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A RESULT ABOUT THE HILBERT TRANSFORM ALONG CURVES

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ABSTRACT. Let G be a connected and simply connected, nilpotent Lie group and let $\gamma\colon (-1, 1) \to G$ be a (connected) analytic curve such that $\gamma(0) = 0$. Then the Hilbert transform along γ,

$$Tf(x) = p.v. \int_{0<|t|<1} f(x\gamma(t)^{-1})dt/t,$$

is bounded on $L^p(G)$, $1 < p < \infty$.

INTRODUCTION

Let G be a simply connected, nilpotent Lie group. From [Ch] we know that if γ is a homogeneous curve with respect to a group of dilations of G, $\gamma(0) = 0$, then the Hilbert transform along γ is a bounded operator on $L^p(G)$, for $1 < p < \infty$; moreover $Tf(x)$ exists a.e. for all $f \in L^p(G)$. In [RS2] singular integral operators whose kernels are homogeneous distributions of critical degree with mean value zero and supported on a homogeneous, analytic submanifold of G are considered.

The purpose of this note is to prove that if $\gamma\colon (-1, 1) \to G$ is a connected, analytic curve such that $\gamma(0) = 0$ then the

$$Tf(x) = p.v. \int_{0<|t|<1} f(x\gamma(t)^{-1})dt/t$$

is a bounded operator on $L^p(G)$, $1 < p < \infty$.

For the proof we first assume that a certain group of dilations associated to γ are automorphisms of G and also that γ generates G. We use the iteration argument of [Ch]. The general case follows from a transference theorem [RS2].

Finally the boundedness of the maximal operator

$$T^* f(x) = \sup_{r>0} \left| \int_{r<|t|<1} f(x\gamma(t)^{-1})dt/t \right|$$

on $L^p(G)$, implies that $Tf(x)$ exists a.e. for all $f \in L^p$.

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The proof of the theorem. Let G be a connected and simply connected, nilpotent Lie group. Let $\gamma\colon (-1, 1) \to G$ be a (connected) analytic curve that generates G in the sense that γ is not contained in any proper closed subgroup of G. Also assume that $\gamma(0)$ is the identity of G. Denote by $\exp\colon \mathscr{g} \to G$ the exponential map. First of all we remark the following facts:

(i) γ generates G if and only if $\{d^j\bar{\gamma}/dt^j|_{t=0}\}_{j\in N}$ generate $\mathscr{g}$ as a Lie algebra, with $\bar{\gamma} = \gamma\circ\exp^{-1}$. Indeed, let $\mathscr{h}$ be the Lie subalgebra of $\mathscr{g}$ generated by $\{d^j\bar{\gamma}/dt^j|_{t=0}\}$ and $H = \exp\mathscr{h}$. Since $\bar{\gamma}$ is analytic, $\bar{\gamma} \subset \mathscr{h}$.

(ii) We can choose a coordinate system on G such that

$$\gamma(t) = \left(\frac{t^{a_1}}{a_1!}\varphi_1(t), \dots, \frac{t^{a_k}}{a_k!}\varphi_k(t), 0, \dots, 0\right)$$

with $1 \le a_1 < a_2 < \cdots < a_k$, φ_i are analytic functions and $\varphi_i(0) = 1$. Indeed, following [SW], we choose $a_1 = \inf\{j|d^j\bar{\gamma}/dt^j|_{t=0} \neq 0\}$. Given $a_1, \dots, a_i$, define

$$a_{i+1} = \inf\left\{j|j > a_i \text{ and } \left\{\frac{d^{a_1}\bar{\gamma}}{dt^{a_1}}\bigg|_{t=0}, \dots, \frac{d^{a_i}\bar{\gamma}}{dt^{a_i}}\bigg|_{t=0}, \frac{d^j\bar{\gamma}}{dt^j}\bigg|_{t=0}\right\} \text{ are linearly independent}\right\}.$$

Define $e_i = d^{a_i}\bar{\gamma}/dt^{a_i}|_{t=0}$. We thus obtain a set $\{e_1, \dots, e_k\}$, $k \le n$, maximal with respect to be L.I. such that

$$\bar{\gamma}(t) = \frac{t^{a_1}}{a_1!}\varphi_1(t)e_1 + \cdots + \frac{t^{a_k}}{a_k!}\varphi_k(t)e_k.$$

We extend $\{e_1, \dots, e_k\}$ to a basis of $\mathscr{g}$, $\{e_1, \dots, e_k, e_{k+1}, \dots, e_n\}$. If A denotes the change of basis matrix, the coordinates of $\gamma(t)$ with respect to $\psi = \exp\circ A$ are $((t^{a_1}/a_1!)\varphi_1(t), \dots, (t^{a_k}/a_k!)\varphi_k(t), 0, \dots, 0)$. From now on we fix this coordinate system on G.

