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ULTRAWEAKLY CLOSED ALGEBRAS AND PREANNIHILATORS

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ABSTRACT. We give an alternate description of algebras in the class of ultraweakly closed subspaces of $\mathscr{B}(\mathscr{H})$ via the preannihilator. We then apply this result to show that proper ultraweakly closed algebras of bounded operators on an infinite-dimensional Hilbert space $\mathscr{H}$ have infinite codimension. We also use this alternate description of algebras to say something the structure of rank-one operators in unicellular algebras.

We begin with some basic definitions and notation. For $\mathscr{H}$, an infinite-dimensional Hilbert space, let $\mathscr{B}(\mathscr{H})$ denote the set of all bounded linear operators on $\mathscr{H}$ and let $\mathscr{T}(\mathscr{H})$ denote the set of all trace-class operators on $\mathscr{H}$. Then $\mathscr{T}(\mathscr{H})$, equipped with the trace norm, is a Banach space whose dual is $\mathscr{B}(\mathscr{H})$. The duality is given by the linear functional on $\mathscr{T}(\mathscr{H}) \times \mathscr{B}(\mathscr{H})$ defined by

$$(t, x) \rightarrow \operatorname{tr}(tx) \text{ for } t \in \mathscr{T}(\mathscr{H}),\ x \in \mathscr{B}(\mathscr{H}),$$

where tr denotes the trace.

Thus we get a w^*-topology on $\mathscr{B}(\mathscr{H})$, which is also known as the ultraweak topology. The weak-operator topology (which we shall refer to as the weak topology) is actually weaker than the ultraweak topology, so all the results stated in this paper for ultraweakly closed subspaces are true for weakly closed subspaces. An excellent exposition of the role of this duality theory in invariant subspace theory is [1]. We shall follow the notation of [1], which the reader can consult for more background.

As usual, we can use the above duality to define preannihilators. For $\mathscr{M}$ an ultraweakly closed subspace of $\mathscr{B}(\mathscr{H})$, the preannihilator is

$$\mathscr{M}_\perp = \{t \in \mathscr{T}(\mathscr{H}) \mid \operatorname{tr}(tm) = 0 \text{ for all } m \in \mathscr{M}\}.$$

The codimension of $\mathscr{M}(\operatorname{codim}(\mathscr{M}))$ is the vector space dimension of $\{\mathscr{B}(\mathscr{H})/\mathscr{M}\}$. If we identify all infinite cardinals, then this is also equal to the vector space dimension of $\mathscr{M}_\perp$.

Given $x, y \in \mathscr{H}$, the operator $x \otimes y$ is the rank-one operator defined by

$$x \otimes y(z) = \langle z, y \rangle x \text{ for } z \in \mathscr{H}.$$

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Every rank-one operator is of this form, and $\operatorname{tr}(x \otimes y) = \langle x, y\rangle$.

Theorem 1. *An ultraweakly closed subspace $\mathscr{M} \subseteq \mathscr{B}(\mathscr{H})$ is an algebra if and only if*

$$\mathscr{M} \cdot \mathscr{M}_\perp \subseteq \mathscr{M}_\perp .$$

Proof. First, suppose $\mathscr{M}$ is an algebra. Let $a \in \mathscr{M}$ and $t \in \mathscr{M}_\perp$; then we must show that $at \in \mathscr{M}_\perp$. If $b \in \mathscr{M}$, then since $ba \in \mathscr{M}$,

$$\operatorname{tr}((at)b) = \operatorname{tr}(t(ba)) = 0.$$

Hence, $at \in \mathscr{M}_\perp$ so $\mathscr{M} \cdot \mathscr{M}_\perp \subseteq \mathscr{M}_\perp$.

Second, suppose $\mathscr{M} \cdot \mathscr{M}_\perp \subseteq \mathscr{M}_\perp$. Let $a, b \in \mathscr{M}$; then we must show that $ab \in \mathscr{M}$. If $t \in \mathscr{M}_\perp$, then since $bt \in \mathscr{M}_\perp$,

$$\operatorname{tr}(t(ab)) = \operatorname{tr}((bt)a) = 0.$$

Thus $ab \in (\mathscr{M}_\perp)^\perp = \mathscr{M}$, so $\mathscr{M}$ is an algebra. □

Note that another condition equivalent to those in Theorem 1 is that $\mathscr{M}_\perp \cdot \mathscr{M} \subseteq \mathscr{M}_\perp$ and that if $\mathscr{M}$ is unital then the conditions are equivalent to $\mathscr{M} \cdot \mathscr{M}_\perp = \mathscr{M}_\perp$.

