DIVISIBILITY BY 2 OF STIRLING-LIKE NUMBERS

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Abstract. We give a characterization of functions of the form \( f(n) = \nu(n - E) \), where \( \nu(-) \) denotes the exponent of 2, and \( E \) is a 2-adic integer. We show that it applies to the restriction to even or odd integers of the function \( f(n) = \nu(a \cdot 5^n + b \cdot 3^n + c) \), with mild restrictions on \( a, b, \) and \( c \). This function is closely related to divisibility of certain Stirling numbers of the second kind.

Let \( \nu(m) \) denote the exponent of 2 in \( m \), and let \( \mathbb{N} \) denote the set of nonnegative integers. Our first result is a characterization of a certain class of functions. We think of a 2-adic integer as a possibly infinite sum of distinct 2-powers.

Theorem 1. Let \( f \) be a function \( \mathbb{N} \to \mathbb{N} \cup \{\infty\} \). Then the following are equivalent:

(i) There is a 2-adic integer \( E \) such that \( f(n) = \nu(n - E) \) for all \( n \);

(ii) For all \( n \) and \( d \),

\[
\begin{align*}
f(n + 2^d) &= d & \text{if } d < f(n) \\
&> f(n) & \text{if } d = f(n) \\
&= f(n) & \text{if } d > f(n);
\end{align*}
\]

(iii) \( f \) satisfies

(a) for all \( n \), \( f(n + 2^{f(n)}) > f(n) \), and

(b) if \( d < f(n) \) and \( \alpha \) is odd, then \( f(n + \alpha 2^d) = d \).

\( E \) is defined by \( E = \sum_{i=1}^{\infty} 2^e_i \) with \( e_i < e_{i+1} \), where \( e_1 = f(0) \), \( E(k) = \sum_{i=1}^{k} 2^e_i \), and \( e_{k+1} = f(E(k)) \). Finally, \( E \) is finite if and only if \( \infty \in f(\mathbb{N}) \).

Proof. (i)\(\Rightarrow\)(ii): \( f(n) \) equals the exponent of the smallest 2-power in which \( n \) and \( E \) differ. Then \( n + 2^{f(n)} \) and \( E \) agree in that 2-power as well, while \( n + 2^d \) and \( E \) first differ at \( 2^d \) if \( d < f(n) \) and at \( 2^{f(n)} \) if \( d > f(n) \).

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(ii)⇒(iii): Write \( \alpha 2^d = 2^d + \Sigma 2^{e_i} \), with \( e_i > d \). Then \( f(n + 2^d) = d \), and adding each \( 2^{e_i} \) to the argument does not change the value of \( f \).

(iii)⇒(i): Define \( E \) as in the theorem. Hypothesis (a) applied to \( E(i-1) \) guarantees that \( e_i < e_{i+1} \). For any \( n \), let \( d = \nu(n-E) \), and choose \( k \) so that \( e_k \leq d < e_{k+1} \). Then \( n-E(k) = \alpha 2^d \), with \( \alpha \) odd, and so we apply (b) with \( n = E(k) \), noting that the hypothesis is satisfied since \( f(E(k)) = e_{k+1} > d \). We deduce \( f(n) = f(E(k) + \alpha 2^d) = d \).

Our second result is an application of Theorem 1.

**Theorem 2.** If \( a \) is odd, \( b \equiv 2 \mod 4 \), and \( a+b+c \equiv 0 \mod 8 \), then there exist 2-adic integers \( E \) and \( E' \) such that

\[
\nu(a \cdot 5^n + b \cdot 3^n + c) = 2 + \left\{ \begin{array}{ll}
\nu(n-E) & \text{if } n \text{ even} \\
\nu(n-E') & \text{if } n \text{ odd}
\end{array} \right.
\]

Let \( g(n) = -2 + \nu(a \cdot 5^n + b \cdot 3^n + c) \). If \( E = \Sigma 2^{e_i} \) with \( e_i < e_{i+1} \) and \( E' = \Sigma 2^{e_i'} \) with \( e_i' < e_{i+1}' \), then \( e_1 = g(0) \), \( e_{k+1} = g(E(k)) \), \( e_1' = 0 \), and \( e_{k+1}' = g(E'(k)) \).

**Proof.** We use the well-known and easily proved fact that

\[
\nu(p^i - 1) = \left\{ \begin{array}{ll}
\nu(4i) & \text{if } p = 5, \text{ or } p = 3 \text{ and } i \text{ even} \\
1 & \text{if } p = 3 \text{ and } i \text{ odd}
\end{array} \right.
\]

to observe that if

\[
\nu(a \cdot 5^n + b \cdot 3^n + c) = e + 2,
\]

and \( \alpha \) is odd and \( d > 0 \), then

\[
\nu(a \cdot 5^{n+d} + b \cdot 3^{n+d} + c) = \left\{ \begin{array}{ll}
= d + 2 & \text{if } d < e \\
> e + 2 & \text{if } d = e \\
= e + 2 & \text{if } d > e.
\end{array} \right.
\]

Here we have used the decomposition

\[
a \cdot 5^{n+d} + b \cdot 3^{n+d} + c = 2^{e+2} \text{ odd} + a5^n(5^{n+2d} - 1) + b3^n(3^{a2d} - 1).
\]

