ON SOME EQUIVALENT METRICS FOR BOUNDED OPERATORS ON HILBERT SPACE

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Abstract. Several operator norm inequalities concerning the equivalence of some metrics for bounded linear operators on Hilbert space are given. In addition, some related inequalities for the Hilbert-Schmidt norm are presented.

1. Introduction

Let $H$ be a complex Hilbert space and let $CD(H)$ denote the family of closed, densely defined linear operators on $H$, and $B(H)$ the bounded members of $CD(H)$. For each $T \in CD(H)$, let $\Pi(T)$ denote the orthogonal projection of $H \oplus H$ onto the graph of $T$, and let $\alpha(T)$ denote the pure contraction defined by $\alpha(T) = T(1 + T^*T)^{-1/2}$. The gap metric on $CD(H)$ is defined by $d(S, T) = \|\Pi(S) - \Pi(T)\|$ for all $S, T \in CD(H)$ (see [8, p. 197]). In [11], W. E. Kaufman introduced a metric $\delta$ on $CD(H)$ defined by $\delta(S, T) = \|\alpha(S) - \alpha(T)\|$ for all $S, T \in CD(H)$ and he showed that this metric is stronger than the gap metric $d$ and not equivalent to it. He also stated that on $B(H)$ the gap metric $d$ is equivalent to the metric generated by the usual operator norm and he proved that the metric $\delta$ has this property.

The purpose of this paper is to present quantitative estimates to this effect. In §2, we present several operator norm inequalities to compare the metric $\delta$, the gap metric $d$, and the usual operator norm metric. In §3, we obtain $\delta$ estimates involving positive operators and operator monotone functions, and finally in §4, we remark how the inequalities of §§2, 3 can be extended to more general norms with an emphasis on the Hilbert–Schmidt norm.

To achieve our goal, we need the following lemmas. Lemmas 1 and 2 are concerned with the continuity of the square root function defined on positive operators. Lemma 3 is concerned with the continuity of the absolute value map, and Lemma 4 addresses the norm of operator matrices.

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Lemma 1 [14]. If $A, B \in B(H)$ are positive, then

$$\|A - B\| \leq \|A^2 - B^2\|.$$  

Lemma 2 [13]. If $A, B \in B(H)$ are positive and $A + B \geq c \geq 0$, then

$$c\|A - B\| \leq \|A^2 - B^2\|.$$  

Lemma 3 [16]. If $A, B \in B(H)$, then

$$\||A| - |B|| \leq \|A - B\|^{1/2}\|A + B\|^{1/2},$$

where $|A| = (A^*A)^{1/2}$ is the absolute value of $A$.

Lemma 4 [3]. If $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in B(H \oplus H)$, then

$$\|T\|^2 \leq \|A\|^2 + \|B\|^2 + \|C\|^2 + \|D\|^2.$$  

In addition to these lemmas we make repeated use of the operator identity:


Consequently, for $S, T \in B(H)$, we have

$$\|S^*S - T^*T\| \leq \|S - T\|\|S + T\|$$

(see [16, Proof of Theorem 2.1]).

2. Equivalent metrics on $B(H)$

In [11, Theorem 2], Kaufman proved that on $B(H)$ the metric $\delta$ and the usual operator norm metric are equivalent. There is a minor gap in the proof of Theorem 2 in [11] that can be filled by employing the continuity of the inversion map as demonstrated in the following two theorems.

Theorem 1. If $S, T \in B(H)$, then

$$\delta(S, T) \leq \|S - T\|(1 + \frac{1}{4}\|S + T\|^2).$$

Proof. For each $T \in B(H)$, let $\beta(T) = (1 + T^*T)^{-1/2}$. Then $\beta(T)$ is a positive definite contraction; that is, $0 < \beta(T) \leq 1$, $\alpha(T) = T\beta(T)$, and $\beta(T) = (1 - \alpha(T)^*\alpha(T))^{1/2}$ (see [9, 11]).
For $S, T \in B(H)$, we have
\[
\delta(S, T) = \|\alpha(S) - \alpha(T)\|
= \|S \beta(S) - T \beta(T)\|
= \left\| \frac{1}{2} (S - T)(\beta(S) + \beta(T)) + \frac{1}{2} (S + T)(\beta(S) - \beta(T)) \right\|
\leq \|S - T\| + \frac{1}{2} \|S + T\| \|\beta(S) - \beta(T)\|
= \|S - T\| + \frac{1}{2} \|S + T\| \|\beta(S)(\beta(S)^{-1} - \beta(T)^{-1})\beta(T)\|
\leq \|S - T\| + \frac{1}{2} \|S + T\| \|\beta(S)^{-1} - \beta(T)^{-1}\|
\leq \|S - T\| + \frac{1}{4} \|S + T\|^2 \|\beta(S)^{-2} - \beta(T)^{-2}\|^2 \quad (\text{Lemma 2})
= \|S - T\| \left(1 + \frac{1}{4} \|S + T\|^2\right), \quad \text{as required.}
\]