Our first application of Theorem 1 is the following, which says that, among ultraweakly closed subspaces, algebras are small.

Theorem 2. *If $\mathscr{A}$ is a proper ultraweakly closed algebra in $\mathscr{B}(\mathscr{H})$ then*

$$\operatorname{codim}(\mathscr{A}) = \infty .$$

To prove this theorem, we consider the two cases, $\mathscr{A}$ intransitive and $\mathscr{A}$ transitive, separately. Since, intuitively, intransitive algebras are "smaller than" transitive algebras, the second case should be easier to prove. The existence of proper ultraweakly closed transitive algebras is as of now unknown. However, in [2], Loebl and Muhly produced an example of an ultraweakly non-selfadjoint reductive algebra which lends credence to the proposition that proper ultraweakly closed transitive algebras exist.

Proposition 3. *If $\mathscr{A}$ is an ultraweakly closed intransitive algebra then*

$$\operatorname{codim}(\mathscr{A}) = \infty .$$

Proof. The hypothesis of the theorem implies that there exists a nontrivial subspace $\mathscr{N}$ of $\mathscr{H}$ such that $\mathscr{A}\mathscr{N} \subseteq \mathscr{N}$. One of $\mathscr{N}$ or $\mathscr{N}^\perp$ is infinite-dimensional; first suppose it is $\mathscr{N}$. Then there exists an orthonormal sequence $\{x_i\}_{i=1}^\infty \subset \mathscr{N}$ and a unit vector $y \in \mathscr{N}^\perp$. Thus for $a \in \mathscr{A}$,

$$\operatorname{tr}((x_i \otimes y)a) = \operatorname{tr}(a(x_i \otimes y)) = \operatorname{tr}((ax_i) \otimes y) = \langle ax_i, y\rangle .$$

Since $\mathscr{N}$ is invariant for $\mathscr{A}$, we get that $\langle ax_i, y\rangle$ is zero. Hence $x_i \otimes y$ is in $\mathscr{A}_\perp$ for all $i = 1, 2 \ldots$. Clearly this is an infinite independent set in $\mathscr{A}_\perp$, so the proposition follows. In the case where $\mathscr{N}^\perp$ is infinite-dimensional, we can choose a unit vector x in $\mathscr{N}$ and an orthonormal sequence $\{y_i\}_{i=1}^\infty$ in $\mathscr{N}^\perp$, and the result follows similarly. □

Let $[\![\ ]\!]$ denote the closed linear span.

Proposition 4. *If $\mathscr{A}$ is a proper ultraweakly closed transitive algebra, then*

$$[\![\mathscr{A}_\perp x]\!] = \mathscr{H} \text{ for all nonzero } x \in \mathscr{H}.$$

Proof. By Theorem 1, $\mathscr{A} \cdot \mathscr{A}_\perp \subseteq \mathscr{A}_\perp$, so for all $x \in \mathscr{H}$

$$\mathscr{A}[\![\mathscr{A}_\perp x]\!] \subseteq [\![\mathscr{A}_\perp x]\!].$$

Thus $[\![\mathscr{A}_\perp x]\!]$ is an invariant subspace for $\mathscr{A}$. The transitivity of $\mathscr{A}$ implies that $[\![\mathscr{A}_\perp x]\!]$ is either 0 or $\mathscr{H}$. Also $\{x \in \mathscr{H} | [\![\mathscr{A}_\perp x]\!] = 0\}$ is an invariant subspace for $\mathscr{A}$ (since $\mathscr{A}_\perp \cdot \mathscr{A} \subseteq \mathscr{A}_\perp$). Thus, either $[\![\mathscr{A}_\perp x]\!] = 0$ for all $x \in \mathscr{H}$, or $[\![\mathscr{A}_\perp x]\!] = \mathscr{H}$ for all nonzero $x \in \mathscr{H}$. If $[\![\mathscr{A}_\perp x]\!] = 0$ for all $x \in \mathscr{H}$, then $\mathscr{A}_\perp x = 0$ for all $x \in \mathscr{H}$, so $\mathscr{A}_\perp = 0$. This implies that $\mathscr{A} = \mathscr{B}(\mathscr{H})$ and the proposition is established. □

