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DERIVATIVES OF HARDY FUNCTIONS

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ABSTRACT. Let B be the open unit ball of $\mathbb{C}^n$, and set $S = \partial B$. It is shown that if $\varphi \in L^p(S)$, $\varphi > 0$, is a lower semicontinuous function on S and $1/q > 1 + 1/p$, then, for a given $\varepsilon > 0$, there exists a function $f \in H^p(B)$ with $f(0) = 0$ such that $|f^*| = \varphi$ almost everywhere on S and $\int_B |\nabla f|^q dV < \varepsilon$ where V denotes the normalized volume measure on B.

1. INTRODUCTION

Let B be the open unit ball of $\mathbb{C}^n$, and set $S = \partial B$. The rotation-invariant probability measure on S will be denoted by σ. The Hardy space $H^p(B)$, $0 < p < \infty$, is then the space of holomorphic functions on B for which

$$\sup_{0<r<1} \int_S |f(r\zeta)|^p d\sigma(\zeta) < \infty.$$

For $p = \infty$, $H^\infty(B)$ denotes the space of bounded holomorphic functions on B. As is well known, to each H^p-function f corresponds its boundary value f^* defined almost everywhere on S by $f^*(\zeta) = \lim_{r \nearrow 1} f(r\zeta)$; the term "almost every" will always refer to the measure σ. A remarkable fact about these boundary values of H^p-functions is that they can be prescribed as in the following.

Theorem A (Alexandrov [1]). *Suppose $\varphi > 0$ is a lower semicontinuous function on S and $\varphi \in L^p(\sigma)$ for some $0 < p \leq \infty$. Then there exists a function $f \in H^p(B)$ with $f(0) = 0$ such that $|f^*| = \varphi$ almost everywhere on S.*

At least for $n > 1$, it seems to be the price we pay for prescribing boundary values of H^p-functions as above that their derivatives may get out of control in general. Roughly speaking, when $n > 1$, a theorem of Rudin [5] shows that if $f \in H^\infty(B)$ is nonconstant and $|f^*| = 1$ on some open subset Γ of S, then f must be extremely oscillatory near a G_δ dense subset Ω of Γ, so that its derivatives may get huge near Ω. One such example (when $n > 1$) is the fact [2] that if f is inner (i.e., $f \in H^\infty(B)$ and $|f^*| = 1$ almost everywhere on

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S) and if $\int_B |\nabla f|^2 dV < \infty$, then f must be constant. Here and elsewhere, $\nabla f = (\partial f/\partial z_1, \dots, \partial f/\partial z_n)$ and V denotes the normalized volume measure on B. In some special cases, however, one may have those derivatives under very good control. For example, one can choose f in Theorem A so that $\int_B |\nabla f|^q dV$ is arbitrarily small whenever $1/q > 1 + n/p$. This follows from Theorem A itself.

Proof (sketch). Let f be as in Theorem A, fix an inner function u on B such that $u(0) = 0$, and define $f_m = u^m f$ for $m = 1, 2, \dots$. Then Theorem A holds with f_m in place of f. Note that f_m converges to 0 uniformly on compact subsets of B and therefore so does $|\nabla f_m|$. Since

$$|f_m(z)| \le C_{n,p} \|\varphi\|_{L^p(\sigma)} \left(\frac{1}{1-|z|}\right)^{n/p} \qquad (z \in B),$$

we have by the Cauchy estimates

$$|\nabla f_m(z)| \le C_{n,p} \|\varphi\|_{L^p(\sigma)} \left(\frac{1}{1-|z|}\right)^{1+n/p} \qquad (z \in B).$$

It follows that the sequence $|\nabla f_m|^q$ is dominated by an integrable function on B whenever $1/q > 1 + n/p$, and thus $\int_B |\nabla f_m|^q dV$ converges to 0 as $m \to \infty$. □

In the present paper we obtain a better upper bound for q in the above observation, which is the main result of the paper.

