JONES AND $Q$ POLYNOMIALS FOR 2-BRIDGE KNOTS AND LINKS

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Dedicated to Professor Yoko Tao on her sixtieth birthday

Abstract. It is known that the $Q$ polynomial of a 2-bridge knot or link can be obtained from the Jones polynomial. We construct arbitrarily many 2-bridge knots or links with the same $Q$ polynomial but distinct Jones polynomials.

The Jones polynomial $V_L(t) \in \mathbb{Z}[t^{\pm \frac{1}{2}}]$ [5] is an invariant of the isotopy type of an oriented knot or link $L$ in the 3-sphere. The writhe $w(D)$ of an oriented planar diagram $D$ of $L$ is the sum of the signs at all the crossings of $D$, according to the convention explained in Figure 1 (p. 836). Let $|D|$ be a diagram $D$ with its orientation unknown. Then Kauffman's bracket polynomial $\langle D \rangle \in \mathbb{Z}[A^{\pm 1}]$ [10] of $|D|$, which is a regular isotopy invariant, is defined by

$$
\langle \emptyset \rangle = 1 \text{ for a simple closed curve } \emptyset ,
$$
$$
\langle D' \rangle = (-A^2 - A^{-2})\langle D \rangle ,
$$
$$
\langle D_{\pm} \rangle = A^{\pm 1}\langle D_0 \rangle + A^{-1}\langle D_{\infty} \rangle ,
$$

where $|D'|$ is the disjoint union of $|D|$ and a simple closed curve and $|D_0|$ are identical diagrams except near one point, where they are as in Figure 2 (p. 836). Then the Jones polynomial can be defined using the formula

$$
V_L(A^4) = (-A^3)^{-w(D)}\langle D \rangle .
$$

The $Q$ polynomial $Q_L(x) \in \mathbb{Z}[x^{\pm 1}]$ [1, 4] is an invariant of the isotopy type of an unoriented knot or link $|L|$ in the 3-sphere, which is defined by the following formulas:

$$
Q_U(x) = 1 \text{ for the unknot } U ,
$$
$$
Q_{L_+}(x) + Q_{L_-}(x) = x(Q_{L_0}(x) + Q_{L_{\infty}}(x)) ,
$$

where the links $|L_i|$ have diagrams $|D_i|$ which are as in the above.

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The author [7] discovered a relation between the Jones and $Q$ polynomials of a 2-bridge knot or link (see, for example, [2, Chapter 12]).

**Theorem** *. If $L$ is a 2-bridge knot or link, then it holds that

$$Q_L(x) = 2x^{-1}V_L(t)V_L(t^{-1}) + 1 - 2x^{-1},$$

where $x = -t - t^{-1}$. Identically,

$$Q_L(x) = 2x^{-1} \langle D \rangle \langle D' \rangle + 1 - 2x^{-1},$$

where $x = -A^4 - A^{-4}$, $D$ is a diagram for $L$, and $D'$ is a mirror image of $D$, so that $\langle D' \rangle (A) = \langle D \rangle (A^{-1})$.

This theorem implies that the $Q$ polynomial of a 2-bridge knot or link can be deduced from the Jones polynomial. Conversely, even if a $Q$ polynomial of some 2-bridge knot or link is given, we cannot necessarily infer its Jones polynomial. In fact, through a computer calculation of polynomial invariants of 2-bridge knots and links [9], except for a reflection such as right- and left-handed trefoils, we found many pairs of 2-bridge knots and links with the same $Q$ polynomial but distinct Jones polynomials; \{10_{14}, 10_{31}\}, \{10_{19}, 10_{36}\}, and \{9_4^2, 9_{10}^2\} are such pairs in the table of [15]. Also it is known that there exist arbitrarily many skein-equivalent 2-bridge knots [8] and links [6], which thus have the same 2-variable Jones, Jones, Alexander, and $Q$ polynomials. (See [12] for the definition of skein equivalence and 2-variable Jones polynomial.)
Generalizing these examples, we prove