Thus \( g \) satisfies (a) of Theorem 1(iii) and (b) restricted to \( d > 0 \). This implies that the functions \( f_0 \) and \( f_1 \) defined by

\[
f_{\epsilon}(n) = g(2n + \epsilon) - 1, \quad \epsilon = 0, 1
\]

satisfy the hypotheses of Theorem 1(iii). Indeed,

\[
f_{\epsilon}(n + 2f_{\epsilon}(n)) = g(2n + 2f_{\epsilon}(n+1) + \epsilon) - 1
\]

\[
= g(2n + 2g(2n+\epsilon) + \epsilon) - 1 > g(2n+\epsilon) - 1 = f_{\epsilon}(n),
\]

and if \( 0 \leq d < f_{\epsilon}(n), \) then \( 0 < d + 1 < g(2n+\epsilon) \); hence

\[
f_{\epsilon}(n + \alpha 2^d) = g(2n + \alpha 2^{d+1} + \epsilon) - 1 = (d+1) - 1.
\]
In order that $f_\varepsilon(m) \geq 0$, we need $g(2m + \varepsilon) > 0$, which follows from

$$g(2m + \varepsilon) = -2 + \nu(5^\varepsilon a(5^{2m} - 1) + 3^\varepsilon b(3^{2m} - 1) + (5^\varepsilon a + 3^\varepsilon b + c)),$$

since each of the three terms is divisible by 8, the last since the hypotheses imply $5a + 3b + c \equiv 0 \mod 8$.

Thus by Theorem 1, $f_\varepsilon(n) = \nu(n - E_\varepsilon)$ for some 2-adic $E_\varepsilon$, and

$$g(2n) = \nu(n - E_0) + 1 = \nu(2n - E)$$

with $E = 2E_0$, and

$$g(2n + 1) = \nu(n - E_1) + 1 = \nu(2n + 1 - E')$$

with $E' = 2E_1 + 1$.

Similar manipulations imply that $E$ and $E'$ satisfy the asserted defining property. \[\square\]

We remark that if $\nu(a + b + c) = \nu < 3$, then $\nu(a5^n + b3^n + c) = \nu$ for all even $n$, and if $\nu(5a + 3b + c) = \nu' < 3$, then $\nu(a5^n + b3^n + c) = \nu'$ for all odd $n$. This is immediate from (3). If $a + b$ is even, $a + b + c \equiv 0 \mod 8$, and $5a + 3b + c \equiv 0 \mod 8$, then $\nu(a \cdot 5^n + b \cdot 3^n + c)$ is more complicated.

This work was motivated by a desire to determine the exponent of 2 in some Stirling numbers of the second kind. These exponents are important in algebraic topology [3–5]. We take as our definition

$$S(n, k) = \sum_{i=0}^{k} \binom{k}{i} (k - i)^n, 1 \leq k \leq n.$$ 

See [2] for other formulas and combinatorial descriptions. In particular,

$$S(n, 5) = \frac{1}{5!}(5^n - 5 \cdot 4^n + 10 \cdot 3^n - 10 \cdot 2^n + 5).$$

One would expect the second and fourth terms to be much more highly 2-divisible than the combination of the others, in which case $\nu(S(n, 5))$ would equal $-1 + g(n)$, where

$$g(n) = -2 + \nu(5^n + 10 \cdot 3^n + 5).$$

Theorem 2 applies to this $g$, and a REDUCE program easily calculates the values of the exponents which are less than 100 to be

$$e_i : 2, 3, 4, 7, 12, 16, 17, 18, 19, 21, 22, 23, 25, 26, 28, 29, 30, 31, 34, 38, 41, 42, 45, 50, 51, 52, 53, 55, 57, 58, 60, 61, 62, 63, 64, 66, 67, 71, 73, 74, 75, 76, 77, 78, 79, 80, 81, 83, 87, 91, 94, 97, 98, 99.$$ 

$$e'_i : 0, 1, 2, 3, 4, 8, 11, 14, 16, 19, 20, 25, 27, 28, 29, 30, 35, 37, 39, 40, 44, 47, 48, 50, 53, 54, 57, 58, 60, 61, 62, 66, 68, 69, 70, 71, 73, 76, 78, 79, 83, 85, 89, 91, 94.$$ 

Now, if $E$ is defined from (4) as in Theorem 2, so that the $e_i$’s less than 100 are as above, then

$$\nu(S(n, 5)) = -1 + \nu(n - E)$$

for even $n$, provided $\nu(n - E) < n - 1$. 

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and similarly for odd $n$. The first failure of this would be for $n$ equal to the smallest $E(k)$ such that $e_{k+1} \geq E(k) - 1$. It can be observed that this never happens for $5 \leq n < 2^{100}$, and for it to happen subsequently would require the unlikely occurrence of more than $2^{94}$ consecutive 0's in the binary expansion of $E$ or $E'$.

It is then a simple matter to read off $\nu(S(n, 5))$ for $n$ even and $n < 2^{100}$ from the smallest 2-power in which $n$ differs from $E$, as determined from the list of $e_i$'s, and similarly for $n$ odd. For example, since

$$1989 = 2^0 + 2^2 + 2^6 + 2^7 + 2^8 + 2^9 + 2^{10},$$

we obtain $\nu(S(1989, 5)) = -1 + \nu(1989 - E') = 0$.

A similar discussion to all of this applies to $S(n, 6)$. If $k < 5$, then $\nu(S(n, k)) = 0$ or 1 depending on the parity of $n$, and for $k > 7$, $\nu(S(n, k))$ is somewhat more complicated to analyze. In work [1] stimulated by an earlier version of this paper, Francis Clarke has generalized this work in a number of directions (larger $k$, odd primes, and a more general context). I thank him for several useful comments on this work.

References

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