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NOTES ON RENEWAL SYSTEMS

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ABSTRACT. A renewal system is a symbolic dynamical system generated by free concatenations of a finite set of words. Renewal systems are sofic, but not every sofic shift is topologically conjugate to a renewal system.

For background on symbolic dynamical systems and sofic shifts see, for example, [M] or [BMT]. Let A be a finite alphabet. A subshift R of A^Z is called a *renewal system* if there is a finite set W of finite strings (*words*) over A such that each element of R can be obtained as an infinite bilateral concatenation of elements of W. This term is due to Roy Adler, whom I thank for suggesting this area of study. The set W^* of finite concatenations of words of W is extensively studied in automata theory, but the dynamical properties of renewal systems are in general not well understood.

It is easy to see that every renewal system is sofic. A subshift S of A^Z is sofic if and only if its *language* (the set of words appearing in elements of S) is regular [W]; the language of R is the set of subwords of W^*, which is regular. Alternatively, let the words of W be $a_1^{(i)} \dots a_{l_i}^{(i)}$, $i = 1, \dots, k$ and let $x_j^{(i)}$, $i = 1, \dots, k$, $j = 1, \dots, l_i$ be distinct symbols. The renewal system x generated by the words $x_1^{(i)} \dots x_{l_i}^{(i)}$, $i = 1, \dots, k$ is a shift of finite type of a special sort we call a *loop system* because of the appearance of the associated directed graph (Figure 1). R is the image of x under the 1-block map $x_j^{(i)} \to a_j^{(i)}$ and hence is sofic.

It is also easy to see that not every sofic shift, or even every shift of finite type, is a renewal system. For example, the finite type shift Σ given by the directed graph in Figure 2 (here we take the vertices rather than the edges as the alphabet of our shift) is not a renewal system. A generating set would have to contain words of the form a^m, b^n to produce the sequences a^∞, b^∞ (here powers denote concatenation); but the word $b^n a^m$ may not appear in Σ. However, Σ is topologically conjugate to a renewal system: the 1-block map $a \to a$, $b \to b$,

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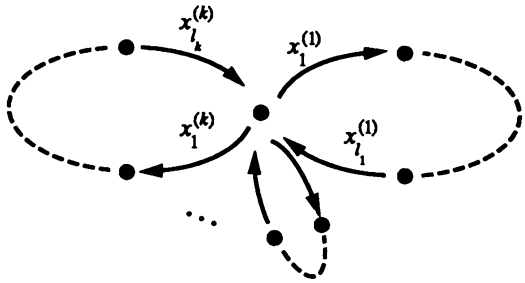

FIGURE 1

$c \to b$ has a 2-block inverse, and the image of Σ is the renewal system generated by $\{a, bb, bbb\}$.

The main result of this paper is that not every sofic shift is topologically conjugate to a renewal system. The example given is not exotic: similar arguments will apply to many sofic shifts. But it is not known if every irreducible shift of finite type is conjugate to a renewal system. (For recent work in this direction see [GLS].)

Example. The sofic shift S given by the labeled directed graph in Figure 3 is not conjugate to a renewal system.

Proof. Suppose S is conjugate to a renewal system R. Let the conjugacy φ and its inverse be given by block codes with memory and anticipation n, which we will also denote by φ and φ^{-1}. Thus $(\varphi(x))_0 = \varphi(x_{-n} \dots x_n)$ for $(x_i) \in S$. The image of the fixed point a^∞ in S is a fixed point $\hat{a}^\infty$ in R, so $\varphi(a^m) = \hat{a}^{m-2n}$ for $m > 2n$, and $\varphi(a^m b a^m)$ has the form $\hat{a}^{m-2n} b_{-n} \dots b_n \hat{a}^{m-2n}$. Since the point $a^\infty b a^\infty$ in S is the unique preimage of $\hat{a}^\infty b_{-n} \dots b_n \hat{a}^\infty$ (where for definiteness we may take these points to have zero coordinate b, b_0 respectively), we must have $\varphi^{-1}(\hat{a}^k b_{-n} \dots b_n \hat{a}^k) = a^k b a^k$ for $k \geq 0$.

Now, the generating set W of R must contain the word $\hat{a}^r$ for some $r \geq 1$. Also, some concatenation of words in W must have the form $\hat{a}^s b_{-n} \dots b_n \hat{a}^t$, s, $t \geq 0$, or it would be impossible to produce the sequence $\hat{a}^\infty b_{-n} \dots b_n \hat{a}^\infty$.

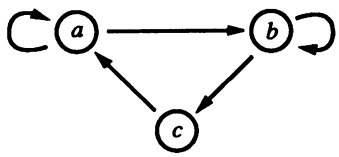

FIGURE 2

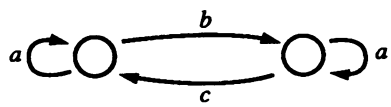

FIGURE 3

So, R must contain a point

$$x = \hat{a}^{\infty} b_{-n} \dots b_n \hat{a}^m b_{-n} \dots b_n \hat{a}^{\infty}$$

with $m \geq 2n$. But then

$$\varphi^{-1}(x) = a^{\infty} b a^{m+2n} b a^{\infty},$$

which is not a point in S. □

Note added in proof. Renewal systems are called *finitely generated systems* by A. Restivo [R], who shows it is decidable when a sofic shift is itself a renewal system.

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