**Theorem.** For any positive integer $N$, there exist $N$ sets of $2^N$ 2-bridge knots (resp. links) $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_N$ with $\mathcal{H}_i = \{K_{i1}, K_{i2}, \ldots, K_{i2^n}\}$ such that

(i) all the knots (resp. links) in the union $\bigcup_{i=1}^{N} \mathcal{H}_i$ share the same $Q$ and Alexander (resp. 2-variable Alexander) polynomials;

(ii) all the knots (resp. links) in each $\mathcal{H}_i$ are skein-equivalent, and so they have the same 2-variable Jones, Jones, and Alexander (resp. reduced Alexander) polynomials; and

(iii) all the knots (resp. links) $K_{11}, K_{21}, \ldots, K_{N1}$ have mutually distinct Jones polynomials.

1. Preliminaries

Let $\alpha$ be a 3-braid. We denote a 3-braid $\alpha S^n_2 \alpha^{-1}$ and $\alpha S^n_2 \alpha^{-1} S^n_1 \alpha$, $m$, $n \in \mathbb{Z}$, by $\alpha(n)$ and $\alpha(m, n)$, respectively, where $S_1$ and $S_2$ are elementary 3-braids as shown in Figure 3. Let $G_\alpha$ and $H_\alpha$ be unoriented 2-bridge knot or link diagrams as shown in Figure 4. From [8, Proposition 2.4], we have

**Lemma 1.**

(i) $\langle H_{\alpha(n)} \rangle = A^n [d - (1 - (A^{-4})^n) d^{-1} \langle G_\alpha \rangle \langle G_\alpha \rangle]$,  

(ii) $\langle G_{\alpha(m, n)} \rangle = A^{m+n} \langle G_\alpha \rangle [(A^{-4})^m + (A^{-4})^n - 1 + \{1 - (A^{-4})^m\} \times \{1 - (A^{-4})^n\} d^{-2} \langle G_\alpha \rangle \langle G_\alpha \rangle]$,  

where $d = A^2 - A^{-2}$ is the bracket polynomial of a trivial 2-component link diagram without any crossing.

Let $Q_\alpha(x)$, $Q_{\alpha(n)}(x)$, and $Q_{\alpha(m, n)}(x)$ be the $Q$ polynomials of unoriented 2-bridge knots or links with diagrams $G_\alpha$, $H_\alpha(n)$, and $G_{\alpha(m, n)}$, respectively.

![Figure 3](https://example.com/figure3.png)
Then we have

Lemma 2.

(i) \( Q_{\alpha(n)}(x) = \frac{1}{2}(\sigma_{n+1} - \sigma_{n-1})(\mu - \mu^{-1}Q_{\alpha}(x)^2) + \mu^{-1}Q_{\alpha}(x)^2, \)

(ii) \( Q_{\alpha(1,-1)}(x) = (x^2/4)(Q_{\alpha}(x) + 1)^2(Q_{\alpha}(x) + \mu) - \mu, \)

where \( \sigma_n \in \mathbb{Z}[x^{\pm1}] \) is the polynomial defined by \( \sigma_{n-1} + \sigma_{n+1} = x\sigma_n, \sigma_0 = 0, \sigma_1 = 1, \) and \( \mu = 2x^{-1} - 1 \) is the \( Q \) polynomial of a trivial 2-component link.

Proof.

(i) is Lemma 6.1 of [6].

(ii) By Lemma 1 (ii),

\[ \langle G_{\alpha(1,-1)} \rangle = \langle G_{\alpha} \rangle(-A^{-4} - 1 - A^4 + \langle G_{\alpha} \rangle\langle G_{\alpha} \rangle). \]

By substituting \( \langle G_{\alpha} \rangle\langle G_{\alpha} \rangle = \frac{x}{2}(Q_{\alpha}(x) + \mu) \) (Theorem *) and \( A^{-4} + A^4 = -x, \)

this becomes

\[ \langle G_{\alpha(1,-1)} \rangle = \frac{x}{2}\langle G_{\alpha} \rangle(Q_{\alpha}(x) + 1), \]

from which we have

\[ \langle G_{\alpha(1,-1)} \rangle = \frac{x}{2}\langle G_{\alpha} \rangle(Q_{\alpha}(x) + 1). \]

Substituting these formulas into \( \langle G_{\alpha(1,-1)} \rangle\langle G_{\alpha(1,-1)} \rangle = \frac{x}{2}(Q_{\alpha(1,-1)}(x) + \mu) \)

(Theorem *), we obtain (ii).

