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A NULLSET FOR NORMAL FUNCTIONS IN SEVERAL VARIABLES

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ABSTRACT. Suppose that Ω is a domain in C^n, $E \subset \Omega$ is closed in Ω, and $f: \Omega\backslash E \to \mathbf{C}^*$ is a meromorphic function. We show that if f is normal and E is an analytic subvariety or, more generally, of locally finite $(2n-2)$-dimensional Hausdorff measure in Ω satisfying a certain geometric condition, then f can be extended to a meromorphic function (= holomorphic mapping) $f^*: \Omega \to \mathbf{C}^*$. In the case of a subvariety sufficient, but not necessary, for the geometric condition is that the singularities of E are normal crossings. As a digression, we give a new proof for the following result, due to Parreau in the case $n = 1$: if f is in the Nevanlinna class and E is polar (in R^{2n}), then f has a meromorphic extension f^* to Ω.

1. INTRODUCTION

The following is a result of Lehto and Virtanen:

1.1. **Theorem** [LV, Theorem 9, p. 62]. *Suppose that G is a domain in $\mathbf{C}$ and $a \in G$. If $f: G\backslash\{a\} \to \mathbf{C}^*$ is a normal meromorphic function, then f has a meromorphic extension $f^*: G \to \mathbf{C}^*$.*

For a slightly different proof see [Jä, Lemma 1, p. 1172]. Using the above result and induction, Järvi [Jä, Theorem 2, p. 1174] generalized Lehto's and Virtanen's result to several variables by showing that an analytic subvariety is removable for normal meromorphic functions. Järvi further showed [Jä, Theorem 1, p. 1173] that if the exceptional analytic subvariety is of codimension 1 and its singularities are normal crossings, then the extension is in fact a meromorphic *function* to $\mathbf{C}^*$; i.e., the indeterminancy set of the extended function is empty. (Järvi uses in this connection the term "holomorphic mapping to $\mathbf{C}^*$.")

In Theorems 3.5 and 3.9 below we generalize Järvi's results by pointing out that it is sufficient that the exceptional set is of locally finite $(2n-2)$-dimensional Hausdorff measure and moreover, in the generalization of Järvi's Theorem 1, satisfies a certain geometric condition. In the proof we need (implicitly) another removable singularity result, Theorem 3.1 below. As another application of

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this theorem, we obtain in Corollary 3.2 below a new, unified proof for the Nevanlinna class results [Pa, Théorème 20, p. 182] and [HR1, Theorem 3.4, p. 477]. For a previous result see [CG1, Theorem C, p. 241].

1.2. We use Cima's and Krantz's definition for normal meromorphic functions [CK, pp. 305–306], see also [Jä, p. 1171]. Note that other definitions exist also. We use mainly the same notation as in [HR1], [HR2] and [Ri2]. Nevertheless, for the sake of convenience of the reader, we recall the following.

The spherical metric in the extended complex plane $\mathbf{C}^*$ is denoted by q. We identify $C^n(=\mathbf{C}^n)$ with $R^{2n}(=\mathbf{R}^{2n})$, $n \geq 1$. We write $B^{2n}(z_0, r)$ for the open ball in C^n with center z_0 and radius r. If $z=(z_1, \dots, z_n) \in C^n$, $n \geq 2$, $A \subset C^n$, and $1 \leq j \leq n$, then we write

$$Z_j = (z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n), \quad z = (z_j, Z_j)$$

and

$$A(Z_j) = \{z_j \in \mathbf{C}\colon z = (z_j, Z_j) \in A\},$$
$$A(z_j) = \{Z_j \in C^{n-1}\colon z = (z_j, Z_j) \in A\}$$