**Theorem 2.** If $S, T \in B(H)$, then
\[
\|S - T\| \leq \frac{1}{4} \delta(S, T) (2 \|\beta(S)^{-1} + \beta(T)^{-1}\| + \|\beta(S)^{-2}\| \|\beta(T)^{-2}\| \|\alpha(S) + \alpha(T)\|^2).
\]
**Proof.** Since $T = \alpha(T) \beta(T)^{-1}$, it follows that
\[
\|S - T\| = \|\alpha(S) \beta(S)^{-1} - \alpha(T) \beta(T)^{-1}\|
= \left\| \frac{1}{2} (\alpha(S) - \alpha(T)) (\beta(S)^{-1} + \beta(T)^{-1})
+ \frac{1}{2} (\alpha(S) + \alpha(T)) (\beta(S)^{-1} - \beta(T)^{-1}) \right\|
\leq \frac{1}{2} \delta(S, T) \|\beta(S)^{-1} + \beta(T)^{-1}\|
+ \frac{1}{2} \|\alpha(S) + \alpha(T)\| \|\beta(S)^{-1} - \beta(T)^{-1}\|
\leq \frac{1}{2} \delta(S, T) \|\beta(S)^{-1} + \beta(T)^{-1}\|
+ \frac{1}{4} \|\alpha(S) + \alpha(T)\|^2 \|\beta(S)^{-2} - \beta(T)^{-2}\|^2 \quad (\text{Lemma 2})
\leq \frac{1}{2} \delta(S, T) \|\beta(S)^{-1} + \beta(T)^{-1}\|
+ \frac{1}{4} \|\beta(S)^{-2}\| \|\beta(T)^{-2}\| \|\alpha(S) + \alpha(T)\| \|\beta(S)^{-2} - \beta(T)^{-2}\|^2.
\]
Since $\|\beta(S)^{-2} - \beta(T)^{-2}\| = \|\alpha(S)^* \alpha(S) - \alpha(T)^* \alpha(T)\| \leq \delta(S, T) \|\alpha(S) + \alpha(T)\|$, it follows that
\[
\|S - T\| \leq \frac{1}{2} \delta(S, T) \|\beta(S)^{-1} + \beta(T)^{-1}\|
+ \frac{1}{4} \|\beta(S)^{-2}\| \|\beta(T)^{-2}\| \|\alpha(S) + \alpha(T)\|^2 \delta(S, T).
\]
This completes the proof of the theorem.

It has been stated (without proof) in [11] that on $B(H)$ the gap metric $d$ is equivalent to the usual operator norm metric. This is an immediate consequence of the results of this section.
Theorem 3. If $S, T \in B(H)$, then
\[
d^2(S, T) \leq 2\delta^2(S, T)\|\alpha(S) + \alpha(T)\|^2 \\
+ 2(\delta(S, T) + \frac{1}{2}\|\alpha(S) + \alpha(T)\|^{3/2}\delta^{1/2}(S, T))^2.
\]

Proof. It is known [9, Remark, p. 532] that the projection-function $\Pi$ has an operator matrix representation given by
\[
\Pi(T) = \begin{bmatrix}
\beta(T)^2 & \beta(T)\alpha(T)^* \\
\alpha(T)\beta(T) & \alpha(T)\alpha(T)^*
\end{bmatrix}.
\]
Using this representation together with Lemma 4, we get
\[
d^2(S, T) = \|\Pi(S) - \Pi(T)\|^2 \\
\leq \|\beta(S) - \beta(T)\|^2 + 2\|\alpha(S)\beta(S) - \alpha(T)\beta(T)\|^2 \\
+ \|\alpha(S)\alpha(S)^* - \alpha(T)\alpha(T)^*\|^2 \\
= \|\alpha(S)\alpha(S) - \alpha(T)\alpha(T)^*\|^2 + 2\|\alpha(S)\beta(S) - \alpha(T)\beta(T)\|^2 \\
+ \|\alpha(S)\alpha(S)^* - \alpha(T)\alpha(T)^*\|^2.
\]