Using Lemma 2, we can prove the following by induction:

Lemma 3.

\[ Q_{\alpha(m,n)}(x) = \frac{1}{4\mu^2}(\sigma_{m+1} - \sigma_{m-1} - 2)(\sigma_{n+1} - \sigma_{n-1} - 2)Q_{\alpha}(x)^3 \]

\[ - \frac{x}{4}(x + 2)\sigma_m \sigma_n Q_{\alpha}(x)^2 \]

\[ + \left\{ 1 - \frac{1}{4}(\sigma_{m+1} - \sigma_{m-1} - 2)(\sigma_{n+1} - \sigma_{n-1} - 2) \right\} Q_{\alpha}(x) \]

\[ + (\mu^2/4)(x + 2)\sigma_m \sigma_n. \]
Since $\sigma_{n+1} - \sigma_{n-1} = \sigma_{-n+1} - \sigma_{-n-1}$ and $\sigma_m \sigma_n = \sigma_{-m} \sigma_{-n}$, we have from this

**Lemma 4.** Let $\alpha$ and $\beta$ be 3-braids. If $Q_\alpha(x) = Q_\beta(x)$, then $Q_{\alpha(m,n)}(x) = Q_{\alpha(-m,-n)}(x) = Q_{\beta(m,n)}(x) = Q_{\beta(-m,-n)}(x)$.

**Remark 1.** We can also prove the following by induction:

$$(Q_\alpha(x) + \mu)(Q_{\alpha(m,n)}(x) + Q_{\alpha(m+n)}(x)) = (Q_{\alpha(m)}(x) + Q_\alpha(x))(Q_{\alpha(n)}(x) + Q_\alpha(x)).$$

**Remark 2.** Let $\alpha = S_2^2 S_1^2$. Then the 2-bridge knots and links $10_{14}, 10_{31}, 10_{19}, 10_{36}, 9_4^2, 9_2^2$ given above have diagrams $G_{\alpha(-2,-1)}, G_{\alpha(2,-1)}, G_{\alpha(2,1)}, G_{\alpha(-2,-1)}, G_{\alpha(1,1)}, G_{\alpha(-1,-1)}$, respectively.

Suppose that $\alpha$ is a pure 3-braid and $m, n$ are even integers. Let $\nabla_\alpha(z)$ and $\nabla_{\alpha(m,n)}(z)$ be the Conway polynomials of the 2-bridge knots with diagrams $G_\alpha$ and $G_{\alpha(m,n)}$, respectively. (See [3]. Substituting $z = t^{1/2} - t^{-1/2}$, we obtain the Alexander polynomials.) Then we can prove the following by induction:

**Lemma 5.** $\nabla_{\alpha(m,n)}(z) = \frac{mn}{4} \nabla_\alpha(z)^3 + \nabla_\alpha(z)$.

Suppose that $\alpha$ is a pure 3-braid and $m, n$ are even integers. Let $\nabla_\alpha(t_1, t_2)$ and $\nabla_{\alpha(m,n)}(t_1, t_2)$ be the Conway potential function of the 2-bridge links with diagrams $H_\alpha$ and $H_{\alpha S_\alpha^m S_{-\alpha}^n}$, respectively, where the links are oriented so that they coincide if $mn = 0$. (See [3]. Substituting $t_i$ for $t_i^{1/2}, i = 1, 2$, we obtain the 2-variable Alexander polynomial.) Then we can prove the following by induction:

**Lemma 6.** $\nabla_{\alpha(m,n)}(t_1, t_2) = \frac{mn}{4}(t_1 - t_1^{-1})^2(t_2 - t_2^{-1})\nabla_\alpha(t_1, t_2)^3 + \nabla_\alpha(t_1, t_2)$.

