DERIVATIVES OF HARDY FUNCTIONS

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ABSTRACT. Let $B$ be the open unit ball of $\mathbb{C}^n$, and set $S = \partial B$. It is shown that if $\varphi \in L^p(S)$, $\varphi > 0$, is a lower semicontinuous function on $S$ and $1/q > 1 + 1/p$, then, for a given $\varepsilon > 0$, there exists a function $f \in H^p(B)$ with $f(0) = 0$ such that $|f^*| = \varphi$ almost everywhere on $S$ and $\int_B |\nabla f|^q dV < \varepsilon$ where $V$ denotes the normalized volume measure on $B$.

1. Introduction

Let $B$ be the open unit ball of $\mathbb{C}^n$, and set $S = \partial B$. The rotation-invariant probability measure on $S$ will be denoted by $\sigma$. The Hardy space $H^p(B)$, $0 < p < \infty$, is then the space of holomorphic functions on $B$ for which

$$\sup_{0<r<1} \int_S |f(r\zeta)|^p d\sigma(\zeta) < \infty.$$ 

For $p = \infty$, $H^\infty(B)$ denotes the space of bounded holomorphic functions on $B$. As is well known, to each $H^p$-function $f$ corresponds its boundary value $f^*$ defined almost everywhere on $S$ by $f^*(\zeta) = \lim_{r \to 1} f(r\zeta)$; the term “almost every” will always refer to the measure $\sigma$. A remarkable fact about these boundary values of $H^p$-functions is that they can be prescribed as in the following.

Theorem A (Alexandrov [1]). Suppose $\varphi > 0$ is a lower semicontinuous function on $S$ and $\varphi \in L^p(\sigma)$ for some $0 < p \leq \infty$. Then there exists a function $f \in H^p(B)$ with $f(0) = 0$ such that $|f^*| = \varphi$ almost everywhere on $S$.

At least for $n > 1$, it seems to be the price we pay for prescribing boundary values of $H^p$-functions as above that their derivatives may get out of control in general. Roughly speaking, when $n > 1$, a theorem of Rudin [5] shows that if $f \in H^\infty(B)$ is nonconstant and $|f^*| = 1$ on some open subset $\Gamma$ of $S$, then $f$ must be extremely oscillatory near a $G_\delta$ dense subset $\Omega$ of $\Gamma$, so that its derivatives may get huge near $\Omega$. One such example (when $n > 1$) is the fact [2] that if $f$ is inner (i.e., $f \in H^\infty(B)$ and $|f^*| = 1$ almost everywhere on
and if \( \int_B |\nabla f|^2 dV < \infty \), then \( f \) must be constant. Here and elsewhere, \( \nabla f = (\partial f/\partial z_1, \ldots, \partial f/\partial z_n) \) and \( V \) denotes the normalized volume measure on \( B \). In some special cases, however, one may have those derivatives under very good control. For example, one can choose \( f \) in Theorem A so that \( \int_B |\nabla f|^q dV \) is arbitrarily small whenever \( 1/q > 1 + n/p \). This follows from Theorem A itself.

**Proof (sketch).** Let \( f \) be as in Theorem A, fix an inner function \( u \) on \( B \) such that \( u(0) = 0 \), and define \( f_m = u^m f \) for \( m = 1, 2, \ldots \). Then Theorem A holds with \( f_m \) in place of \( f \). Note that \( f_m \) converges to 0 uniformly on compact subsets of \( B \) and therefore so does \( |\nabla f_m| \). Since

\[
|f_m(z)| \leq C_{n,p,\|\varphi\|_L^p(\sigma)} \left( \frac{1}{1-|z|} \right)^{n/p} \quad (z \in B),
\]

we have by the Cauchy estimates

\[
|\nabla f_m(z)| \leq C_{n,p,\|\varphi\|_L^p(\sigma)} \left( \frac{1}{1-|z|} \right)^{1+n/p} \quad (z \in B).
\]

It follows that the sequence \( |\nabla f_m|^q \) is dominated by an integrable function on \( B \) whenever \( 1/q > 1 + n/p \), and thus \( \int_B |\nabla f_m|^q dV \) converges to 0 as \( m \to \infty \). \( \square \)

In the present paper we obtain a better upper bound for \( q \) in the above observation, which is the main result of the paper.