Now observe that both
\[
\|\alpha(S)\alpha(S) - \alpha(T)\alpha(T)^*\|^2
\]
and
\[
\|\alpha(S)\alpha(S)^* - \alpha(T)\alpha(T)^*\|^2
\]
are majorized by $\delta^2(S, T)\|\alpha(S) + \alpha(T)\|^2$. Observe also that
\[
\|\alpha(S)\beta(S) - \alpha(T)\beta(T)\|^2 \\
\leq \frac{1}{2}\delta(S, T)\|\beta(S) + \beta(T)\|^2 + \frac{1}{2}\|\alpha(S) + \alpha(T)\|\|\beta(S) - \beta(T)\| \\
\leq \delta(S, T) + \frac{1}{2}\|\alpha(S) + \alpha(T)\|\|\beta(S) - \beta(T)\| \\
\leq \delta(S, T) + \frac{1}{2}\|\alpha(S) + \alpha(T)\|\|\beta(S)^2 - \beta(T)^2\|^{1/2} (\text{Lemma 1}) \\
\leq \delta(S, T) + \frac{1}{2}\|\alpha(S) + \alpha(T)\|^{3/2}\delta^{1/2}(S, T).
\]

In view of these observations we have
\[
d^2(S, T) \leq 2\delta^2(S, T)\|\alpha(S) + \alpha(T)\|^2 + 2(\delta(S, T) \\
+ \frac{1}{2}\|\alpha(S) + \alpha(T)\|^{3/2}\delta^{1/2}(S, T))^2,
\]
as required.

It can easily be seen that for $S, T \in B(H)$ we have $\alpha(S)^*\alpha(S) - \alpha(T)^*\alpha(T) = \beta(S)^2(S^*S - T^*T)\beta(T)^2$. It follows from this identity that
\[
\|\alpha(S)^*\alpha(S) - \alpha(T)^*\alpha(T)\| \leq \|S^*S - T^*T\|.
\]

This estimate enables us to show that for $S, T \in B(H)$ we have
\[
d^2(S, T) \leq \|S - T\|^2(2 + 4\|S + T\|^2 + \frac{1}{2}\|S + T\|^4).
\]
For $T \in B(H)$ we have $\|\beta(T)\| \leq 1$ and $\|\alpha(T)\| < 1$ (see [9]). However, $\|\alpha(T)\beta(T)\| \leq \frac{1}{2}$, which is proved as follows.

$$|\alpha(T)\beta(T)|^2 = \beta(T)\alpha(T)^*\alpha(T)\beta(T)$$

$$= T^*T(1 + T^*T)^{-2} \leq \frac{1}{4}.$$ 

Hence $\|\alpha(T)\beta(T)\| \leq \frac{1}{2}$. This estimate is needed for the proof of the following theorem.

**Theorem 4.** If $S, T \in B(H)$, then

$$\delta(S, T) \leq \frac{1}{4}d(S, T)(2\|\beta(S)^{-1} + \beta(T)^{-1}\| + \|\beta(S)^{-2}\|\|\beta(T)^{-2}\|).$$

**Proof.** Since $\alpha(T) = \alpha(T)\beta(T)\beta(T)^{-1}$, it follows that

$$\delta(S, T) = \|\alpha(S)\beta(S)\beta(S)^{-1} - \alpha(T)\beta(T)\beta(T)^{-1}\|$$

$$\leq \frac{1}{2}\|\alpha(S)\beta(S) - \alpha(T)\beta(T)\|\|\beta(S)^{-1} + \beta(T)^{-1}\|$$

$$+ \frac{1}{2}\|\alpha(S)\beta(S) + \alpha(T)\beta(T)\|\|\beta(S)^{-1} - \beta(T)^{-1}\|$$

$$\leq \frac{1}{2}d(S, T)\|\beta(S)^{-1} + \beta(T)^{-1}\| + \frac{1}{2}\|\beta(S)^{-1} - \beta(T)^{-1}\|$$

$$\leq \frac{1}{2}d(S, T)\|\beta(S)^{-1} + \beta(T)^{-1}\| + \frac{1}{2}\|\beta(S)^{-2} - \beta(T)^{-2}\| \quad \text{(Lemma 2)}$$

$$\leq \frac{1}{2}d(S, T)\|\beta(S)^{-1} + \beta(T)^{-1}\| + \frac{1}{2}\|\beta(S)^{-2}\|\|\beta(T)^{-2}\|d(S, T),$$

which completes the proof of the theorem.