## 2. Proof of theorem

**Knot Case.** Let $\alpha = S_2^2 S_1^4$ and $\varepsilon_i u_i$ be either $(2\varepsilon_i, 4\varepsilon_i)$ or $(4\varepsilon_i, 2\varepsilon_i)$, $\varepsilon_i = \pm 1$. We define a pure 3-braid $\alpha[e_1 u_1, e_2 u_2, \ldots, e_p u_p]$ by

$$\alpha[e_1 u_1, e_2 u_2, \ldots, e_i u_i, e_{i+1} u_{i+1}] = (\alpha[e_1 u_1, e_2 u_2, \ldots, e_i u_i])(e_{i+1} u_{i+1}),$$

where we interpret $\alpha[e_1 u_1, e_2 u_2, \ldots, e_p u_p]$ as $\alpha$ if $p = 0$. Let $K_\alpha[e_1 u_1, e_2 u_2, \ldots, e_p u_p]$ be the oriented 2-bridge knot with diagram $G_\alpha[e_1 u_1, e_2 u_2, \ldots, e_p u_p]$, so $K_\alpha$ is $S_2$ in the table of [15]. Let $K_\alpha[e_1, e_2, \ldots, e_p]$ be the set of $2^p$ knots $K_\alpha[e_1, e_2, \ldots, e_p]$, $\varepsilon_i u_i = (2\varepsilon_i, 4\varepsilon_i)$ or $(4\varepsilon_i, 2\varepsilon_i)$, and $K_\alpha$ be the union $\bigcup_{\varepsilon_i = \pm 1} K_\alpha[e_1, e_2, \ldots, e_p]$, so $K_\alpha$ are skinequivalent by [8, Propositions 3.1 and 3.2], and mutually nonisotopic by [8, Lemma 3.1]. All the knots in $K_\alpha$ share the same $Q$ and Alexander polynomials by Lemmas 4 and 5. Since for a 2-bridge knot the minimal crossing number equals the maximal degree of the $Q$ polynomial plus one [11, 13], they have the same
minimal crossing number, which we denote by \( c_{\alpha,p} \), so \( c_{\alpha,0} = 5 \). We denote the Jones polynomial of \( K_{\alpha[e_1u_1, e_2u_2, \ldots, e_p u_p]} \) (resp. \( K_{\alpha} \)) by \( V_{\alpha; e_1, e_2, \ldots, e_p}(t) \) (resp. \( V_{\alpha}(t) = t - t^2 + 2t^3 - t^4 + t^5 - t^6 \)) and the writhe of \( G_{\alpha[e_1u_1, e_2u_2, \ldots, e_p u_p]} \) (resp. \( G_{\alpha} \)) by \( \omega_{\alpha; e_1, e_2, \ldots, e_p} \) (resp. \( \omega_{\alpha} = -6 \)). From Lemma 1 (ii), we have

\[
\langle G_{\alpha[e_1u_1, \ldots, e_p u_p, e u]} \rangle = \langle G_{\alpha[e_1, \ldots, e_p]} \rangle \left( A^{-16e} + A^{-8e} - 1 + (A^{-16e} - 1)(A^{-8e} - 1) \right)_{d=2}
\]

Because \( V_{\alpha; e_1, \ldots, e_p}(A^4) = (-A^3)^{-\omega_{\alpha; e_1, \ldots, e_p}} \langle G_{\alpha[e_1u_1, \ldots, e_p u_p]} \rangle \) and \( \omega_{\alpha; e_1, \ldots, e_p} = \omega_{\alpha} - 4(e_1 + \cdots + e_p) \), we have

\[
V_{\alpha; e_1, \ldots, e_p}(t) = t^e V_{\alpha; e_1, \ldots, e_p}(t) \left( t^e + t^{3e} - t^{5e} + (t^{-1})^2(t^{2e} + 1) V_{\alpha; e_1, \ldots, e_p}(t) V_{\alpha; e_1, \ldots, e_p}(t^{-1}) \right).
\]

