \). Simply renumerating them, we have a countable family of disks \( \{ D_k \} \) which satisfy the three conditions stated above.

**Lemma 1.** If \( f \) belongs to the disk algebra \( A(\mathbb{D}) \), then for any \( \epsilon > 0 \) there exists sequences \( \{ a_n \}_{n=1}^{\infty} \subseteq \mathbb{C} \) and \( \{ \lambda_n \} \subseteq \mathbb{D} \) such that \( \sum_{n=1}^{\infty} |a_n||k_{\lambda_n}|_1 < \infty \) and \( \| f - \sum_{n=1}^{\infty} a_nk_{\lambda_n} \| < \epsilon \).

**Proof.** Elementary but central to our proof is the computation of \( \| k_{\alpha} \|_1 \) where \( k_{\alpha} \) is the reproducing kernel for the Bergman space. Note that

\[
\| k_{\alpha} \|_1 = \left( \frac{1}{1 - \alpha} \right), \ \text{where} \ \langle \cdot, \cdot \rangle \text{ denotes the usual inner product in } \mathcal{S}^{a}_{\infty}(\mathbb{D}).
\]

So

\[
\| k_{\alpha} \|_1 = \left\| \sum_{n \geq 0} \alpha^n z^n \right\|_{\mathcal{S}^{a}_{\infty}}^2 = \sum_{n \geq 0} |\alpha|^n \| z^n \|_{\mathcal{S}^{a}_{\infty}}^2 = \sum_{n \geq 0} \frac{|\alpha|^{2n}}{n+1},
\]

and

\[
(1) \quad \| k_{\alpha} \|_1 \sum_{n \geq 0} \frac{|\alpha|^{2n}}{n+1} = |\alpha|^{-2} \ln(1 - |\alpha|^2)^{-1}, \quad |\alpha| \neq 0,
\]

Consequently,

\[
M = \int_{\mathbb{D}} \| k_{\alpha} \|_1 \, dA(\alpha) = \sum_{n \geq 0} \frac{1}{(n+1)(2n+1)} < \infty.
\]

Now let \( f \in A(\mathbb{D}) \). For given \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( z_1, z_2 \in \mathbb{D} \) and \( |z_1 - z_2| < \delta \) implies \( |f(z_1) - f(z_2)| < \epsilon/M \). For this \( \epsilon \) we choose a family of disks \( D_k \) centred at \( \lambda_k \) and having radius \( r_k \) satisfying the three conditions stated in Corollary 1. Note that \( f \in \mathcal{S}^{a}_{\infty}(\mathbb{D}) \), and hence

\[
f(z) = \int_{\mathbb{D}} f(\alpha)k_{\alpha}(z) \, dA(\alpha) = \sum_{k=1}^{\infty} \int_{D_k} k_{\alpha}(z)f(\alpha) \, dA(\alpha) \quad \text{(by condition (1) of the previous corollary)}
\]

\[
= \sum_{k=1}^{\infty} f(\lambda_k) \int_{D_k} k_{\alpha}(z) \, dA(\alpha) + \sum_{k=1}^{\infty} f(\lambda_k) \int_{D_k} k_{\alpha}(z) \, dA(\alpha).
\]

The second term clearly = \( \sum_{k=1}^{\infty} (f(\lambda_k)r_k^2)k_{\lambda_k}(z) \).
If we write \( a_k = f(\lambda_k) r_k^2 \), then
\[
 f(z) - \sum_{k=1}^{\infty} a_k k_{\lambda_k}(z) = \sum_{k=1}^{\infty} \int_{D_k} (f(\alpha) - f(\lambda_k)) k_{\alpha}(z) \, dA(\alpha).
\]