(iii) We now associate to γ a group of dilations D_r, $r > 0$, by

$$D_r(x_1, \dots, x_n) = (r^{a_1}x_1, \dots, r^{a_k}x_k, r^{a_{k+1}}x_{k+1}, \dots, r^{a_n}x_n),$$

where $a_1, \dots, a_k$ are as in (ii) and $a_{k+1}, \dots, a_n$ are arbitrary integers. Let $Q = a_1 + \cdots + a_n$.

(iv) Let γ_0 be the curve in G with coordinates

$$(t^{a_1}/a_1!, \dots, t^{a_k}/a_k!, 0, \dots, 0).$$

Then γ_0 is analytic, homogeneous with respect to D_r and generates G. Indeed, if $\bar{\gamma} = \psi^{-1}\circ\gamma$ and $\bar{\gamma}_0 = \psi^{-1}\circ\gamma$, then $d^{a_i}\gamma_0/dt^{a_i}|_{t=0} = d^{a_i}\gamma/dt^{a_i}|_{t=0} = e_i$, $1 \le i \le k$.

We have associated to γ a group of dilations $\{D_r\}_{r>0}$ and a curve γ_0 that is homogeneous with respect to D_r and also generates G. For the moment we also assume that $\{D_r\}$ are automorphisms of G and fix a homogeneous norm

with respect to D_r, e.g., $|x| = \max_{1\le i\le n}\{|x_i|^{1/a_i}\}$. Denote by $\overline{B}$ the closure of the unit ball in $\mathbf{R}^n$.

Lemma 1. *Let ϕ, ϕ_0 be the functions on $\overline{B}$ defined by $\phi(t_1,\dots,t_n) = \gamma(t_1)\dots\gamma(t_n)$; $\phi_0(t_1,\dots,t_n) = \gamma_0(t_1)\dots\gamma_0(t_n)$. For each $j\in\mathbf{N}$, $j\ge 0$, define ϕ_j on $\{t\in\mathbf{R}^n \mid \frac{1}{2}\le |t_i|\le 1\}$ by $\phi_j(t) = D_{2^j}\phi(2^{-j}t)$. If $J_j(t) = \det d\phi_j(t)$ and $J_0(t) = \det d\phi_0(t)$, then $J_j(t) \xrightarrow[j\to+\infty]{} J_0(t)$ uniformly on $\{t\in\mathbf{R}^n \mid \frac{1}{2}\le|t_i|\le 1\}$.*

Proof. Let $\bar\phi_j(t) = \exp^{-1}\phi_j(t)$, $\bar\phi_0(t) = \exp^{-1}\phi_0(t)$.

$$\begin{aligned}\bar\phi_j(t) &= \exp^{-1}(D_{2^j}\phi(2^{-j}t)) = D_{2^j}\exp^{-1}\phi(2^{-j}t)\\ &= D_{2^j}\exp^{-1}(\gamma(2^{-j}t_1)\cdots\gamma(2^{-j}t_n)).\end{aligned}$$

Then

$$(*)\qquad \bar\phi_j(t) = D_{2^j}(\bar\gamma(2^{-j}t_1)+\cdots+\bar\gamma(2^{-j}t_n)) + D_{2^j}(P_1(\cdots),\dots,P_n(\cdots))$$

where $P_j(\cdots)$ denotes $P_j(\bar\gamma(2^{-j}t_1),\dots,\bar\gamma(2^{-j}t_n))$ and P_j are polynomial functions on the coordinates of $\gamma(t_1),\dots,\gamma(t_n)$.