Proposition 5. *If $\mathscr{A}$ is a proper ultraweakly closed transitive algebra, then*

$$\operatorname{codim}(\mathscr{A}) = \infty.$$

Proof. By Proposition 4, $[\![\mathscr{A}_\perp x]\!] = \mathscr{H}$ for all nonzero $x \in \mathscr{H}$. Thus $\mathscr{A}_\perp$ must be infinite-dimensional □

Theorem 2 now is a direct consequence of Propositions 3 and 5.

We give an application of Theorem 1 to unicellular algebras.

Definition. An algebra $\mathscr{A}$ is *unicellular* if the lattice of invariant subspace of $\mathscr{A}$ form a totally ordered set.

Usually the term *nest* is used to describe a totally ordered lattice of invariant subspaces of an algebra, and an algebra is called a *nest algebra* if it is unicellular and reflexive.

Definition. A subset $\mathscr{S}$ of operators has Property (**U**) if $x_1 \otimes y_1 \in \mathscr{S}$ and $x_2 \otimes y_2 \in \mathscr{S}$ implies that either $x_1 \otimes y_2 \in \mathscr{S}$ or $x_2 \otimes y_1 \in \mathscr{S}$.

Theorem 6. *Let $\mathscr{A}$ be a unital ultraweakly closed algebra in $\mathscr{B}(\mathscr{H})$. Then $\mathscr{A}$ is unicellular if and only if $\mathscr{A}_\perp$ has Property* (**U**).

Proof. Suppose $\mathscr{A}$ is unicellular. Let $x_1 \otimes y_1$ and $x_2 \otimes y_2$ be in $\mathscr{A}_\perp$. Then

$$\operatorname{tr}((x_i \otimes y_i)a) = \langle ax_i, y_i \rangle = 0 \text{ for all } a \in \mathscr{A}, \quad i = 1, 2.$$

Thus $[\![\mathscr{A} x_i]\!] \perp y_i$ for $i = 1, 2$. The unicellularity of $\mathscr{A}$ implies that either $[\![\mathscr{A} x_1]\!] \subseteq [\![\mathscr{A} x_2]\!]$ or $[\![\mathscr{A} x_2]\!] \subseteq [\![\mathscr{A} x_1]\!]$. If the former is true, then $[\![Ax_1]\!] \perp y_2$ so $x_1 \otimes y_2 \in \mathscr{A}_\perp$. If the latter is true, then $[\![\mathscr{A} x_2]\!] \perp y_1$, so $x_2 \otimes y_1 \in \mathscr{A}_\perp$.

Now suppose that $\mathscr{A}_\perp$ has Property (**U**). To show that $\mathscr{A}$ is unicellular, it is enough to show that the lattice of cyclic subspaces for $\mathscr{A}$ is totally ordered. (The cyclic subspaces of $\mathscr{A}$ are those of the form $[\![\mathscr{A} x]\!]$ where $x \in \mathscr{H}$). Let $[\![\mathscr{A} x_1]\!]$ and $[\![\mathscr{A} x_2]\!]$ be two cyclic subspaces for $\mathscr{A}$. If $[\![\mathscr{A} x_2]\!] \not\subseteq [\![\mathscr{A} x_1]\!]$, then there exists $y_1 \in [\![\mathscr{A} x_1]\!]^\perp \backslash [\![\mathscr{A} x_2]\!]^\perp$. For all $y \in [\![\mathscr{A} x_2]\!]^\perp$, $x_1 \otimes y_1$ and $x_2 \otimes y$ are in $\mathscr{A}_\perp$. However, $x_2 \otimes y_1 \notin \mathscr{A}_\perp$, and $\mathscr{A}_\perp$ has Property (**U**), so $x_1 \otimes y$ is in $\mathscr{A}_\perp$. Therefore $[\![\mathscr{A} x_1]\!] \perp y$, hence $[\![\mathscr{A} x_2]\!]^\perp \subseteq [\![\mathscr{A} x_1]\!]^\perp$, which implies that