First we show that \( \sum_{k=1}^{\infty} |a_k||k_{\lambda_k}|_1 < \infty \). Observe that by condition (3) of Corollary 1 we may choose constants \( C_1, C_2 \) \((C_1, C_2 > 0, C_1 < 1, C_2 > 1)\) such that
\[
 C_1 < \frac{1 - |\lambda_k|^2}{1 - |z|^2} < C_2
\]
for \( z \in D_k \), whenever \( k \geq 1 \). It follows that \( |\ln(1 - |\lambda_k|^2)| \leq |\ln(1 - |z|^2)| + C \)
for \( z \in D_k \), \( k \geq 1 \). Hence
\[
\sum_{k=1}^{\infty} |a_k||k_{\lambda_k}|_1 = \sum_{k=1}^{\infty} |f(\lambda_k)| r_k^2 |\lambda_k|^{-2} |\ln(1 - |\lambda_k|^2)|
\]
\[
\leq \|f\|_\infty \left[ \sum_{k \geq 1, |\lambda_k| < \frac{1}{2}} r_k^2 |\lambda_k|^{-2} |\ln(1 - |\lambda_k|^2)| + \sum_{k \geq 1, |\lambda_k| \geq \frac{1}{2}} r_k^2 |\lambda_k|^{-2} |\ln(1 - |\lambda_k|^2)| \right]
\]
\[
\leq \|f\|_\infty \left[ \sup_{0 < x < 1/2} \frac{1 - x^2}{x^2} \sum_{k=1}^{\infty} r_k^2 + 4 \sum_{k=1}^{\infty} \int_{D_k} (|\ln(1 - |z|^2)| + C) \, dA(z) \right]
\]
\[
< \infty.
\]

It remains to show that
\[
g(z) = \sum_{k=1}^{\infty} \int_{D_k} (f(\alpha) - f(\lambda_k)) k_{\alpha}(z) \, dA(\alpha) \in Y \quad \text{and} \quad \|g\|_* < \epsilon.
\]

Note \( g(z) = \int_D k_\alpha(z) \, d\mu(\alpha) \) where \( d\mu(\alpha) = \sum_{k \geq 1} (f(\alpha) - f(\lambda_k)) x_{D_k}(\alpha) \, dA(\alpha) \).
Hence \( \int_D |k_\alpha|_1 \, d|\mu(\alpha) \leq \frac{1}{M} \int_D ||k_\alpha||_1 \, dA(\alpha) = \epsilon \). We remark that, as a trivial consequence of this argument, we have \( A(\mathbb{D}) \subseteq Y \).

The next lemma shows that \( A(\mathbb{D}) \) is in fact dense in \( Y \).

**Lemma 3.** Suppose \( f(z) = \int_{\mathbb{D}} k_\alpha(z) \, d\mu(\alpha) \in Y \) and, for \( \delta > 0 \), let \( f_\delta(z) = \int_{|z| \leq 1 - \delta} k_\alpha(z) \, d\mu(\alpha) \). Then \( f_\delta \in A(\mathbb{D}) \) and \( \|f - f_\delta\|_* \to 0 \) as \( \delta \to 0 \).

**Proof.** \( f(z) - f_\delta(z) = \int_{1 - \delta < |z| < 1} k_\alpha(z) \, d\mu(\alpha) \). Clearly \( f_\delta \in Y \) and \( \|f - f_\delta\|_* \to 0 \) as \( \delta \to 0 \). In order to show that \( f_\delta \in A(\mathbb{D}) \), simply note that \( k_\alpha(z) = \sum_{n \geq 0} (n + 1) \alpha^n z^n \) and
\[
|f_\delta(z) - \sum_{n=0}^{N} (n + 1) \alpha^n d\mu(\alpha) z^n| \leq \sum_{n=0}^{N} \int_{|z| \leq 1 - \delta} (n + 1) |\alpha|^n \, d|\mu(\alpha) \]
\[
\leq C \sum_{n=N+1}^{\infty} (n + 1)(1 - \delta)^n
\]
which is the tail-end of a series converging to \(1/\delta(1-\delta)\) and \(C = \int_{|\alpha| \leq 1-\delta} d|\mu|(|\alpha|)\). Thus \(f_\delta\) is a uniform limit of polynomials and hence belongs to \(A(\mathbb{D})\).