If γ is homogeneous, $\bar\phi_j(t) = \bar\phi(t)$ and the first term of the right side of $(*)$ is also independent of j. Thus either $P_l\equiv 0$ or $P_l(t_1,\dots,t_n)$ is homogeneous of degree a_l, $l = 1,\dots,n$,

$$\begin{aligned}P_l(t_1,\dots,t_n) &= \sum \beta^{i_1,\ \ ,i_k}_{j_1,\ \ ,j_k}\varphi_{j_1}(t_{i_1})\cdots\varphi_{j_k}(t_{i_k})t_{i_1}^{a_{j_1}}\cdots t_{i_k}^{a_{j_k}}\\ &= \sum \beta^{i_1,\ \ ,i_k}_{j_1,\ \ ,j_k}F_{j_1,\dots,j_k}(t_{i_1},\ \ ,t_{i_k})t_{i_1}^{a_{j_1}}\cdots t_{i_k}^{a_{j_k}},\end{aligned}$$

with $a_{j_1}+\cdots+a_{j_k} = a_l$, $\beta^{i_1,\ \ ,i_k}_{j_1,\ \ ,j_k}\in\mathbf{C}$ and $F_{j_1},\dots,j_k(0) = 1$. Then

$$\begin{aligned}\bar\phi_j(t) = &\left(\dots,\sum_{i=1}^n \varphi_l(2^{-j}t_i)t_i^{a_l},\dots\right)\\ &+\left(\dots,\sum \beta'^s F_{j_1,\ \ ,j_k}(2^{-j}t_{i_1},\dots,2^{-j}t_{i_k})t_{i_1}^{a_{j_1}}\cdots t_{i_k}^{a_{j_k}},\dots\right).\end{aligned}$$

By the mean value theorem applied to $\varphi_l'^s$ and $F'^s_{j_1,\ \ ,j_k}$, we have that

$$\begin{aligned}\bar\phi_j(t) &= \left(\dots,\sum_{i=1}^n t_i^{a_l} + \sum \beta'^s t_{i_1}^{a_{j_1}}\cdots t_{i_k}^{a_{j_1}},\dots\right)\\ &\quad + 2^{-j}(G_1(t_1,\dots,t_n),\dots,G_n(t_1,\dots,t_n))\\ &= \bar\phi_0(t) + 2^{-j}(G_1(t_1,\dots,t_n),\dots,G_n(t_1,\dots,t_n)),\end{aligned}$$

with G_k bounded on $\{t \mid \frac{1}{2}\le|t_i|\le 1\}$, $k = 1,\dots,n$. This proves the lemma since $\det d_x\exp = 1$.

Remark. (v) If γ is an analytic curve that generates G then $\phi\colon \bar B\to G$ given by $\phi(t_1,\dots,t_n) = \gamma(t_1)\cdots\gamma(t_n)$ has rank n almost everywhere. Indeed, since γ_0 is a homogeneous curve that generates G, $J_0\not\equiv 0$ [Ch]. Since γ_0 is analytic,

$J_0(t) \neq 0$ almost everywhere. Let $J(t) = \det \phi(t)$. Since $\phi_j(t) = D_{2^j}\phi(2^{-j}t)$, $2^{j(Q-n)}J(2^{-j}t) = J_j(t) \xrightarrow[j\to\infty]{} J_0(t)$ on $\{t|\frac{1}{2} \leq |t_i| \leq 1\}$. Then $J \not\equiv 0$. (Furthermore $J(t) \neq 0$ almost everywhere).

By a result of [RS2] we know that if $\phi\colon \overline{B} \to \mathbf{R}^n$ is an analytic function having rank n at almost everywhere point and ψ is a C^1-function supported on $\overline{B}$ then the transported measure by ϕ of $\psi(t)\,dt$ is absolutely continuous and its density ρ satisfies and L^1-Hölder condition $\int |\rho(x+y) - \rho(x)|\,dx \leq c\|y\|^\delta$, for some $\delta > 0$ depending on ϕ, $c > 0$ that depends also on ψ.

In the next lemma we say a little more about c.

Lemma 2. *Let* $J(t) = \det d\phi(t)$. *Then*

$$\int |\rho(x+y) - \rho(x)|\,dx \leq \widetilde{C}\|y\|^\delta \left(\int_{\overline{B}} |\nabla\psi| + |\psi|\right)^\delta \left(\int_{\overline{B}} |\psi|/2\delta/|J|(1-\delta)\right)^{1-\delta},$$

for some $\delta > 0$, $\widetilde{\mathbf{C}} > 0$, *depending only on* ϕ.

Proof. Following the proof of [RS2] we denote by Z the set of zeros of $J(t)$ and consider a covering $\{B_j(z_j, r_j)\}_j$ of $\overline{B} - Z$, satisfying,

(1) $r_j = c_0|J(z_j)|$;

(2) the balls $B_j^* = B(z_j, 2r_j)$ are disjoint from Z and have the bounded overlapping property; i.e. there is an integer N such that no point belongs to more that N of the B_j^*.