$[\![\mathscr{A}x_1]\!] \subseteq [\![\mathscr{A}x_2]\!]$. If $[\![\mathscr{A}x_1]\!] \nsubseteq [\![\mathscr{A}x_2]\!]$ the result follows similarly. Thus the cyclic subspaces for $\mathscr{A}$, and hence all the invariant subspaces for $\mathscr{A}$, form a totally ordered lattice. So $\mathscr{A}$ is unicellular. □

Theorem 7. *If $\mathscr{A}$ is an ultraweakly closed unital unicellular algebra then $\mathscr{A}$ satisfies Property* $(\mathbf{U})$.

Proof. If $x_i \otimes y_i$ are elements of $\mathscr{A}$ for $i = 1, 2$, then $[\![\mathscr{A}_\perp x_i]\!] \perp y_i$ for $i = 1, 2$. As noted in the proof of Proposition 4, Theorem 1 implies that $[\![\mathscr{A}_\perp x_i]\!]$ is an invariant subspace for $\mathscr{A}$ for $i = 1, 2$. Since $\mathscr{A}$ is unicellular, either $[\![\mathscr{A}_\perp x_1]\!] \subseteq [\![\mathscr{A}_\perp x_2]\!]$, or vice versa. If $[\![\mathscr{A}_\perp x_1]\!] \subseteq [\![\mathscr{A}_\perp x_2]\!]$, then $[\![\mathscr{A}_\perp x_1]\!] \perp y_2$, so $x_1 \otimes y_2 \in (\mathscr{A}_\perp)^\perp = \mathscr{A}$. If the reverse inclusion is true, then $x_2 \otimes y_1 \in \mathscr{A}$. Thus $\mathscr{A}$ has Property $(\mathbf{U})$. □

Using this theorem we can give a simple proof of the following known result.

Theorem 8. *If an ultraweakly closed algebra $\mathscr{A}$ is such that every operator in $\mathscr{A}$ is upper triangular with respect to some fixed orthonormal basis $\mathscr{B} = \{e_n\}_{n \in \mathbf{N}}$, $\mathscr{A}$ contains the operators which are diagonal with respect to $\mathscr{B}$ and*

$$Lat\,\mathscr{A} = \{\{\vee e_j : j \leq n\} : \text{for all } n \in \mathbf{N}\}$$

then $\mathscr{A}$ is the set of all operators which are upper triangular with respect to $\mathscr{B}$.

Proof. Since $e_j \otimes e_j$ is in $\mathscr{A}$ for all $j \in \mathbb{N}$, by Theorem 7, given $i \leq j$ either $e_i \otimes e_j$ or $e_j \otimes e_i$ is in $\mathscr{A}$. The upper triangularity of $\mathscr{A}$ implies that it must be that $e_j \otimes e_i \in \mathscr{A}$ for all $i \leq j$. But the span of these rank-one operators is ultraweakly dense in the set of all upper triangular operators with respect to $\mathscr{B}$. □

The above theorem is a special case of the known result that any unicellular algebra containing a maximal abelian self-adjoint subalgebra must be reflexive. Theorem 7 can be used to give a simple proof, which is similar to the proof of Theorem 8, of this result in the case where the Hilbert space is spanned by the atoms of the nest.

Comment. The main reason that Theorem 7 is true is that the lattice property of being totally ordered is inherited by sublattices. In general, given an ultraweakly closed algebra $\mathscr{A}$, any property of $Lat(\mathscr{A})$ which is inherited by sublattices should give a condition on the rank-one operators in $\mathscr{A}$. For example, a result of [3] which implies that a transitive algebra $\mathscr{A}$ cannot contain any rank-one operators follows easily from Proposition 4 and the fact that the only sublattice of $Lat(\mathscr{A}) = \{0, \mathscr{H}\}$ which arises as the lattice of a nontrivial subspace of $\mathscr{B}(\mathscr{H})$ is $\{0, \mathscr{H}\}$.

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