**Proposition 2.** The dual of \(X_0\) can be identified with \(Y\). More precisely, every bounded linear functional on \(X_0\) is the form \(h \mapsto (h, f) - \int_{\mathbb{D}} h(z)\overline{f}(z)\,dA(z)\) for a unique \(f \in Y\), and the norm of the linear functional on \(X_0\) induced by \(f\) is equivalent to \(\|f\|_*\).

**Proof.** Suppose \(f(z) = \int_{\mathbb{D}} k_\alpha(z)\,d\mu(\alpha) \in Y\) and \(h \in X\). Note that if \(h \in X\) and \(\alpha \in \mathbb{D}\) is fixed, \(|h(z)k_\alpha(z)| \leq C|h(z)| \leq C\|k_\alpha\|_1\) and, as shown at the beginning of the proof of Lemma 1 \(\|k_\alpha\|_1 \in L^1(|d\mu|)\). Hence by the standard argument \(h(\alpha) = (h, k_\alpha) = \int_{\mathbb{D}} h(z)k_\alpha(z)\,dA(z)\). Now

\[
(h, f) = \int_{\mathbb{D}} \int_{\mathbb{D}} h(z)k_\alpha(z)\,dA(z)\,d\overline{\mu}(\alpha)
= \int_{\mathbb{D}} h(\alpha)\,d\overline{\mu}(\alpha),
\]

and we have \(|(h, f)| \leq \int_{\mathbb{D}} |h(\alpha)|\|k_\alpha\|_1\,d|\mu|(\alpha)\) for all \(\mu\) which represent \(f\). Hence \(|(h, f)| \leq \|h\|_*\|f\|_*\). Conversely, suppose that \(L\) is a bounded linear functional on \(X_0\). Define \(S: X_0 \rightarrow C(\mathbb{D})\) by \(Sh(\alpha) = h(\alpha)\|k_\alpha\|_1^{-1}\) if \(\alpha \in \mathbb{D}\) and \(Sh(\alpha) = 0\) if \(\alpha \in \partial \mathbb{D}\), and let \(L_1(g) = L \circ S^{-1}(g)\) whenever \(g \in S(X_0)\). Note that by definition of \(X_0\) and \(\|\cdot\|_*\), \(S\) is an isometry and hence \(\|L_1\| = \|L\|\). By the standard applications of Hahn–Banach and Riesz representation theorems, we have a measure \(\mu\) on \(\mathbb{D}\) (cutting it down to \(\mathbb{D}\) if necessary) such that \(L(h) = \int_{\mathbb{D}} \|k_\alpha\|_1^{-1}h(\alpha)\,d\mu(\alpha)\) and \(|\mu|(|\mathbb{D}|) \leq \|L\|\).

If we define \(f(z) = \int_{\mathbb{D}} k_\alpha(z)\|k_\alpha\|_1^{-1}\,d\overline{\mu}(\alpha)\), then \(f \in Y\), \(\|f\|_* \leq \|L\|\), and \(L(h) = \int_{\mathbb{D}} h(z)\overline{f}(z)\,dA(z)\). Uniqueness follows from the fact that \(f\) is analytic and \(X_0\) contains polynomials.

**Corollary 2.** \(X_0\) is the closure of the little Bloch space \(B_0\) (in fact, of polynomials) in the \(\|\cdot\|_*\) norm.

**Proof.** If \(L\) is in \(X_0^*\) and \(L(z^n) = 0\) for \(n \geq 0\), then the unique function \(f \in Y\) which represents \(L\) is clearly identically zero. This shows that \(X_0\) is the closure of polynomials, and hence of \(B_0\).

**Remark 1.** \(B_0\) is clearly a proper subset of \(X_0\). The easiest way to see this is to note that \(\|k_\alpha\|_1^{-1} \rightarrow 0\) as \(|\alpha| \rightarrow 1\), and hence \(H^\infty(\mathbb{D}) \subseteq X_0\). However there exists an infinite Blaschke product which does not belong to \(B_0\) [6]. A natural question is whether the Bloch space \(B\) is dense in \(X\). That remains open.