Using the observation that $T = \alpha(T)\beta(T)\beta(T)^{-2}$, it can easily be shown that for $S, T \in B(H)$, we have

$$\|S - T\| \leq \frac{1}{2}d(S, T)(\|\beta(S)^{-2} + \beta(T)^{-2}\| + \|\beta(S)^{-2}\|\|\beta(T)^{-2}\|).$$

In [11, Theorem 5], Kaufman proved that if $\{S_n\}$ is a sequence in $CD(H)$ and $T \in CD(H)$ is such that $\delta(S_n, T) \to 0$ as $n \to \infty$, then $\delta(|S_n|, |T|) \to 0$ as $n \to \infty$. For bounded operators this is an immediate consequence of the following estimate.

**Theorem 5.** If $S, T \in B(H)$, then

$$\delta(|S|, |T|) \leq \|\alpha(S) - \alpha(T)\|^{1/2}\|\alpha(S) + \alpha(T)\|^{1/2}.$$ 

**Proof.** Since $\alpha(|T|) = |\alpha(T)|$ (see [9, Lemma 3]), it follows that

$$\delta(|S|, |T|) = \|\alpha(|S|) - \alpha(|T|)\|$$

$$= \|\|\alpha(S)\| - |\alpha(T)|\|$$

$$\leq \|\alpha(S) - \alpha(T)\|^{1/2}\|\alpha(S) + \alpha(T)\|^{1/2} \quad \text{(Lemma 3)}.$$
3. δ-ESTIMATES FOR POSITIVE OPERATORS
AND OPERATOR MONOTONE FUNCTIONS

Recall that a real-valued continuous function \( f \) on \([0, \infty)\) is said to be operator monotone if for any positive operators \( A, B \in B(H) \), the relation \( A \leq B \) always implies \( f(A) \leq f(B) \). It is well known that \( f(t) = t^r \) is operator monotone for \( 0 < r \leq 1 \) (see [1]).

It has been shown in [16, Theorem 2.3] that if \( A, B \in B(H) \) are positive, then for any operator monotone function \( f \) with \( f(0) = 0 \) we have

\[
\|f(A) - f(B)\| \leq f(\|A - B\|).
\]

In the same spirit we have the following related result for the metric \( \delta \).

**Theorem 6.** If \( A, B \in B(H) \) are positive, then for any real-valued continuous function \( f \) on \([0, \infty)\) with \( f(0) = 0 \) and \( f^2 \) operator monotone, we have

\[
\delta(f(A), f(B)) \leq f(\|A - B\|).
\]

**Proof.** We have

\[
\begin{align*}
\delta(f(A), f(B)) &= \|f(A)(1 + f(A)^2)^{-1/2} - f(B)(1 + f(B)^2)^{-1/2}\| \\
&\leq \|f(A)^2(1 + f(A)^2)^{-1} - f(B)^2(1 + f(B)^2)^{-1}\|^{1/2} & \text{(Lemma 1)} \\
&= \|(1 + f(A)^2)^{-1}(f(A)^2 - f(B)^2)(1 + f(B)^2)^{-1}\|^{1/2} \\
&\leq \|f(A)^2 - f(B)^2\|^{1/2} \\
&\leq f(\|A - B\|) & \text{([16, Theorem 2.3] applied to } f^2).}
\end{align*}
\]

The following corollary is an important special case of Theorem 6.

**Corollary 1.** If \( A, B \in B(H) \) are positive, then for any real number \( r, 0 < r \leq \frac{1}{2} \), we have

\[
\delta(A^r, B^r) \leq \|A - B\|^r.
\]

It should be noted that if \( f \) is a nonnegative continuous function on \([0, \infty)\) such that \( f^2 \) is operator monotone, then \( f \) is also operator monotone. Of course, the converse is not true. If we merely assume in Theorem 6 that \( f \) is operator monotone, then we have the following weaker result.