Let $\{n_j\}$ be a smooth partition of unity on $\overline{B} - Z$ subordinated to $\{B_j\}$ and such that $\|\nabla n_j\|_\infty \leq \widetilde{\mathbf{C}} r_j^{-1}$. Let $\psi_j = \psi n_j$ and let ρ_j be the transported measure by ϕ of ψ_j. Since $\phi|_{B_j}$ is invertible [RS2] $\rho_j(x) = |J\phi^{-1}(x)|^{-1}\psi_j(\phi^{-1}(x))$ and thus

$$\int |\rho_j(x)|\,dx = \|\psi_j\|_1 = \int |\psi_j|$$

and

$$\begin{aligned}
&\int |\nabla\rho_j(x)|\,dx \\
&\leq \widetilde{\mathbf{C}}\left(\int |\nabla\psi(t)|n_j(t)|J(t)|^{-1}\,dt + \int |\psi|\,|\nabla n_j|\,|J(t)|^{-1}\,dt + \int |J(t)|^{-2}|\psi|\,dt\right) \\
&\leq \widetilde{\mathbf{C}} r_j^{-2}\left(\int_{B_j} |\nabla\psi| + \int_{B_j} |\psi|\right).
\end{aligned}$$

This implies that if $0 < \delta \leq 1$,

$$\int |\rho_j(x+y) - \rho_j(x)|\,dx \leq \widetilde{\mathbf{C}}\|y\|^\delta r_j^{-2\delta}\left(\int_{B_j} |\nabla\psi| + |\psi|\right)^\delta \left(\int_{B_j} |\psi|\right)^{1-\delta}.$$

Now by Hölder's inequality

$$\sum_j \left(\int_{B_j} |\nabla\psi| + |\psi|\right)^{\delta} r_j^{-2\delta} \left(\int_{B_j} |\psi|\right)^{1-\delta}$$
$$\leq \left(\sum_j \left(\int_{B_j} |\nabla\psi| + |\psi|\right)\right)^{\delta} \left(\sum_j r_j^{-2\delta/(1-\delta)} \int_{B_j} |\psi|\right)^{1-\delta}.$$

The lemma follows from the fact that $J|_{B_j^*}$ is comparable with r_j and from the bounded overlapping property.

Theorem. *Let γ be an analytic curve that generates G and such that the group of dilations associated to γ are automorphisms of G. Then*

$$Tf(x) = p.v. \int_{0<|t|<1} f(x\gamma(t)^{-1})\, dt/t$$

defines a bounded operator on $L^P(G)$, $1 < p < \infty$.

Proof. (We follow [RS2] for the proof.) Denoted by μ the distribution given by $\mu(f) = p.v. \int_{0<|t|\leq 1} f(\gamma(t))\, dt/t$. Let $\tilde{\varphi}$ be in $\mathscr{C}^{\infty}(\frac{1}{2}, 2)$ such that $\sum_{j=0} \tilde{\varphi}(2^j|t|) = 1$. Let $\varphi(x) = \varphi(|x|)$ where $|\ |$ denote the homogeneous norm defined in Remark (iv). Let $\mu_j = \tilde{\varphi}(2^j|t|)\mu$.

Let $\varphi_j(x) = 2^{jQ}\varphi(D_{2^j}x)$ and $\eta_j = \frac{1}{c}(\varphi_{j+1} - \varphi_j)$ with $c = \int \varphi$.

Since $\{c^{-1}\varphi_j\}$ is an approximate identity in G we can write

$$\delta = c^{-1} \lim_{j\to+\infty} \varphi_j = \sum_{j\geq j_0} \eta_j + c^{-1}\varphi_{j_0}$$

for each j_0 fixed. Also $\eta = \sum_{j\geq 0} \mu_j$. Then

$$\mu = \delta * \sum_j \mu_j = \sum_j \left(\sum_{k\geq j_0} \eta_k * \mu_j\right) + c^{-1} \sum_j \varphi_{j_0} * \mu_j.$$

For each j fixed, take $j_0 = j + 1$. Then

$$\mu = \sum_{k\geq 1} \left(\sum_{j\geq 0} \eta_{k+j} * \mu_j\right) + c^{-1} \sum_j (\varphi_{j+1} * \mu_j)$$
$$= \sum_{k\geq 1} M_k + N.$$

If we prove

(1) For $0 < \epsilon \leq 1$, $\|M_k\|_{p,p} \leq \mathbf{C}_{\epsilon,p} 2^{\epsilon k}$, $\|N\|_{p,p} \leq \mathbf{C}$.