**Corollary 3.** The unit ball of \(Y = (Y)_1 = \Gamma^{-w^*}\) where

\[
\Gamma = \left\{ \sum c_i k_{\alpha_i}/\|k_{\alpha_i}\|_1 : \sum |c_i| \leq 1 \right\}
\]

**Proof.** Recall that \(S: X_0 \rightarrow C_0(\overline{\mathbb{D}})\) is an isometry from \(X_0\) into the set of continuous functions on \(\overline{\mathbb{D}}\) which vanish on \(\partial \mathbb{D}\), and hence \(S^*: C_0(\mathbb{D})^* \rightarrow X_0^* = Y\) is a partial isometry onto \(Y\). It is well known that \(((C_0(\overline{\mathbb{D}}))^*)_1\) is
Proposition 3. The dual of $Y$ can be identified with $X$. More precisely, every bounded linear functional on $Y$ is of the form $f \mapsto \langle f, h \rangle = \int_{\Omega} f(z) \overline{h}(z) \, dA(z)$ for a unique $h \in X$, and the norm of the linear functional induced by $h$ is equivalent to $\|h\|_*$. 

Proof. It is clear from the first part of the proof of Proposition 2 that any function $h \in X$ induces a bounded linear functional $L$ on $Y$ and $\|L\| \leq \|h\|_*$. 

Conversely, suppose that $L$ is a bounded linear functional on $Y$, and define $h(\alpha) = \overline{L}(k_{\alpha})$ for $\alpha \in \mathbb{D}$. Note that $\|k_{\alpha}\|_* = \|k_{\alpha}\|_1$; hence $\sup_{\alpha \in D} \{\|k_{\alpha}\|_1^{-1} \|h(\alpha)\|\} \leq \|L\|$. Next we prove that $h$ is analytic. As a first step in this proof we claim that $h$ is continuous. Fix $\alpha \in \mathbb{D}$ and choose a positive $r < (1 - |\alpha|/2)$. If $D(\alpha, r)$ is the Euclidean disk of radius $r$ centred at $\alpha$, then $L(\int_{D(\alpha, r)} k_{\lambda}(z) \, dA(\lambda)) = r^2 h(\alpha)$. To see this, let $p = X - s$, write

$$L(\int_{D(\alpha, r)} k_{\lambda}(z) \, dA(\lambda)) = \overline{L}(r^2 k_{\alpha}(z)) = r^2 h(\alpha),$$

hence, $L(\int_{D(\alpha, r)} k_{\lambda}(z) \, dA(\lambda)) = \overline{L}(\int_{D(\alpha, r)} k_{\lambda}(z) \, dA(\lambda)) = \overline{L}(r^2 k_{\alpha}(z)) = r^2 h(\alpha)$. 

Now if $\beta \in \mathbb{D}$, choose $r$ such that $0 < r < (1 - |\alpha|)/2, (1 - |\beta|)/2$, and write $D_1 = D(\alpha, r), D_2 = D(\beta, r)$. Then

$$r^2(h(\alpha) - h(\beta)) = \overline{L} \left( \int_{D_1} k_{\lambda}(z) \, dA(\lambda) - \int_{D_2} k_{\lambda}(z) \, dA(\lambda) \right)$$

$$= \overline{L} \left( \int_{D_1 - D_2} k_{\lambda}(z) \, dA(\lambda) - \int_{D_1 - D_1} k_{\lambda}(z) \, dA(\lambda) \right)$$

$$= r^2 |h(\alpha) - h(\beta)|$$

$$\leq \|L\| \left( \left\| \int_{D_1 - D_2} k_{\lambda}(z) \, dA(\lambda) \right\|_* + \left\| \int_{D_1 - D_1} k_{\lambda}(z) \, dA(\lambda) \right\|_* \right)$$

