ISOMETRICALLY INVARIANT EXTENSIONS
OF LEBESGUE MEASURE

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Abstract. The purpose of this note is to give a very short prove of the theorem
that every isometrically invariant measure extending Lebesgue measure on \( \mathbb{R}^n \)
has a proper isometrically invariant extension, i.e., that there is no maximal
isometrically invariant extension of Lebesgue measure on \( \mathbb{R}^n \).

All the measures that we will consider in this note will be countable additive
isometrically invariant extensions of Lebesgue measure on \( n \)-dimensional
Euclidean space \( \mathbb{R}^n \). By isometries we will understand bijections of \( \mathbb{R}^n \) that
preserve standard Euclidean distance. All the algebraic and measure theoretical
terminology that will be used is standard and follows from [La, Ru] respectively.

The first construction of a proper isometrically invariant extension of Lebesgue
measure goes back to Szpilrajn’s paper [Sz] of 1935. In the same paper,
Szpilrajn stated Sierpinski’s question: “Does there exist a maximal isometrically
invariant extension of Lebesgue measure on \( \mathbb{R}^n \)?” A negative answer to this
question, i.e., the theorem “every isometrically invariant measure that extends
Lebesgue measure on \( \mathbb{R}^n \) has a proper isometrically invariant extension,” was
proved by several mathematicians under different additional assumptions and
restrictions (see [Pk, Hu, Ha]). Without any assumption the theorem was proved
in 1983 by Ciesielski and Pele (see [CP]). For more historical details of this
issue see also [Ci1]. The purpose of this note is to give a very short proof of
the theorem that is different from that of [CP] and follows from the general
technique introduced by the author in [Ci2].

Theorem. Let \( \mu: \mathcal{M} \to [0, \infty] \) be an isometrically invariant extension of Lebesgue
measure \( \mathbb{R}^n \). Then there exists a proper isometrically invariant extension of
\( \mu \).

The proof will be based on the following easy and well-known lemmas.

Lemma 1 (Szpilrajn). Let \( \mu: \mathcal{M} \to [0, \infty] \) be an isometrically invariant measure on \( \mathbb{R}^n \). If a family \( \mathcal{A} \) of subsets of \( \mathbb{R}^n \) is closed under countable union,
closed under isometries action (i.e., \( g[A] \in \mathcal{A} \) for every \( A \in \mathcal{A} \) and every isometry \( g \)) and such that every \( A \in \mathcal{A} \) has \( \mu \) inner measure zero, then \( \mu \) has an isometrically invariant extension \( \nu : \mathcal{N} \to [0, \infty] \) such that \( \mathcal{A} \subset \mathcal{N} \) and \( \nu(A) = 0 \) for every \( A \in \mathcal{A} \).

Proof. If \( \mathcal{F} \) is an ideal of subsets of \( \mathbb{R}^n \) generated by the family \( \mathcal{A} \), and \( \mathcal{N} \) stands for a \( \sigma \)-algebra generated by \( \mathcal{A} \cup \mathcal{F} \) then all elements of \( \mathcal{N} \) are of the form \((M \cup I_1) \setminus I_2\) where \( M \in \mathcal{A} \) and \( I_1, I_2 \in \mathcal{F} \). It is easy to see that \( \nu : \mathcal{N} \to [0, \infty] \) such that \( \nu((M \cup I_1) \setminus I_2) = \mu(M) \) is a well-defined isometrically invariant measure on \( \mathbb{R}^n \) extending \( \mu \).

In the proof of the next lemma we use a method which goes back to Harazivili’s paper [Ha] (see also [Ci2]).

Lemma 2. Let \( \mathbb{R}^n = \bigcup\{N_k : k = 0, 1, 2, \ldots\} \). If each \( N_k \) satisfies the condition

for every countable set \( \{g_r : r = 0, 1, 2, \ldots\} \) of isometries there is an uncountable set \( H \) of isometries such that for every distinct \( h_1, h_2 \in H \)

\[
\bigcap\{g_r[N_k] : r = 0, 1, 2, \ldots\}
\]

then every isometrically invariant extension \( \mu : \mathcal{M} \to [0, \infty] \) of Lebesgue measure on \( \mathbb{R}^n \) has a proper isometrically invariant extension.