(2) $\exists \sigma > 0$ such that $\|M_k\|_{2,2} \leq \mathbf{C} 2^{-\sigma k}$.

Then the theorem follows by interpolation and duality. Indeed, fix $1 < p < 2$ and take $1 < p_0 < p$. Then $\frac{1}{p} = \frac{1}{2}s + (1-s)\frac{1}{p_0}$ for some $0 < s < 1$. Thus

$\|M_k\|_{p,p} \le \mathbf{C}_\epsilon 2^{-\sigma k s} 2^{\epsilon k(1-s)}$ for $0 < \epsilon \le 1$ and some $\sigma > 0$. Choosing ϵ small enough we obtain $\sum_k \|M_k\|_{p,p} < +\infty$.

(1) We observe that $M_k = \sum_{j\ge 0} \eta_{k+j} * \mu_j = \sum_{j\ge 0} f_j^k$ where f_j^k has the following properties

(a) $f_j^k = 0$,

(b) $\int |f_j^k(xy) - f_j^k(x)|\, dx \le \mathbf{C} 2^{(k+j)\epsilon} |y|^\epsilon$,

(c) $\sup f_j^k \subset \{x \mid |x| \le \mathbf{C} 2^{-j}|)$,

(d) $\int |f_j^k| \le \mathbf{C}$.

Then Cotlar's lemma implies that $\|M_k\|_{2,2} \le \mathbf{C} 2^{\epsilon k}$ and the weak type (1.1) of M_k follows by checking that $\int_{|x|>2|y|} |M_k(xy) - M_k(x)|\, dx \le \mathbf{C} 2^{k\epsilon}$. The same argument holds for N. For a proof see [ChNSW].

(2) By Cotlar's lemma it is enough to prove that for some $\sigma > 0$ $\|f_j^k * f_l^k\|_{2,2} \le \mathbf{C} 2^{-\sigma k} 2^{-|j-l|\sigma}$. We check this for $j > l$, by using the iteration argument of [Ch].

$$\begin{aligned}\|f_j^k * f_l^k\|_{2,2} &\le \mathbf{C}\|\eta_{k+j} * \mu_j * \mu_l^*\|_{2,2}\\ &\le \mathbf{C}\|\eta_{k+j} * \mu_j\|^{1/2} \|\mu_l^* * \mu_l * \mu_j^* * \eta_{k+j}\|^{1/2}\\ &\le \cdots \le \mathbf{C}\|\eta_{k+j} * \mu_j\|^{1-2^{-r}} \|(\mu_l^* * \mu_l)^{2^{r-1}} * \mu_j^* * \eta_{k+j}\|^{2^{-r}}.\end{aligned}$$

Let $\psi_l(t_1, \dots, t_n) = \prod_{i=1}^n \frac{1}{t_i} \tilde{\varphi}(2^l t_i)$ and $\phi(t_1, \dots, t_n) = \gamma(t_1) \cdots \gamma(t_n)$ as in Lemma 1.

Since $\mu_l * \cdots * \mu_l$ is the transported measure by ϕ of $\psi_l(t)dt$ and since ϕ is analytic with rank n almost everywhere, $\mu_l * \cdots * \mu_l$ is absolutely continuous [RS2].

If ρ_l denotes its density, we have to prove that

$$\|\rho_l * \mu_j^* * \eta_{k+j}\|_1 \le \mathbf{C} 2^{-\sigma k} 2^{-|j-l|\sigma}$$

for some $\sigma > 0$. (Recall that we have assumed $l < j$.)

Let $\tilde{\rho}_l(x) = 2^{-lQ} \rho_l(D_{2^l} x)$. Then $\tilde{\rho}_l$ is the transported measure by $\phi_l(t) = D_{2^l}\phi(2^{-l}t)$ of $\tilde{\psi}(t) = 2^{-ln}\psi_l(2^{-l}t)$ $(\sup \tilde{\psi} \subset \{t \mid \frac{1}{2} \le |t_i| \le 2\})$.

If we prove that

(A) $\displaystyle\int_G |\tilde{\rho}_l(xy) - \tilde{\rho}_l(x)|\, dx \le \mathbf{C}|y|^\epsilon$, for some $\mathbf{C}, \epsilon > 0$, independent of l

then $\int_G |\rho_l(xy) - \rho_l(x)| dx \le \mathbf{C} 2^{l\epsilon} |y|^\epsilon$. From this and the fact that η_{k+j} has mean value zero and sop $\eta_{k+j} \subset \{x \| x| \le \mathbf{C} 2^{-(k+j)}\}$ we obtain the inequality desired.