**Main theorem.** Let \( \varphi > 0 \) be a lower semicontinuous function on \( S \). Suppose \( \varphi \in L^p(\sigma) \) for some \( 0 < p \leq \infty \) and \( 1/q > 1 + 1/p \). Then, for a given \( \varepsilon > 0 \), there exists a function \( f \in H^p(B) \) with \( f(0) = 0 \) such that \( |f^*| = \varphi \) almost everywhere on \( S \) and \( \int_B |\nabla f|^q dV < \varepsilon \).

The author does not know whether the upper bound \( (1+1/p)^{-1} \) for \( q \) in the above theorem is sharp in general. There is, however, a reason why the upper bound cannot be bigger than 1, as the following remark shows.

**Remark.** Suppose \( f \in H^p(B) \) and \( f(0) = 0 \). Then

\[
|f^*(\zeta)| \leq \int_0^1 \left| \frac{\partial}{\partial t} f(t\zeta) \right| dt \leq \int_0^1 |\nabla f(t\zeta)| dt
\]

for almost every \( \zeta \in S \). Hence, for \( q \geq 1 \), we have

\[
\int_S |f^*|^q d\sigma \leq \int_S \int_0^1 |\nabla f(t\zeta)|^q dt d\sigma(\zeta) \leq C_{n,q} \int_B |\nabla f|^q dV.
\]

It follows that if the conclusion of the main theorem holds for some \( q \), then \( q < 1 \). Note that the upper bound for \( q \) in the main theorem is sharp for \( p = \infty \).

2. **Proof of main theorem**

We need prove the main theorem only for higher-dimensional cases. From now on, \( n > 1 \) will therefore be fixed. The main idea is to modify Rudin's
proof [5] of Theorem A so that the derivatives in question are all under control at the same time. Before going further, let us introduce some notation first. There is a nonisotropic metric \( d \) on \( S \) defined by
\[
d(\zeta, \eta) = |1 - \zeta, \eta |^{1/2} \quad (\zeta, \eta \in S).
\]
Since \( \sigma \) is rotation-invariant, the volume \( \sigma(Q_\delta(\eta)) \) of the corresponding balls
\[
Q_\delta(\eta) = \{ \zeta \in S : d(\zeta, \eta) < \delta \} \quad (0 < \delta < \sqrt{2})
\]
is independent of \( \eta \) and thus will be denoted by \( A(\delta) \). The notation \( A(B) \) will denote the class of functions holomorphic on \( B \) and continuous on \( B \).
Finally, \( \chi_E \) will denote the characteristic function of \( E \subset S \).

**Lemma 1.** Let \( 0 < p < 1 \). Then there are constants \( \alpha = \alpha(n, p) < 1 \) and \( \beta = \beta(n, p) > 10 \) with the following property: If \( Q = Q_\delta(\eta) \), \( 0 < \delta < 1 \), and
\[
h(z) = \frac{i \langle z, \eta \rangle}{[2 + \beta \delta^{-2}(1 - \langle z, \eta \rangle)]^{4n/p}} \quad (z \in \bar{B}),
\]
then
\[
\begin{align*}
(1) \quad \int_S |h|^{1/2} d\sigma &< A(\delta), \\
(2) \quad \int_S \chi_Q - \text{Re} \ h|^{1/2} d\sigma &< \alpha \cdot A(\delta), \\
(3) \quad \int_B |\nabla h|^p dV &< \delta^{2n+2-2p}.
\end{align*}
\]