for the sections of A. The α-dimensional Hausdorff outer measure is denoted by H^α. Recall that $H^0(A) = \operatorname{card} A$ is the number of points of A. Suppose that Ω is a domain in C^n, $n \geq 1$. It is well-known that if V is a k-dimensional analytic subvariety in Ω, then $H^{2k}(V \cap K) < \infty$ for each compact set $K \subset \Omega$. Suppose that $f \not\equiv \infty$ is meromorphic in Ω. Then there are analytic subvarieties P_f, the pole set of f, and I_f, the indeterminancy set of f, in Ω such that $I_f \subset P_f$, $H^{2n-4}(I_f \cap K) < \infty$, $H^{2n-2}(P_f \cap K) < \infty$ for each compact set $K \subset \Omega$, f is holomorphic in $\Omega \setminus P_f$, f is spherically continuous in $\Omega \setminus I_f$, and $f | P_f \setminus I_f = \infty$. In the sequel we use the following convention: When we write "... $f\colon \Omega \to \mathbf{C}^*$ is a meromorphic function ..." or "... a meromorphic function with values in $\mathbf{C}^*$ ..." or something equivalent, then it is always meant that f is meromorphic in Ω *and* $I_f = \varnothing$, i.e., locally in Ω either f or $1/f$ is holomorphic. Our terminology (probably nonstandard) here follows that used by Cima and Krantz [CK, p. 305]. As noted above, the term "holomorphic mapping to $\mathbf{C}^*$" is also used; see, e.g., [Jä and Ki].

2. Lemmas

In this section we give the lemmas we need in the sequel.

2.1. **Lemma** (cf. [Fe, 2.10.25, 2.10.11, pp. 188, 176] and [Sh, Corollary 4, p. 114]). *Suppose that $E \subset C^n$, $n \geq 2$, and $1 \leq j \leq n$.*

(a) *If $H^{2n-1}(E) = 0$, then $H^1(E(Z_j)) = 0$ for H^{2n-2}-almost all $Z_j \in C^{n-1}$.*

(b) *If $H^{2n-2}(E) < \infty$, then $H^{2n-4}(E(z_j)) < \infty$ for H^2-almost all $z_j \in \mathbf{C}$ and $H^0(E(Z_j)) < \infty$ for H^{2n-2}-almost all $Z_j \in C^{n-1}$.*

The following is a special case of Cima's and Krantz's result [CK, Corollary 1.7, p. 309]; the statement concerning the order of normality (see [CK, definition for normality, pp. 305–306]) is implicit in this cited result. See also [Jä, Lemma 2, p. 1173].

2.2. **Lemma.** *Suppose that Ω is a domain in C^n, $n \geq 2$, and $1 \leq j \leq n$. Suppose that $f\colon \Omega \to \mathbf{C}^*$ is a normal meromorphic function. If $Z_j \in C^{n-1}$ is such that $\Omega(Z_j)$ is a nonempty domain in $\mathbf{C}$, then the meromorphic function $f_{Z_j}\colon \Omega(Z_j) \to \mathbf{C}^*$,*

$$f_{Z_j}(z_j) = f(z_j, Z_j),$$

is normal. Similarly, if $z_j \in \mathbf{C}$ is such that $\Omega(z_j)$ is a nonempty domain in C^{n-1} and $f_{z_j}\colon \Omega(z_j) \to \mathbf{C}^*$,

$$f_{z_j}(Z_j) = f(z_j, Z_j),$$

is a meromorphic function, then f_{z_j} is normal. Moreover, $C_{f_{Z_j}}$, $C_{f_{z_j}} \leq C_f$, where $C_{f_{Z_j}}$, $C_{f_{z_j}}$, and C_f are the orders of normality of f_{Z_j}, f_{z_j}, and f, respectively.

For the next lemma see, e.g., [HR2, Lemma 3.4, p. 299].

2.3. **Lemma.** *Suppose that Ω is a domain in C^n, $n \geq 2$, $E \subset \Omega$ is closed in Ω and f is holomorphic in $\Omega \backslash E$. If for each j, $1 \leq j \leq n$, and for H^{2n-2}-almost all $Z_j \in C^{n-1}$, the section $E(Z_j)$ is totally disconnected and the holomorphic function $f_{Z_j}\colon (\Omega\backslash E)(Z_j) \to \mathbf{C}$,*

$$f_{Z_j}(z_j) = f(z_j, Z_j),$$

has a meromorphic extension to $\Omega(Z_j)$, then f has a meromorphic extension to Ω.