But if $\bar{\rho}_l = \tilde{\rho}_l \circ \exp$ and $\bar{\phi}_l = \exp^{-1} \circ \phi_l$ then $\bar{\rho}_l$ is the transported measure of $\tilde{\psi}(t)dt$ by $\bar{\phi}_l$ and (A) is equivalent to

(B)

$\displaystyle\int_{\mathbf{R}^n} |\bar{\rho}_l(x+y) - \bar{\rho}_l(x)| dx \le \mathbf{C}\|y\|^\delta$, $\mathbf{C}, \delta > 0$, independent of l (see [RS2]).

Finally we prove (B). Since by Lemma 2,

$$\int_{\mathbf{R}^n} |\bar{\rho}_l(x+y) - \bar{\rho}_l(x)| dx \le \mathbf{C}\|y\|^\delta \left(\int |\nabla\tilde{\psi}| + |\tilde{\psi}|\right)^\delta \left(\int |\tilde{\psi}|/|\bar{J}_l| 2\delta/1-\delta\right)^{1-\delta}$$

for some $0 < \delta < 1$, we only have to check that $\int_{\sup\tilde{\psi}} |\bar{J}_l^{(t)}|^{-\alpha} dt \le \mathbf{C}$, with $\mathbf{C}$ independent of l, for some $0 < \alpha < 1$. By Lemma 1, $2^{l(Q-n)}\bar{J}(2^{-l}t) = \bar{J}_l(t)$ converges uniformly to $\bar{J}_0(t)$, which is a homogeneous polynomial of degree $Q-n$. Thus if we develop $\bar{J}(t)$ about 0 in the Taylor expansion, $\bar{J}(t) = P(t) + R(t)$, where P is a homogeneous polynomial of degree $Q-n$ and $\frac{R(t)}{|t|^{Q-n}} \to_{|t|\to 0} 0$. By a linear change of variables we can normalize P in the t_n-direction, i.e., $P(t) = t_n^{Q+n} + \sum_{j=0}^{Q-n-1} b_j(t_1, \dots, t_{n-1}) t_n^j$ with $b_j(0) = 0$. The Weierstrass preparation theorem now implies that

$$\bar{J}(t) = (t_n^{Q-n} + a_1(t')t_n^{Q-n-1} + \cdots + a_{Q-n}(t'))h(t), \qquad t' = (t_1, \dots, t_{n-1}),$$

where $a_1, \dots, a_{Q-n}$ and h are analytic functions in a neighborhood of 0, and $h(0) \ne 0$. (See [H].) Then

$$\begin{aligned}
\int_{\sup\tilde{\psi}} |\bar{J}_l(t)|^{-\alpha} dt &= 2^{-l(Q-n)\alpha} \int_{\sup\tilde{\psi}} |\bar{J}(2^{-l}t)|^{-\alpha} dt \\
&\le \mathbf{C} 2^{-l(Q-n)\alpha} \int_{\frac{1}{2}\le|t|\le 2} 2^{l(n-Q)\alpha} |t_n^{Q-n} + 2^l a_1(t')t_n^{Q-n-1} \\
&\qquad + \cdots + 2^{l(Q-n)} a_{Q-n}(t')|^{-\alpha} dt_n dt' \\
&\le \mathbf{C} A_\alpha \int_{1\le|t'|\le 2} (1 + 2^l|a_1(t')| + \cdots + 2^{l(Q-n)}|a_{Q-n}(t')|)^{-\alpha} dt'
\end{aligned}$$

which is bounded independent of l. The last inequality follows form [RS1] and the proof of the theorem is complete.

The general case where γ is any connected analytic curve in G, follows by transference [CW, RS2] as is shown in the following remarks.

Remark. (vi) Assume now that γ generates a proper subgroup of G. As in Remark (ii) we choose $\{e_1, \dots, e_l\}$ that generate a Lie subalgebra $\mathscr{k}$ of $\mathscr{g}$. Extend it to a basis $\{e_1, \dots, e_l, \dots, e_n\}$ of $\mathscr{g}$ and define for $r > 0$, $D_r(x_1, \dots, x_l, \dots, x_n) = (r^{a_1}x_1, \dots, r^{a_l}x_l, \dots, r^{a_n}x_n)$ where $a_1, \dots, a_l$ are determined by γ and $a_{l+1}, \dots, a_n$ are arbitrary integers. Assume that D_r are automorphisms of G, let γ_0 be the homogeneous curve (w.r.t. D_n) associated to γ, and let H be the subgroup of G generated by γ_0.