**Proof.** For \( t \geq 1 \) and \( z \in \bar{B} \), define
\[
h_t(z) = \frac{i < z, \eta >}{[2 + t(1 - < z, \eta >)]^{4n/p}}.
\]
Then
\[
(4) \quad |\nabla h_t(z)|^p \leq \frac{1}{|2 + t(1 - < z, \eta >)|^{4n}} + \frac{8nt^p}{|2 + t(1 - < z, \eta >)|^{4n+p}}.
\]
By [3, Proposition 17] we have the following integral formula:
\[
\int_B \frac{dV(z)}{|1 - < z, w >|^l} = \left( \frac{1}{1 - |w|^2} \right)^{l-1} \int_B \frac{dV(z)}{|1 - < z, w >|^{2n+2-l}}
\]
for \( l \) real and \( w \in \bar{B} \). From this it is easily verified that
\[
\int_B \frac{dV(z)}{|2 + t(1 - < z, \eta >)|^l} < \left( \frac{1}{2t} \right)^{n+1}
\]
holds for every \( l > 2n + 2 \). Thus, by (4),
\[
\int_B |\nabla h_t|^p dV < \frac{10n}{2^{n+1}} \left( \frac{1}{t} \right)^{n+1-p}.
\]
which in turn implies (3) whenever $\beta > 10$. We need find such $\beta$ for which (1) and (2) also hold.

Assume $t$ is sufficiently large so that $t\delta^2 \geq 2$. Then the set $E_t = \{\zeta \in \mathcal{S} : |1 - t(1 - \langle \zeta, \eta \rangle)| < 1\}$ is contained in $Q$, and hence

$$\int_Q |\text{Re} \, h_i|^2 \, d\sigma \geq \int_{E_t} |\text{Re} \, h_i(\zeta)|^2 \, d\sigma(\zeta).$$

On the right side of the above, make successive changes of variables, first $\lambda = \langle \zeta, \eta \rangle$ and then $w = t(1 - \lambda)$, to obtain

$$\frac{n - 1}{\pi t^n} \int_D \left| \text{Im} \, \frac{1 - t^{-1}w}{(2 + w)^{4n/p}} \right|^2 \left( 2 \text{Re} \, w - \frac{|w|^2}{t} \right)^{n-2} \, dm(w)$$

where $D = \{w \in \mathbb{C} : |1 - w| < 1\}$ and $m$ denotes the area measure on $\mathbb{C}$. From this we easily obtain a positive constant $c = c(n, p) < 1$ such that

$$\frac{1}{8} \int_Q |\text{Re} \, h_i|^2 \, d\sigma \geq \left( \frac{c}{t} \right)^n$$

whenever $t\delta^2 \geq 2$.

We now define $\beta = 10/c$. Then $\beta > 10$ and [5, §3.7] shows that (1) and (2) hold with $\alpha = 1 - (c^2/20)^n + (c^2/25)^n$. This completes the proof. \qed

Lemma 2. Let $0 < p < 1$. Then there is a constant $\gamma = \gamma(n, p) < 1$ with the following property: If $Q = Q_r(\eta)$, $0 < r < 1$, and $\tau_i > 0$, $i = 1, 2$, then there exists a function $f \in A(B)$ with $f(0) = 0$ such that

(5) $|f| < 1$ on $Q$,

(6) $|f| < \tau_1$ on $S \setminus Q$,

(7) $\int_S |f|^{1/2} \, d\sigma < A(r)$,

(8) $\int_S |x \cdot \text{Re} \, f|^{1/2} \, d\sigma < \gamma \cdot A(r)$,

(9) $\int_B |\nabla f|^p \, dV < \tau_2$.

Proof. Put $\tau = \min\{1/2, \tau_1, \tau_2^{1/(1-p)}\}$ and let $\delta = \tau r$. Pick a maximal collection $\{\zeta_1, \ldots, \zeta_N\}$ of points in $Q_{r/2}(\eta)$ subject to the condition $d(\zeta_i, \zeta_j) \geq 2\delta$ for $i \neq j$. Then the $d$-balls $Q_j = Q_{\delta}(\zeta_j)$ are pairwise disjoint and contained in $Q$. Thus, by [4, Proposition 5.1.4]

$$N \leq \frac{A(r)}{A(\delta)} \leq \left( \frac{r}{\delta} \right)^{2n} = \tau^{-2n}.$$

Associate $h_j$ to $Q_j$ as in Lemma 1 and define

$$f = h_1 + \cdots + h_N.$$
Obviously \( f \in A(B) \), \( f(0) = 0 \), and [5, §3.8] shows that \( f \) satisfies (5) \( \sim \) (8) with \( \gamma = [1 - (1 - \alpha)16^{-n}] \), where \( \alpha \) is the same constant as in Lemma 1.