Main theorem. *Let $\varphi > 0$ be a lower semicontinuous function on S. Suppose $\varphi \in L^p(\sigma)$ for some $0 < p \le \infty$ and $1/q > 1 + 1/p$. Then, for a given $\varepsilon > 0$, there exists a function $f \in H^p(B)$ with $f(0) = 0$ such that $|f^*| = \varphi$ almost everywhere on S and $\int_B |\nabla f|^q dV < \varepsilon$.*

The author does not know whether the upper bound $(1+1/p)^{-1}$ for q in the above theorem is sharp in general. There is, however, a reason why the upper bound cannot be bigger than 1, as the following remark shows.

Remark. Suppose $f \in H^p(B)$ and $f(0) = 0$. Then

$$|f^*(\zeta)| \le \int_0^1 \left|\frac{\partial}{\partial t} f(t\zeta)\right| dt \le \int_0^1 |\nabla f(t\zeta)|\, dt$$

for almost every $\zeta \in S$. Hence, for $q \ge 1$, we have

$$\int_S |f^*|^q d\sigma \le \int_S \int_0^1 |\nabla f(t\zeta)|^q dt\, d\sigma(\zeta) \le C_{n,q} \int_B |\nabla f|^q dV.$$

It follows that if the conclusion of the main theorem holds for some q, then $q < 1$. Note that the upper bound for q in the main theorm is sharp for $p = \infty$.

2. Proof of main theorem

We need prove the main theorem only for higher-dimensional cases. From now on, $n > 1$ will therefore be fixed. The main idea is to modify Rudin's

proof [5] of Theorem A so that the derivatives in question are all under control at the same time. Before going further, let us introduce some notation first. There is a nonisotropic metric d on S defined by

$$d(\zeta, \eta) = |1 - < \zeta, \eta >|^{1/2} \qquad (\zeta, \eta \in S).$$

Since σ is rotation-invariant, the volume $\sigma(Q_\delta(\eta))$ of the corresponding balls

$$Q_\delta(\eta) = \{\zeta \in S : d(\zeta, \eta) < \delta\} \qquad (0 < \delta < \sqrt{2})$$

is independent of η and thus will be denoted by $A(\delta)$. The notation $A(B)$ will denote the class of functions holomorphic on B and continuous on $\bar{B}$. Finally, χ_E will denote the characteristic function of $E \subset S$.

Lemma 1. *Let* $0 < p < 1$. *Then there are constants* $\alpha = \alpha(n, p) < 1$ *and* $\beta = \beta(n, p) > 10$ *with the following property: If* $Q = Q_\delta(\eta)$, $0 < \delta < 1$, *and*

$$h(z) = \frac{i\langle z, \eta\rangle}{[2 + \beta\delta^{-2}(1 - \langle z, \eta\rangle)]^{4n/p}} \qquad (z \in \overline{B}),$$

then

$$\int_S |h|^{1/2} d\sigma < A(\delta), \tag{1}$$

$$\int_S |\chi_Q - Re\, h|^{1/2} d\sigma < \alpha \cdot A(\delta), \tag{2}$$

$$\int_B |\nabla h|^p dV < \delta^{2n+2-2p}. \tag{3}$$

Proof. For $t \geq 1$ and $z \in \bar{B}$, define

$$h_t(z) = \frac{i < z, \eta >}{[2 + t(1 - < z, \eta >)]^{4n/p}}.$$

Then

$$|\nabla h_t(z)|^p \leq \frac{1}{|2 + t(1 - < z, \eta >)|^{4n}} + \frac{8nt^p}{|2 + t(1 - < z, \eta >)|^{4n+p}}. \tag{4}$$

By [3, Proposition 17] we have the following integral formula:

$$\int_B \frac{dV(z)}{|1 - < z, w >|^l} = \left(\frac{1}{1 - |w|^2}\right)^{l-n-1} \int_B \frac{dV(z)}{|1 - < z, w >|^{2n+2-l}}$$

for l real and $w \in B$. From this it is easily verified that

$$\int_B \frac{dV(z)}{|2 + t(1 - < z, \eta >)|^l} < \left(\frac{1}{2t}\right)^{n+1}$$

holds for every $l > 2n + 2$. Thus, by (4),

$$\int_B |\nabla h_t|^p dV < \frac{10n}{2^{n+1}} \left(\frac{1}{t}\right)^{n+1-p},$$

which in turn implies (3) whenever $\beta > 10$. We need find such β for which (1) and (2) also hold.