Then H is homogeneous with the dilations induced and γ generates H. By transference, $Tf = f * \mu$ is bounded on $L^p(G)$.

Remark. (vii) Let γ be an analytic curve that generates a connected and simply connected, nilpotent Lie Group N. Let D_r be a group of dilation associated to

γ, which *are not necessarily automorphisms of* N and let γ_0 be as in Remark (ii). As in [RS2] let $\mathscr{g}$ be the step m free Lie algebra generated by $\mathscr{n}$, for m large enough and let G be the simply connected group with $\mathscr{L}(G) = \mathscr{g}$. Then there exists H a normal subgroup of G such that G/H is isomorphic to N. Denote by $\sigma: \mathscr{g} \to \mathscr{n}$ the quotient morphism. By construction of $\mathscr{g}$ the dilations D_r on $\mathscr{n}$ extend to automorphisms $\tilde{D}_r$ of $\mathscr{g}$.

Let $\bar{\gamma} = \exp_N^{-1} \circ \gamma$, $\tilde{\gamma} = \exp_G \circ \bar{\gamma}$ and $\bar{\gamma}_0 = \exp_N^{-1} \circ \gamma_\circ$, $\tilde{\gamma}_\circ = \exp_G \bar{\gamma}_\circ$. Then $\tilde{\gamma}_\circ$ is a homogeneous curve w.r.t. $\tilde{D}_r$ and the coordinates of $\tilde{\gamma}_\circ$ (resp. $\tilde{\gamma}$) with respect to $\exp_G$ are those of γ_0 (resp. γ). Thus $Tf = f * \mu$ is bounded on $L^p(G)$, $1 < p < \infty$ and by transference on $L^p(N)$, $1 < p < \infty$.

Remark. (viii) The maximal function along γ,

$$M_\gamma f(x) = \sup_{r \le 1} \frac{1}{r} \left| \int_{0 \le |t| \le r} |f(x\gamma(t)^{-1})| \right| dt$$

is bounded on $L^p(G)$, $1 < p \le \infty$. Also

$$T^* f(x) = \sup_r \left| \int_{r \le |t| \le 1} f(x\gamma(t)^{-1}) dt/t \right|$$

is bounded on $L^p(G)$, $1 < p < \infty$ and hence $Tf(x)$ exists a.e. for all $f \in L^p(G)$, $1 < p < \infty$.

To see this we first remark that if $Hf = \sup_j |f * f_j|$ with f_j satisfying

(i) $\|f_j\|_1 \le \mathbf{C}_1$,
(ii) $\int |f_j(xy) - f_j(x)| \le \mathbf{C}_1 2^{j\epsilon} |y|^\epsilon \forall \epsilon > 0$,
(iii) $\sup f_j \subset \{|x| \le \mathbf{C} 2^{-j}\}$.

Then H is bounded on L^∞ and of weak type (1.1) with constant $C_1 C_\epsilon$. Indeed, we can assume that $f \ge 0$. For $\lambda > 0$ we decompose $f = g + b$ where $\|g\|_\infty \le \lambda$, $b = \sum_j b_j$, $\sup b_j \subset B_j = B(a_j, r_j)$, $\int b_j = 0$, $\int |b_j| \le A\lambda |B_j^*|$, and $\sum |B_j^*| \le \frac{A}{\lambda} \|f\|_1$, $B_j^* = B(a_j, 2r_j)$. Then it is enough to estimate $|\{x | Hb(x) > \lambda\}|$. But

$$\{x | Hb(x) > \lambda\} \subset \left(\bigcup_j B_j^* \right) \bigcup \left\{ x \in {}^c \left(\bigcup_j B_j^* \right) \middle| Hb(x) > \lambda \right\}$$