Note that \( |\nabla f|^p \leq |\nabla h_1|^p + \cdots + |\nabla h_N|^p \). Therefore, by (10) and the choice of \( \tau \), we have

\[
\int_B |\nabla f|^p \, dV \leq N\delta^{2n+2-2p} < \tau^{2-2p} \leq \tau^2.
\]

This proves (9). The proof is complete. \( \Box \)

In the following lemma, \( P[\psi] \) denotes the Poisson integral of \( \psi \in L^1(\sigma) \). That is,

\[
P[\psi](z) = \int_S \frac{1 - |z|^2}{|z - \zeta|^{2n}} \psi(\zeta) \, d\sigma(\zeta) \quad (z \in B).
\]

**Lemma 3.** Suppose that \( \varphi > 0 \) is a lower semicontinuous function on \( S \) and \( \varphi \in L^1(\sigma) \). Then, for given \( 0 < p < 1 \) and \( \epsilon > 0 \), there exists a function \( F \in H^{1/2}(B) \) with \( F(0) = 0 \) such that

(a) \( \Re F^* = \varphi \) almost everywhere on \( S \)

(b) \( \Re F \leq P[\varphi] \) on \( B \), and

(c) \( \int_B |\nabla F|^p \, dV < \epsilon \).

**Note.** In the proof of the main theorem the fact that \( F(0) = 0 \) will not be used. However, it implies that \( F \) is nonconstant as far as the lemma itself is concerned.

**Proof.** Let \( \gamma < 1 \) be the same constant as in Lemma 2, and assume \( \int_S \varphi^{1/2} \, d\sigma < 1 \) without loss of generality. We will choose by induction a sequence of functions \( F_0, F_1, \ldots \) such that for \( N = 0, 1, \ldots \),

\[ \begin{align*}
(11_N) \quad F_N & \in A(B) \quad \text{and} \quad F_N(0) = 0, \\
(12_N) \quad \varphi - \Re (F_0 + \cdots + F_N) > 0 \quad \text{on} \ S, \\
(13_N) \quad \int_S [\varphi - \Re (F_0 + \cdots + F_N)]^{1/2} \, d\sigma < \left( \frac{1 + \gamma}{2} \right)^N, \\
(14_N) \quad \int_S |F_N|^{1/2} \, d\sigma < \left( \frac{1 + \gamma}{2} \right)^{N-1}, \\
(15_N) \quad \int_B |\nabla F_N|^p \, dV < \frac{\epsilon}{2^{N+1}}.
\end{align*} \]

To begin the induction, put \( F_0 \equiv 0 \); (11_0) \( \sim \) (15_0) are trivial. Suppose now that \( F_0, \ldots, F_N \) have been chosen so that (11_N) \( \sim \) (15_N) hold. Put \( \varphi_N = \varphi - \Re (F_0 + \cdots + F_N) \). Then \( \varphi_N \) is a positive lower semicontinuous function on \( S \) and thus can be approximated by an increasing sequence of positive continuous functions on \( S \). Therefore the proof of [5, Lemma 3.4] shows that there are positive numbers \( c_1, \ldots, c_j \), and a disjoint collection
\[ \{ Q_1, \ldots, Q_J \} = \{ Q_{r_1}(\eta_1), \ldots, Q_{r_J}(\eta_J) \}, \quad 0 < r_j < 1, \] of \( d \)-balls such that
\[ \int_S \left( \phi_n - \sum_{j=1}^J c_j \chi_{Q_j} \right)^{1/2} < \frac{1 - \gamma}{2} \int_S \phi_n^{1/2} d\sigma \]
and
\[ \phi_n - \sum_{j=1}^J c_j \chi_{Q_j} > \tau \quad \text{on} \ S \]
for some positive number \( \tau \). Associate \( f_j \) to \( Q_j \) as in Lemma 2, with \( \tau_1 = \tau/2Jc_j \) and \( \tau_2 = \varepsilon/2^{N+2}Jc_j^p \) for \( j = 1, \ldots, J \) and define
\[ F_{N+1} = c_1 f_1 + \cdots + c_J f_j. \]
Then (11\(_{N+1}\)) is clear, and [5, §3.9] shows that (12\(_{N+1}\)) \((14\(_{N+1}\)) \text{ are satisfied. Finally,}
\[ \int_B |\nabla F_{N+1}|^p dV \leq \sum_{j=1}^J c_j^p \int_B |\nabla f_j|^p dV < \frac{\varepsilon}{2^{N+2}}. \]
This proves (15\(_{N+1}\)), and the construction can therefore proceed. By (14\(_N\))
\[ \sum_{N=0}^\infty \int_S |F_N|^{1/2} d\sigma < \infty, \]
so that \( F = \sum_{N=0}^\infty F_N \in H^{1/2}(B) \) and \( F(0) = 0 \), since the same is true of each \( F_N \). Next, (a) and (c) follow from (13\(_N\)) and (15\(_N\)), respectively. Finally, (b) follows from (12\(_N\)) by the harmonicity of each \( F_0 + \cdots + F_N \). The proof is complete. \( \Box \)