**Theorem 7.** If \( A, B \in B(H) \) are positive, then for any operator monotone function \( f \) on \([0, \infty)\) with \( f(0) = 0 \) we have

\[
\delta(f(A), f(B)) \leq (2f(\|A - B\|)f(\frac{1}{2}\|A + B\|))^{1/2}.
\]

**Proof.** We have

\[
\begin{align*}
\delta(f(A), f(B)) &\leq \|f(A)^2 - f(B)^2\|^{1/2} & \text{(from the proof of Theorem 6)} \\
&\leq \|f(A) - f(B)\|\|f(A) + f(B)\|^{1/2} \\
&\leq f(\|A - B\|)^{1/2}\|f(A) + f(B)\|^{1/2} & \text{[16, Theorem 2.3]}.}
\end{align*}
\]
But it is known that every operator monotone function is operator concave (see [1]), and hence \( \frac{1}{2}(f(A) + f(B)) \leq f(\frac{1}{2}(A + B)) \). Therefore we have
\[
\delta(f(A), f(B)) \leq (2f(||A - B||)||f(\frac{1}{2}(A + B)||))^{1/2} \\
\leq (2f(||A - B||))f(\frac{1}{2}||A + B||))^{1/2}.
\]

**Corollary 2.** If \( A, B \in B(H) \) are positive, then for any real number \( r, 0 < r \leq 1 \), we have
\[
\delta(A^r, B^r) \leq 2^{1-r/2}||A - B||^{r/2}||A + B||^{r/2}.
\]
In particular \( \delta(A, B) \leq ||A - B||^{1/2}||A + B||^{1/2} \).

For arbitrary operators \( S, T \in B(H) \) we have the following generalization of Corollary 2 for the case \( r = 1 \), which is related to Theorem 5.

**Corollary 3.** If \( S, T \in B(H) \), then
\[
\delta(|S|, |T|) \leq ||S - T||^{1/2}||S + T||^{1/2}.
\]

**Proof.** By Corollary 1, applied to the positive operators \( S^*S \) and \( T^*T \), \( r = \frac{1}{2} \), we have
\[
\delta(|S|, |T|) \leq ||S^*S - T^*T||^{1/2} \\
\leq ||S - T||^{1/2}||S + T||^{1/2}.
\]

It should be noted that an alternative proof of Corollary 3 can be provided. This proof is based on the fact that \( \alpha(|T|) = |\alpha(T)| \), Lemma 1, and the relation
\[
\alpha(S)^*\alpha(S) - \alpha(T)^*\alpha(T) = \beta(S)^2(S^*S - T^*T)\beta(T)^2.
\]

4. Inequalities for the Hilbert–Schmidt norm

We remark that inequalities similar to those given in §§2, 3 can be proved for certain unitarily invariant norms including the Schatten \( p \)-norms (see [17]). This is due to the availability of inequalities in these norms similar to those given in Lemmas 1–4 and [16, Theorem 2.3]. For such inequalities the reader is referred to [2, 4–7, 15] and references therein.

Besides the usual operator norm, the Hilbert–Schmidt norm is of particular importance among the unitarily invariant norms. The following improvements of Theorems 1 and 3 are available for the Hilbert–Schmidt norm.

**Theorem 8.** If \( S, T \in B(H) \), then
\[
||\alpha(S) - \alpha(T)||_2 \leq ||S - T||_2,
\]
where \( ||\cdot||_2 \) denotes the Hilbert–Schmidt norm. In particular if \( S - T \) is in the Hilbert–Schmidt class, then so is \( \alpha(S) - \alpha(T) \).

**Proof.** We start by observing that \( |s/\sqrt{1 + s^2} - t/\sqrt{1 + t^2}| \leq |s - t| \) for all real numbers \( s \) and \( t \). Therefore if we first assume that \( S \) and \( T \) are selfadjoint operators, then by [12] we get
\[
||\alpha(S) - \alpha(T)||_2 \leq ||S - T||_2.
\]
The general case follows by applying the selfadjoint case to the selfadjoint operators
\[
\tilde{S} = \begin{bmatrix} 0 & S^* \\ S & 0 \end{bmatrix} \quad \text{and} \quad \tilde{T} = \begin{bmatrix} 0 & T^* \\ T & 0 \end{bmatrix}
\]
defined on \( H \oplus H \). Using the fact (see [10]) that \( \alpha(T^*) = \alpha(T)^* \) together with some matrix computations, we obtain
\[
\alpha(\tilde{S}) = \begin{bmatrix} 0 & \alpha(S)^* \\ \alpha(S) & 0 \end{bmatrix} \quad \text{and} \quad \alpha(\tilde{T}) = \begin{bmatrix} 0 & \alpha(T)^* \\ \alpha(T) & 0 \end{bmatrix}.
\]
Now
\[
\|\tilde{S} - \tilde{T}\|^2 = 2\|S - T\|^2
\]
and
\[
\|\alpha(\tilde{S}) - \alpha(\tilde{T})\|^2 = 2\|\alpha(S) - \alpha(T)\|^2.
\]
Since \( \|\alpha(\tilde{S}) - \alpha(\tilde{T})\|^2 \leq \|\tilde{S} - \tilde{T}\|^2 \), it follows that \( \|\alpha(S) - \alpha(T)\|_2 \leq \|S - T\|_2 \) as required.