Assume t is sufficiently large so that $t\delta^2 \geq 2$. Then the set $E_t = \{\zeta \in S : |1 - t(1 - <\zeta, \eta>)| < 1\}$ is contained in Q, and hence

$$\int_Q |\mathrm{Re}\, h_t|^2 d\sigma \geq \int_{E_t} |\mathrm{Re}\, h_t(\zeta)|^2 d\sigma(\zeta).$$

On the right side of the above, make successive changes of variables, first $\lambda = \langle \zeta, \eta \rangle$ and then $w = t(1 - \lambda)$, to obtain

$$\frac{n-1}{\pi t^n} \int_D \left| \mathrm{Im}\, \frac{1 - t^{-1} w}{(2+w)^{4n/p}} \right|^2 \left(2\mathrm{Re}\, w - \frac{|w|^2}{t} \right)^{n-2} dm(w)$$

where $D = \{w \in \mathrm{C} : |1 - w| < 1\}$ and m denotes the area measure on C. From this we easily obtain a positive constant $c = c(n, p) < 1$ such that

$$\frac{1}{8} \int_Q |\mathrm{Re}\, h_t|^2 d\sigma \geq \left(\frac{c}{t} \right)^n$$

whenever $t\delta^2 \geq 2$.

We now define $\beta = 10/c$. Then $\beta > 10$ and [5, §3.7] shows that (1) and (2) hold with $\alpha = 1 - (c^2/20)^n + (c^2/25)^n$. This completes the proof. □

Lemma 2. *Let $0 < p < 1$. Then there is a constant $\gamma = \gamma(n, p) < 1$ with the following property: If $Q = Q_r(\eta)$, $0 < r < 1$, and $\tau_i > 0$, $i = 1, 2$, then there exists a function $f \in A(B)$ with $f(0) = 0$ such that*

$$|f| < 1 \quad \text{on } Q, \tag{5}$$

$$|f| < \tau_1 \quad \text{on } S \setminus Q, \tag{6}$$

$$\int_S |f|^{1/2} d\sigma < A(r), \tag{7}$$

$$\int_S |\chi_Q - Re\, f|^{1/2} d\sigma < \gamma \cdot A(r), \tag{8}$$

$$\int_B |\nabla f|^p dV < \tau_2. \tag{9}$$

Proof. Put $\tau = \min\{1/2, \tau_1, \tau_2^{1/2(1-p)}\}$ and let $\delta = \tau r$. Pick a maximal collection $\{\zeta_1, \ldots, \zeta_N\}$ of points in $Q_{r/2}(\eta)$ subject to the condition $d(\zeta_i, \zeta_j) \geq 2\delta$ for $i \neq j$. Then the d-balls $Q_j = Q_\delta(\zeta_j)$ are pairwise disjoint and contained in Q. Thus, by [4, Proposition 5.1.4]

$$N \leq \frac{A(r)}{A(\delta)} \leq \left(\frac{r}{\delta} \right)^{2n} = \tau^{-2n}. \tag{10}$$

Associate h_j to Q_j as in Lemma 1 and define

$$f = h_1 + \cdots + h_N.$$

Obviously $f \in A(B)$, $f(0) = 0$, and [5, §3.8] shows that f satisfies (5) $\sim$ (8) with $\gamma = [1 - (1-\alpha)16^{-n}]$, where α is the same constant as in Lemma 1.