**Proof of main theorem.** We may assume \( 0 < p < \infty \). Also, without loss of generality, assume \( \varphi > 1 \) so that \( \log \varphi > 0 \). Note that \( \log \varphi \in L^1(\sigma) \) by Jensen’s inequality and that \( 1/r = 1/q - 1/p > 1 \) by assumption. Thus, by Lemma 3, there is a function \( F \in H^{1/2}(B) \) such that
\[ \text{Re } F^* = \log \varphi \quad \text{almost everywhere on } S, \]
\[ \text{Re } F \leq P[\log \varphi] \quad \text{on } B, \]
and
\[ \left( \int_B |\nabla F|^r dV \right)^{1/r} \left( \int_S \varphi^p d\sigma \right)^{1/p} < \left( \frac{\varphi}{2} \right)^{1/q}. \]
Choose an inner function \( u \) on \( B \) with \( u(0) = 0 \) such that
\[ \left( \int_B |\nabla u|^r dV \right)^{1/r} \left( \int_S \varphi^p d\sigma \right)^{1/p} < \left( \frac{\varphi}{2} \right)^{1/q}. \]
This can be done because $r < 1$ (see the Introduction). We now define $f = u \exp F$. Clearly $f(0) = 0$, $f \in H^p(B)$ by (17), and $|f^*| = \varphi$ almost everywhere on $S$ by (16). Note that $(P[\varphi^q])^{p/q} \leq P[\varphi^p]$ by Jensen's inequality. Thus, by Hölder's inequality and (18), we obtain

$$\int_B |\nabla F|^q P[\varphi^q] \, dV \leq \left( \int_B |\nabla F|^r \, dV \right)^{q/r} \left( \int_B P[\varphi^p] \, dV \right)^{q/p} = \left( \int_B |\nabla F|^r \, dV \right)^{q/r} \left( \int_S \varphi^p \, d\sigma \right)^{q/p} < \frac{\varepsilon}{2}.$$  

Similarly, by (19),

$$\int_B |\nabla u|^q P[\varphi^q] \, dV < \frac{\varepsilon}{2}.$$  

It follows that

$$\int_B |\nabla f|^q \, dV < \varepsilon$$

because $|\nabla f|^q \leq (|\nabla u|^q + |\nabla F|^q) P[\varphi^q]$. The proof is complete. □

**Remark.** Let $N_*(B)$ denote the Smirnov class consisting of holomorphic functions on $B$ for which

$$\sup_{0<r<1} \int_S \log^+ |f_r| \, d\sigma < \infty$$

and $\{\log^+ |f_r|\}$ is uniformly integrable (with respect to the measure $\sigma$), where $f_r(\zeta) = f(r\zeta)$ for $\zeta \in S$. Then a trivial modification of the above proof of the Main Theorem shows the following:

**Theorem 4.** Let $\varphi > 0$ be a lower semicontinuous function on $S$ such that

$$\log \varphi \in L^1(\sigma).$$

Then, for a given $\varepsilon > 0$, there exists a function $f \in N_*(B)$ with $f(0) = 0$ such that $|f^*| = \varphi$ almost everywhere on $S$ and $\exp \int_B \log |\nabla f| \, dV < \varepsilon$. □

**References**


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