and

$$\begin{aligned}
&\left|\left\{x \in {}^c\left(\bigcup B_j^*\right) \mid Hb(x) > \lambda\right\}\right| \\
&\quad \le \frac{1}{\lambda}\int_{{}^c(\cup B_j^*)} Hb(x) \le \frac{1}{\lambda}\sum_i \int_{{}^cB_i^*} Hb_i(x) \\
&\quad = \frac{1}{\lambda}\sum_i \int_{{}^cB_i^*} \sup_j \left|\int_{B_i} b_i(y) f_j(y^{-1}x)\,dy\right| dx \\
&\quad = \frac{1}{\lambda}\sum_i \int_{{}^cB_i^*} \sup_j \left|\int_{B_i} b_i(y)(f_j(y^{-1}x) - f_j(a_i^{-1}x)dy\right| dx \\
&\quad \le \frac{1}{\lambda}\sum_i \int_{B_i} |b_i(y)| \sum_j \int_{C_{B_i^*}} |f_j(y^{-1}x) - f_j(a_i^{-1}x)|\,dx\,dy \\
&\quad \le \frac{1}{\lambda}\sum_i \int_{B_i} |b_i(y)| \sum_j \int_{|z|>2|\tilde{y}|} |f_j(\tilde{y}^{-1}z) - f_j(z)|\,dz \\
&\quad \le \mathbf{C}_\epsilon \cdot \frac{1}{\lambda}\sum_i \int_{B_i} |b_i(y)|\,dy \quad \text{by (ii) and (iii)}, \ \tilde{y} = y^{-1}a_i
\end{aligned}$$

Now

$$M_\gamma f(x) \le \mathbf{C}\sup(f * |\mu_j|)(x)$$

and

$$\sup_j(f * |\mu_j|) \le \sum_{k\ge 1}\sup_j |f * \eta_{k+j} * |\mu_j|| + \sup_j(f * \varphi_j * |\mu_j|) = \sum_k \widetilde{M}_k + \widetilde{N}.$$

By the above $\widetilde{M}_k$ is bounded on L^∞ and for each $\epsilon > 0$ there exists a constant C_ϵ such that $\widetilde{M}_k$ is of weak type (1.1) with constant $C_\epsilon 2^{k\epsilon}$. The same holds for $\widetilde{N}$. We also have that $\|\widetilde{M}_k\|_{2,2} \le \mathbf{C}2^{-\sigma k}$ for some $\sigma > 0$, by using the technique of the square functions (see [Ch]). Indeed we have to check that the operators $S_k f(x) = \sum_j a_j f * \eta_{k+j} * |\mu_j|$, $a_j = \pm 1 \forall j$, have L^2 boundedness $\mathbf{C}2^{-\sigma k}$ but this follows as in the proof of the theorem since n-times $|\mu_j| * \cdots * |\mu_j|$ is absolutely continuous.

The boundedness of T^* on L^p, $1 < p < \infty$, reduces to prove that the operators

$$T_1 f(x) = \sup_{i\ge 0}\left|f * \sum_{j\ge i}\mu_j * \varphi_i\right|$$

and

$$T_2 f(x) = \sup_i \left|\sum_{j=0}^{i} f * \mu_j * (\delta - \varphi_i)\right|$$

are bounded on L^p for $1 < p < \infty$ (see [Ch]). $T_1 f(x)$ is bounded pointwise by the maximal operator on G since the functions $\psi_i = \sum_{j>i}\mu_j * \varphi_i$ satisfy $\sup \psi_i \subset \{x \mid |x| \le \mathbf{C}2^{-i}\}$ and $|\psi_i(x)| \le \mathbf{C}2^{iQ}$.

Now $T_2 f(x) \leq \sum_k \sup_j |f * \mu_j * (\delta - \varphi_{k+j})| = \sum_k \widetilde{M}_k$ and $\widetilde{M}_k f \leq M_\gamma f + \sup_j |f * \mu_j * \varphi_{k+j}|$. Since $\sup_j |f * \mu_j * \varphi_{k+j}|$ is bounded on L^∞ and of weak type (1.1), it is bounded on L^p for $1 < p < \infty$ and so is $\widetilde{M}_k$. To see that $\|\widetilde{M}_k\|_{2,2} \leq \mathbf{C}2^{-\sigma k}$, for some C, $\sigma > 0$, we can argue as in the theorem since n-times $\mu_j * \cdots * \mu_j$ is absolutely continuous with density ρ_j satisfying the equation $\int |\rho_j(xy) - \rho_j(x)| \leq \mathbf{C}2^{j\epsilon}|y|^\epsilon$ and $(\delta - \varphi_{k+j})$ is a measure with bounded L^1-norm, $\operatorname{supp}(\delta - \varphi_{k+j}) \subset \{x \| x| \leq \mathbf{C}2^{-(k+j)}\}$ and it has mean value zero.

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