Let \( R_{\alpha; e_1, \ldots, e_p} \) and \( r_{\alpha; e_1, \ldots, e_p} \) be the maximal and minimal degrees of \( V_{\alpha; e_1, \ldots, e_p}(t) \). Then \( c_{\alpha,p} = R_{\alpha; e_1, \ldots, e_p} - r_{\alpha; e_1, \ldots, e_p} \) \([10, 14, 16]\). It is easy to see that

\[
R_{\alpha; e_1, \ldots, e_p} = R_{\alpha; e_1, \ldots, e_p} + c_{\alpha,p} + 3e + 2, \quad \text{and} \quad r_{\alpha; e_1, \ldots, e_p} = R_{\alpha; e_1, \ldots, e_p} - 2c_{\alpha,p} + 3e - 2.
\]

Thus we have \( c_{\alpha,p+1} = 3c_{\alpha,p} + 4 = 3^{p+1}(c_{\alpha,0} + 2) - 2 = 7 \cdot 3^{p+1} - 2 \), and \( R_{\alpha; e_1, \ldots, e_p} = R_{\alpha} + (c_{\alpha,0} + 2)(3p - 1)/2 + 3(e_1 + \cdots + e_p) = (7 \cdot 3^p + 5)/2 + 3(e_1 + \cdots + e_p) \), which shows that at least \( p + 1 \) polynomials in \( V_{\alpha; e_1, \ldots, e_p}(t) \)'s are distinct.

**Remark 3.** \( \mathcal{H}_{\alpha; e_1} = \{K_{\alpha(2e_1, 4e_1)}, K_{\alpha(4e_1, 2e_1)}\} \), \( c_{\alpha,1} = 19 \), \( R_{\alpha; e_1} = 3e_1 + 13 \), \( r_{\alpha; e_1} = 3e_1 - 6 \).

\[
\mathcal{H}_{\alpha; e_1, e_2} = \{K_{\alpha(2e_1, 4e_1)(2e_2, 4e_2)}, K_{\alpha(2e_1, 4e_1)(4e_2, 2e_2)}, K_{\alpha(4e_1, 2e_1)(2e_2, 4e_2)}, K_{\alpha(4e_1, 2e_1)(4e_2, 2e_2)}\}.
\]

\( c_{\alpha,2} = 65 \), \( R_{\alpha; e_1, e_2} = 34 + 3(e_1 + e_2) \). We can see by a computer calculation that \( V_{\alpha; e_1, e_2}(t) \neq V_{\alpha; e_1, e_2}(t) \).

**Link case.** We change the definition of the pure 3-braid \( \alpha[e_1 u_1, e_2 u_2, \ldots, e_p u_p] \) as follows:

\[
\alpha = S_{S_2}^2 \cdot \ldots \cdot S_{S_2}^2; \alpha[e_1 u_1, \ldots, e_i u_i, e_{i+1} u_{i+1}] = \beta S_{S_2}^{2m} \beta^{-1} S_{S_2}^{2n} \beta,
\]

where \( \beta = [e_1 u_1, \ldots, e_i u_i] \) and \( e_{i+1} u_{i+1} = (2m, 2n) \). Consider the oriented 2-bridge link \( L_{\alpha[e_1 u_1, \ldots, e_i u_i]} \) having diagram \( H_{\alpha[e_1 u_1, \ldots, e_p u_p]} \). Then we can prove this case in the same way as the other, using lemmas similar to Lemmas 1–4 and Lemma 6.
Remark 4. If \( e_iu_i = (2e_i, 2e_i) \) and \( \alpha = S_2^{a_i}S_1^{b_i} \ldots S_2^{a_n}S_1^{b_n}, \ a_i, b_i = \pm 2 \), then each set \( \mathcal{H}_{\alpha; e_1, \ldots, e_n} \) consists of a single fibered 2-bridge knot. See, for example, [6, Lemma 6.3]. Thus we can prove: There exist arbitrarily many fibered 2-bridge knots (resp. links) with the same \( Q \) and Alexander polynomials but mutually distinct Jones polynomials.

REFERENCES


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