EXAMPLE OF AN ALGEBRA WHICH IS NONTOPOLOGIZABLE AS A LOCALLY CONVEX TOPOLOGICAL ALGEBRA

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Abstract. Let $X$ be a real or complex linear space and denote by $L(X)$ the algebra of all its endomorphisms. We prove that $L(X)$ is topologizable as a locally convex topological algebra (with jointly continuous multiplication) if and only if it is topologizable as a Banach algebra and this holds if and only if $X$ is of finite dimension.

A topological algebra is a Hausdorff topological linear space equipped with a jointly continuous associative multiplication. A topological algebra is said to be locally convex if its underlying topological linear space is locally convex. The topology of a locally convex algebra can be given by means of a family $(\|x\|_\alpha)$ of seminorms such that, for each index $\alpha$, there is an index $\beta$ such that

\[
\|xy\|_\alpha \leq \|x\|_\beta \|y\|_\beta
\]

for all $x$ and $y$ in the algebra in question (see [4]). For general information on topological algebras the reader is referred to [1, 2, 3, 4] and the references therein. Note that some authors define topological algebras as topological linear spaces equipped with a separately continuous multiplication (cf. [2]).

In [5] and [6] we posed the following question (Problem 2): Is it true that for every real or complex algebra there is a topology making it a locally convex algebra? In [6] we have shown that the answer is positive if we replace the requirement of joint continuity of multiplication by the weaker assumption of its separate continuity. We also conjectured (cf. [6, Problem 2a]) that the answer to the problem is negative if we take, as the algebra in question, the algebra of all endomorphisms of an infinite-dimensional real or complex linear space. The aim of this paper is to prove this conjecture, which gives the non-trivial implication in the result formulated in the abstract.

Theorem. Let $X$ be a real or complex infinite-dimensional vector space. Then there is no topology on the algebra $L(X)$ of all endomorphisms of $X$ making it a locally convex algebra.

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Proof. Assume that there is a system $(\|x\|_\alpha)$ of seminorms on $L(X)$ satisfying relations (1) and separating between the points of $L(X)$, and try to get a contradiction. Consider on $L(X)$ the one-dimensional operators $f \otimes z$, given by $x \mapsto f(x)z$, where $f$ is a linear functional on $X$ and $z \in X$. Let $g_0$ be a fixed linear functional on $X$, $g_0 \neq 0$, and fix a nonzero element $z_0$ in $X$. Since the seminorms on $L(X)$ separate between its elements we can choose an index $\alpha_0$ so that $\|g_0 \otimes z_0\|_{\alpha_0} \neq 0$. After a suitable normalization of $g_0$ we can assume

\[(2) \quad \|g_0 \otimes z_0\|_{\alpha_0} = 1.\]

For an arbitrary functional $f$ on $X$ and an arbitrary element $x$ in $X$, the operators $f \otimes z_0$ and $g_0 \otimes x$ are in $L(X)$. Applying to these operators relations (1) with $\alpha = \alpha_0$ and a suitable $\beta_0$, we obtain

\[(3) \quad \| (f \otimes z_0)(g_0 \otimes x) \|_{\alpha_0} \leq \| f \otimes z_0\|_{\beta_0} \| g_0 \otimes x\|_{\beta_0}.\]

We have $(f \otimes z_0)(g_0 \otimes x) = f(x)g_0 \otimes z_0$, and so, by (2), the left hand of (3) is $\|f(x)\|$. Put $c(f) = \| f \otimes z_0\|_{\beta_0}$ and $\|x\| = \| g_0 \otimes x\|_{\beta_0}$. With this notation we can rewrite (3) as

\[(4) \quad |f(x)| \leq c(f)\|x\|.\]

Relation (4) holds for all linear functionals $f$ on $X$ and all elements $x$ in $X$; it implies that $\| \cdot \|$ is a norm on $X$. This is because $\| \cdot \|$ is a seminorm on $X$ and, for any nonzero element $x$ in $X$, there is a linear functional $f$ on $X$ with $f(x) \neq 0$. Thus $X$ equipped with the norm $\| \cdot \|$ is a normed space and, by (4), all its linear functionals are continuous. It is obvious that the only normed spaces with this property are finite-dimensional and $X$ is of infinite dimension. The conclusion follows. \qed

Corollary. If $X$ is an infinite-dimensional real or complex vector space, then there is no topology on the algebra $L_{FD}(X)$ of all finite-dimensional endomorphisms of $X$ making of it a locally convex algebra.

The problem still remains open as to whether every commutative algebra over real or complex scalars can be given a topology so that it becomes a locally convex algebra.

References


* Added in proof. In a letter to the author Vladimir Müller gives a construction of a commutative algebra which is nontopologizable as a topological algebra. He shows also that the above theorem is true if we replace “locally convex algebra” by “topological algebra.”


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