**Theorem 9.** If \( S, T \in B(H) \), then
\[
\|\Pi(S) - \Pi(T)\|_2 \leq 2\|S - T\|_2\|\alpha(S) + \alpha(T)\|^2 + 1).
\]
In particular if \( S - T \) is in the Hilbert–Schmidt class, then so is \( \Pi(S) - \Pi(T) \).

**Proof.** Using the matrix representation of \( \Pi \), as in the proof of Theorem 3, we have
\[
\|\Pi(S) - \Pi(T)\|_2^2 = \|\beta(S)^2 - \beta(T)^2\|_2^2 + 2\|\alpha(S)\beta(S) - \alpha(T)\beta(T)\|_2^2
+ \|\alpha(S)\alpha(S)^* - \alpha(T)\alpha(T)^*\|_2^2
= \|\alpha(S)^*\alpha(S) - \alpha(T)^*\alpha(T)\|_2^2 + 2\|S\beta(S)^2 - T\beta(T)^2\|_2^2
+ \|\alpha(S)\alpha(S)^* - \alpha(T)\alpha(T)^*\|_2^2.
\]
Now, note that both
\[
\|\alpha(S)^*\alpha(S) - \alpha(T)^*\alpha(T)\|_2^2
\]
and
\[
\|\alpha(S)\alpha(S)^* - \alpha(T)\alpha(T)^*\|_2^2
\]
are majorized by \( \|\alpha(S) - \alpha(T)\|^2\|\alpha(S) + \alpha(T)\|^2 \), and hence, by Theorem 8, this is in turn majorized by \( \|S - T\|^2\|\alpha(S) + \alpha(T)\|^2 \).

Finally, by noting that \( |s/(1 + s^2) - t/(1 + t^2)| \leq |s - t| \) for all real numbers \( s \) and \( t \), an argument similar to the one used in the proof of Theorem 8 can be presented here to show that \( \|S\beta(S)^2 - T\beta(T)^2\|_2 \leq \|S - T\|_2 \). Therefore, we conclude that
\[
\|\Pi(S) - \Pi(T)\|_2^2 \leq 2\|S - T\|^2\|\alpha(S) + \alpha(T)\|^2 + 2\|S - T\|^2,
\]
which completes the proof of the theorem.
Once again if we use the relation
\[ \alpha(S)^* \alpha(S) - \alpha(T)^* \alpha(T) = \beta(S)^2 (S^* S - T^* T) \beta(T)^2, \]
then we can also show that for \( S, T \in B(H) \), we have
\[ \| \Pi(S) - \Pi(T) \|_2^2 \leq 2 \| S - T \|_2^2 (\| S + T \|^2 + 1). \]
Whenever this estimate is compared with the estimate given in Theorem 9, one should keep in mind that \( \| \alpha(S) + \alpha(T) \| < 2 \).

We have seen from the proof of Theorem 9 that if \( S, T \in B(H) \), then
\[ \| S \beta(S)^2 - T \beta(T)^2 \|_2 \leq \| S - T \|_2. \]
However, for the usual operator norm we have
\[ \| S \beta(S)^2 - T \beta(T)^2 \| \leq \frac{1}{2} \| S - T \|. \]
To prove this inequality it is sufficient to assume that \( S \) and \( T \) are selfadjoint. In this case we have
\[ S \beta(S)^2 - T \beta(T)^2 = S(1 + S^2)^{-1} - T(1 + T^2)^{-1} \]
\[ = (1 + S^2)^{-1}(S(1 + T^2) - (1 + S^2)T)(1 + T^2)^{-1} \]
\[ = (1 + S^2)^{-1}(S - T)(1 + T^2)^{-1} \]
\[ + (1 + S^2)^{-1}S(T - S)T(1 + T^2)^{-1}. \]
Since \( \| (1 + T^2)^{-1} \| \leq 1 \) and \( \| T(1 + T^2)^{-1} \| \leq \frac{1}{2} \), it follows that \( \| S \beta(S)^2 - T \beta(T)^2 \| \leq \frac{1}{2} \| S - T \|. \)

Finally, we remark that this inequality is also true for every unitarily invariant norm.

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