Note that $|\nabla f|^p \le |\nabla h_1|^p + \cdots + |\nabla h_N|^p$. Therefore, by (10) and the choice of τ, we have

$$\int_B |\nabla f|^p dV \le N\delta^{2n+2-2p} < \tau^{2-2p} \le \tau_2 .$$

This proves (9). The proof is complete. □

In the following lemma, $P[\psi]$ denotes the Poisson integral of $\psi \in L^1(\sigma)$. That is,

$$P[\psi](z) = \int_S \frac{1-|z|^2}{|z-\zeta|^{2n}}\, \psi(\zeta)\, d\sigma(\zeta) \qquad (z \in B).$$

Lemma 3. *Suppose that $\varphi > 0$ is a lower semicontinuous function on S and $\varphi \in L^1(\sigma)$. Then, for given $0 < p < 1$ and $\varepsilon > 0$, there exists a function $F \in H^{1/2}(B)$ with $F(0) = 0$ such that*

(a) *$\operatorname{Re} F^* = \varphi$ almost everywhere on S*
(b) *$\operatorname{Re} F \le P[\varphi]$ on B, and*
(c) *$\int_B |\nabla F|^p dV < \varepsilon$.*

Note. In the proof of the main theorem the fact that $F(0) = 0$ will not be used. However, it implies that F is nonconstant as far as the lemma itself is concerned.

Proof. Let $\gamma < 1$ be the same constant as in Lemma 2, and assume $\int_S \varphi^{1/2}\, d\sigma < 1$ without loss of generality. We will choose by induction a sequence of functions $F_0, F_1, \ldots$ such that for $N = 0, 1, \ldots,$

$$(11_N) \qquad F_N \in A(B) \quad \text{and} \quad F_N(0) = 0,$$

$$(12_N) \qquad \varphi - \operatorname{Re}(F_0 + \cdots + F_N) > 0 \quad \text{on } S,$$

$$(13_N) \qquad \int_S [\varphi - \operatorname{Re}(F_0 + \cdots + F_N)]^{1/2} d\sigma < \left(\frac{1+\gamma}{2}\right)^N,$$

$$(14_N) \qquad \int_S |F_N|^{1/2} d\sigma < \left(\frac{1+\gamma}{2}\right)^{N-1},$$

$$(15_N) \qquad \int_B |\nabla F_N|^p dV < \frac{\varepsilon}{2^{N+1}}.$$

To begin the induction, put $F_0 \equiv 0$; $(11_0) \sim (15_0)$ are trivial. Suppose now that $F_0, \ldots, F_N$ have been chosen so that $(11_N) \sim (15_N)$ hold. Put $\varphi_N = \varphi - \operatorname{Re}(F_0 + \cdots + F_N)$. Then φ_N is a positive lower semicontinuous function on S and thus can be approximated by an increasing sequence of positive continuous functions on S. Therefore the proof of [5, Lemma 3.4] shows that there are positive numbers $c_1, \ldots, c_J$, and a disjoint collection

$\{Q_1, \dots, Q_J\} = \{Q_{r_1}(\eta_1), \dots, Q_{r_J}(\eta_J)\}$, $0 < r_j < 1$, of d-balls such that

$$\int_S \Big(\varphi_N - \sum_{j=1}^{J} c_j \chi_{Q_j}\Big)^{1/2} < \frac{1-\gamma}{2} \int_S \varphi_N^{1/2} d\sigma$$

and

$$\varphi_N - \sum_{j=1}^{J} c_j \chi_{Q_j} > \tau \quad \text{on } S$$

for some positive number τ. Associate f_j to Q_j as in Lemma 2, with $\tau_1 = \tau/2Jc_j$ and $\tau_2 = \varepsilon/2^{N+2}Jc_j^p$ for $j = 1, \dots, J$ and define

$$F_{N+1} = c_1 f_1 + \cdots + c_J f_J.$$

Then (11_{N+1}) is clear, and [5, §3.9] shows that $(12_{N+1}) \sim (14_{N+1})$ are satisfied. Finally,

$$\int_B |\nabla F_{N+1}|^p dV \le \sum_{j=1}^{J} c_j^p \int_B |\nabla f_j|^p dV < \frac{\varepsilon}{2^{N+2}}.$$