$$\leq \|L\| |C| D_1 \Delta D_2|$$

where $C = \sum_{\lambda \in D_1 \cup D_2} \{\|k_{\lambda}\|_1, \lambda \in D_1 \cup D_2\}$ and $D_1 \Delta D_2$ is the usual symmetric difference. Clearly $h(\beta) \to h(\alpha)$ as $\beta \to \alpha$, and $h$ is continuous. Now if $\Gamma$ is a closed path lying inside $\mathbb{D}$, then $\int_{\Gamma} k_{\lambda} \, d\lambda = 0$ and $\sup \{\|k_{\lambda}\|_* = \|k_{\lambda}\|_1, \lambda \in \Gamma\} < \infty$. Hence $\int_{\Gamma} L(k_{\lambda}) \, d\lambda = \int h(\lambda) \, d\lambda = 0$, and by Morera's theorem $h$ is analytic. Thus $h \in X$.

It remains to show that $h$ induces $L$. It was shown in Proposition 2 that $h(\alpha) = \int_{\mathbb{D}} h(z) k_{\alpha}(z) \, dA(z) = \langle h, k_{\alpha} \rangle = \langle k_{\alpha}, h \rangle$, so by definition of $h(\alpha)$ we have $L(k_{\alpha}) = \langle k_{\alpha}, h \rangle$. If $f(z) = \sum_{n=1}^{\infty} a_n k_{\lambda_n}$, $\sum_{n \geq 1} |a_n| \|k_{\lambda_n}\| < \infty$, then $L(f) = \sum_{n=1}^{\infty} a_n \langle k_{\lambda_n}, h \rangle$ by continuity of $L$. As we know that $h \in X$ implies
that \( g \mapsto \langle g, h \rangle \) is continuous on \( Y \), \( \langle f, h \rangle = \sum_{n \geq 1} a_n \langle k_{x_n}, h \rangle \), and hence \( L(f) = \langle f, h \rangle \). By Proposition 1 discrete sums like \( f \) are dense in \( Y \) and the proof is complete.

**Remark 2.** One trivial consequence of this is to show that \( Y \neq \mathcal{L}_a^1(dA) \). If this were the case, then we would have \( X = \mathcal{B} \), making the two norms equivalent. This in turn would imply that \( \mathcal{B}_0 = X_0 \), which was shown to be false. (See Remark 1). Another standard consequence of this duality is that \( X_0 \) is \( wk-* \)-dense in \( X \). It now follows from Corollary 3 that \( \mathcal{B}_0 \) and hence \( \mathcal{B} \) are \( wk-* \)-dense in \( X \). As remarked earlier, we have not been able to determine whether \( \mathcal{B} \) is norm-dense in \( X \). Also, is \( X_0 \subseteq \mathcal{B} \)?

As a consequence of this proposition, we now have a stronger version of Corollary 3.

**Corollary 4.** The unit ball of \( Y = (Y)_1 = \overline{\Gamma} \) where

\[
\Gamma = \left\{ \sum_{i=1}^n c_i k_{x_i} / \|k_{x_i}\|_1, \sum_i |c_i| \leq 1 \right\},
\]

and hence

\[
Y = \left\{ \sum_i a_i k_{x_i}, \sum_i |a_i| \|k_{x_i}\|_1 < \infty \right\};
\]

i.e., the discrete sums are complete with respect to the norm \( \| \cdot \|_* \).