Proof. Let \( \mu : \mathcal{M} \to [0, \infty] \) be an isometrically invariant extension of Lebesgue measure on \( \mathbb{R}^n \). Define

\[
\mathcal{A}_k = \left\{ \bigcup\{g_r[N_k] : r = 0, 1, 2, \ldots\} : \text{where all } g_r’s \text{ are isometries of } \mathbb{R}^n \right\}.
\]

If \( M \in \mathcal{A}_k \) is a subset of \( A \in \mathcal{A} \), then \( h_1[M] \cap h_2[M] = \emptyset \) for every distinct \( h_1, h_2 \) from \( H \). But \( \mu(h[M]) = \mu(M) \) for every \( h \) from \( H \). Moreover, measure \( \mu \) is \( \sigma \)-finite as an extension of Lebesgue measure. This implies that \( \mu(M) = 0 \) and so \( A \) has \( \mu \) inner measure zero.

Thus we proved that every \( \mathcal{A}_k \) satisfies the assumptions of Lemma 1. Hence for each \( k = 0, 1, 2, \ldots \) there is an isometrically invariant extension \( \nu_k \) of \( \mu \) such that \( \nu_k(N_k) = 0 \). But all \( N_k \)’s cannot have \( \mu \) measure zero. So some \( \nu_k \) must be a proper extension of \( \mu \).

The following lemma is an elementary geometrical fact and will be left without the proof.

Lemma 3. Every isometry of \( \mathbb{R}^n \) can be represented as a superposition to \( L \) where \( t \) is a translation by a vector \( (t_1, t_2, \ldots, t_n) \) and \( L \) is a linear transformation of \( \mathbb{R}^n \) represented by some \( n \times n \) matrix \( (a_{ij}) \).

Proof of the theorem. By Lemma 2 it is enough to construct \( N_k \)’s such that \( \mathbb{R}^n = \bigcup\{N_k : k = 0, 1, 2, \ldots\} \) and each \( N_k \) satisfies condition (*).
Let $\mathcal{B}$ be a transcendence base of $\mathbf{R}$ over $\mathbf{Q}$ and let us represent $\mathcal{B}$ as $\mathcal{B} = \bigcup\{\mathcal{B}_k : k = 0, 1, 2, \ldots \}$ where $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2 \subset \ldots$ and $\mathcal{B}_{k+1} \setminus \mathcal{B}_k$ is uncountable. Define

$$N_k = [\text{cl}_R(\mathbf{Q}(\mathcal{B}_k))]^n,$$

where $\mathbf{Q}(\mathcal{B}_k)$ is a field generated by $\mathbf{Q}$ and $\mathcal{B}_k$, and $\text{cl}_R(\mathbf{Q}(\mathcal{B}_k))$ is an algebraic closure of $\mathbf{Q}(\mathcal{B}_k)$ in $\mathbf{R}$. We have to prove that $N_k$'s satisfy $(*)$.

So let us choose $k$ and a countable set $\{g_r : r = 0, 1, 2, \ldots \}$ of isometries. There exists a countable set $\mathcal{A} \subset \mathcal{B}$ such that all $g_r$'s are defined over $\text{cl}_R(\mathbf{Q}(\mathcal{A}))$, i.e., that for each $g_r$ the coefficients $t_{ij}$'s and $a_{ij}$'s from Lemma 3 are in $\text{cl}_R(\mathbf{Q}(\mathcal{A}))$. Let $L = \text{cl}_R(\mathbf{Q}(\mathcal{A} \cup \mathcal{B}_k))$. Then

$$\bigcup \{g_r[N_k] : r = 0, 1, 2, \ldots \} \subset L^n.$$

Define

$$H = \{t_{\alpha} : \alpha \in \mathcal{B}_{k+1} \setminus (\mathcal{A} \cup \mathcal{B}_k)\},$$

where $t_{\alpha}$ is a translation by a vector $(\alpha, 0, 0, \ldots, 0)$. Then $H$ is uncountable and for distinct $\alpha, \beta \in H$,

$$t_{\alpha} \left( \bigcup \{g_r[N_k] : r = 0, 1, 2, \ldots \} \right) \cap t_{\beta} \left( \bigcup \{g_r[N_k] : r = 0, 1, 2, \ldots \} \right)$$

$$\subset t_{\alpha}(L^n) \cap t_{\beta}(L^n) = \emptyset$$

as $\alpha - \beta \notin L$. This finishes the proof of the theorem.

**References**


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