This proves (15_{N+1}), and the construction can therefore proceed. By (14_N)

$$\sum_{N=0}^{\infty} \int_S |F_N|^{1/2} d\sigma < \infty,$$

so that $F = \sum_{N=0}^{\infty} F_N \in H^{1/2}(B)$ and $F(0) = 0$, since the same is true of each F_N. Next, (a) and (c) follow from (13_N) and (15_N), respectively. Finally, (b) follows from (12_N) by the harmonicity of each $F_0 + \cdots + F_N$. The proof is complete. □

Proof of main theorem. We may assume $0 < p < \infty$. Also, without loss of generality, assume $\varphi > 1$ so that $\log\varphi > 0$. Note that $\log\varphi \in L^1(\sigma)$ by Jensen's inequality and that $1/r = 1/q - 1/p > 1$ by assumption. Thus, by Lemma 3, there is a function $F \in H^{1/2}(B)$ such that

$$\text{Re } F^* = \log\varphi \quad \text{almost everywhere on } S, \tag{16}$$

$$\text{Re } F \le P[\log\varphi] \quad \text{on } B, \tag{17}$$

and

$$\Big(\int_B |\nabla F|^r dV\Big)^{1/r} \Big(\int_S \varphi^p d\sigma\Big)^{1/p} < \Big(\frac{\varepsilon}{2}\Big)^{1/q}. \tag{18}$$

Choose an inner function u on B with $u(0) = 0$ such that

$$\Big(\int_B |\nabla u|^r dV\Big)^{1/r} \Big(\int_S \varphi^p d\sigma\Big)^{1/p} < \Big(\frac{\varepsilon}{2}\Big)^{1/q}. \tag{19}$$

This can be done because $r < 1$ (see the Introduction). We now define $f = u \exp F$. Clearly $f(0) = 0$, $f \in H^p(B)$ by (17), and $|f^*| = \varphi$ almost everywhere on S by (16). Note that $(P[\varphi^q])^{p/q} \leq P[\varphi^p]$ by Jensen's inequality. Thus, by Hölder's inequality and (18), we obtain

$$\int_B |\nabla F|^q P[\varphi^q]\, dV \leq \left(\int_B |\nabla F|^r dV\right)^{q/r} \left(\int_B P[\varphi^p]\, dV\right)^{q/p}$$
$$= \left(\int_B |\nabla F|^r dV\right)^{q/r} \left(\int_S \varphi^p d\sigma\right)^{q/p} < \frac{\varepsilon}{2}.$$

Similarly, by (19),

$$\int_B |\nabla u|^q P[\varphi^q] dV < \frac{\varepsilon}{2}.$$

It follows that

$$\int_B |\nabla f|^q dV < \varepsilon$$

because $|\nabla f|^q \leq (|\nabla u|^q + |\nabla F|^q)\, P[\varphi^q]$. The proof is complete. □

Remark. Let $N_*(B)$ denote the Smirnov class consisting of holomorphic functions on B for which

$$\sup_{0<r<1} \int_S \log^+ |f_r|\, d\sigma < \infty$$

and $\{\log^+ |f_r|\}$ is uniformly integrable (with respect to the measure σ), where $f_r(\zeta) = f(r\zeta)$ for $\zeta \in S$. Then a trivial modification of the above proof of the Main Theorem shows the following:

Theorem 4. *Let $\varphi > 0$ be a lower semicontinuous function on S such that $\log \varphi \in L^1(\sigma)$. Then, for a given $\varepsilon > 0$, there exists a function $f \in N_*(B)$ with $f(0) = 0$ such that $|f^*| = \varphi$ almost everywhere on S and $\exp \int_B \log |\nabla f|\, dV < \varepsilon$.* □

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