**Proof.** Suppose \( f \in (Y)_1 \). Without loss of generality, we may write \( f(z) = \int_B k_{x}(z) \, d\mu_{\gamma}(\alpha) \) where \( \int_B \|k_{x}\|_1 \, d\mu_{\gamma}(\alpha) \leq 1 \). We claim that whenever \( L \in Y^* \), \( |L(f)| \leq \sup_{\alpha} |L(k_{x}/\|k_{x}\|_1)| \). By Proposition 3, and \( L \in Y^* \) is induced by a function \( h \in X \), \( L(f) = \int_B \int_B k_{x}(z) \, h(z) \, dA(z) \) and \( |L| \sim \|h\|_* \). In particular, \( L(f) = \int_B \int_B k_{x}(z) \, h(z) \, dA(z) \) and \( \|L\| \leq \sup_{\alpha} |L(k_{x}/\|k_{x}\|_1)| \) since \( \int_B \|k_{x}\|_1 \, d\mu_{\gamma}(\alpha) \leq 1 \). This proves the claim. Now if there exists \( f \in (Y)_1 \) which does not belong to \( \overline{\Gamma} \) then by the Hahn–Banach theorem we may choose \( L \in Y^* \) such that \( \Re L(f) > \sup\{\Re L(g), g \in \Gamma\} \). Note that \( g \in \Gamma \) implies \( e^{i\theta} g \in \Gamma \) for all real \( \theta \); hence \( \sup\{\Re L(g), g \in \Gamma\} = \sup\{|L(g)|, g \in \Gamma\} \). Thus \( |L(f)| \leq \sup\{\|L(k_{x})/\|k_{x}\|_1\|, \|L\| \}, \) which is clearly a contradiction. Now if \( f = (Y) \), write \( f = f_1 + g_1, \|f_1\|_* \leq 1/2, g_1 \in \Gamma \), and note that \( 2f_1 \in (Y)_1 \). By induction, we obtain a sequence of \( g_i \)s in \( \Gamma \) such that \( f = \sum_{i=1}^\infty 2^{-i+1} g_i \).

**Remark 3.** It is also natural to ask how the space \( Y \) relates to \( \mathcal{L}_a^1(d\nu) \) where \( \nu \) is the weighted area measure given by \( d\nu(z) = \|k_{x}\|_1 \, dA(z) \). It is easy to see that any function \( h \in \mathcal{L}_a^1(d\nu) \) induces a bounded linear functional \( f \mapsto \langle f, h \rangle \) on \( X_0 \). This coupled with the duality in Proposition 2 clearly implies that \( h \in Y \). Thus \( \mathcal{L}_a^1(d\nu) \subseteq Y \subseteq \mathcal{L}_a^1(dA) \). In particular \( \nu \) is not a Carleson measure in \( D \) [2, p. 26]. Thus, \( \mathcal{L}_a^1(d\nu) = Y \) is equivalent to the seemingly weaker condition \( \sup_{\lambda \in D} \|k_{\lambda}\|_{\mathcal{L}_a^1(d\nu)}/\|k_{\lambda}\|_1 < \infty \). Note that \( \nu \) being
a circular measure, it is easy to show \( \| k_\lambda \|_{L^p(D_
u)} = \sum_{n \geq 0} \left\{ \sum_{k \geq 1} \left( \frac{1}{k(k+n)} |\lambda|^2 n \right) \right\} \), whereas \( \| k_\lambda \|_1 = \sum_{n \geq 0} (n+1)^{-1} |\lambda|^{2n} \). As \( \sum_{k \geq 1} [k(k+n)]^{-1} \sim (\log n)/n \), an elementary computation shows that \( \sup_{\lambda \in D} \| k_\lambda \|_{L^p(D_
u)} / \| k_\lambda \|_1 = \infty \).

More generally, does there exist a positive measure \( \lambda \) on \( D \) s.t. \( H^\lambda \| k^\lambda \|_1 < M \| k \|_1 \) for all \( z \in D \) but \( \lambda \) is not a Carleson measure?

In conclusion, we remark that it is possible to define a proper subspace of \( \mathcal{S}^2 \), suitably renormed so that functions in \( X \) (respectively \( X_0 \)) induce continuous (completely continuous) Hankel operators on it. We intend to pursue this and related questions in a forthcoming paper.

Added in proof. The answer to our question at the end of Remark 3 is negative if \( \lambda \) is a circular measure [8, p. 257]. We thank the referee for bringing [8] to our notice.

References