2.4. **Lemma** [Ca, p. 202]. *Let $\mathscr{F}$ be a family of functions meromorphic in a domain G of $\mathbf{C}$. Suppose that each $f \in \mathscr{F}$ omits three distinct points $a_f, b_f, c_f \in \mathbf{C}^*$. If there is a $\delta > 0$ such that*

$$q(a_f, b_f)q(b_f, c_f)q(c_f, a_f) \geq \delta$$

for all $f \in \mathscr{F}$, then $\mathscr{F}$ is spherically equicontinuous.

3. Singularities

We begin with a partial generalization to [HR2, Theorem 3.5, pp. 300–301]. For a related result see [Ri2, Theorem 2, p. 549].

3.1. **Theorem.** *Suppose that Ω is a domain in C^n, $n \geq 1$, that $E \subset \Omega$ is closed in Ω and $H^{2n-1}(E) = 0$. Suppose that f is holomorphic in $\Omega\backslash E$. If* *for some* $p \in \mathbf{R}$

$$\int_{\Omega\setminus E} \frac{|f(z)|^{p-2}}{(1+|f(z)|^p)^2} \sum_{j=1}^{n} \left| \frac{\partial f}{\partial z_j}(z) \right|^2 dm(z) < \infty,$$

then f *has a meromorphic extension* f^* *to* Ω. *If* $p = 0$, *then* f^* *is holomorphic.*

Proof. The case $p = 0$ is a part of the cited result in [HR2]. Thus we may suppose that $p \neq 0$. Consider first the case in which $n = 1$.

An easy computation shows that

$$\int_{\mathbf{C}} \frac{|w|^{p-2}}{(1+|w|^p)^2} dm(w) = \frac{2\pi}{|p|}.$$

Take $z_0 \in E$ arbitrarily. By assumption, there is an $r_0 = r_0(p, f, z_0) > 0$ such that

$$\int_{B^2(z_0, r_0)\setminus E} \frac{|f(z)|^{p-2}}{(1+|f(z)|^p)^2} |f'(z)|^2 dm(z) < \frac{\pi}{|p|}.$$

Since clearly

$$\int_{f(B^2(z_0, r_0)\setminus E)} \frac{|w|^{p-2}}{(1+|w|^p)^2} dm(w) \leq \int_{B^2(z_0, r_0)\setminus E} \frac{|f(z)|^{p-2}}{(1+|f(z)|^p)^2} |f'(z)|^2 dm(z),$$

$f|B^2(z_0, r_0)\setminus E$ omits a set of positive measure. Thus by a result of Kametani [No, Theorem 2, p. 10, and Remark, p. 11], f has a meromorphic extension to $B^2(z_0, r_0)$.

The case in which $n \geq 2$ is proved using Fubini's theorem, Lemma 2.1(a) and Lemma 2.3. See [HR2, p. 301] and [Ri2, p. 549].

3.2. **Corollary** ([Pa, Théorème 20, p. 182] and [HR1, Theorem 3.4, p. 477]). *Suppose that* Ω *is a domain in* C^n, $n \geq 1$, *and that* $E \subset \Omega$ *is closed in* Ω *and polar in* R^{2n}. *Suppose that* f *is holomorphic in* $\Omega\setminus E$. *If the* (*pluri*)*subharmonic function* $\log^+|f|$ *has a harmonic majorant in* $\Omega\setminus E$, *then* f *has a meromorphic extension to* Ω.

Proof. Because of the inequality $\log(1+x^2) \leq 2\log^+|x| + \log 2$, the (pluri)subharmonic function $\log(1+|f|^2)$ has a harmonic majorant in $\Omega\setminus E$. Proceeding then as in [Ri2, proof of Corollary 1, p. 549], one sees that the measure $\mu = \Delta\log(1+|f|^2)$ has *locally finite mass* near E; i.e., for each domain D relatively compact in Ω, one has $\mu(D\setminus E) < \infty$. See also [Ce, p. 283]. On the other hand, computing the Laplacian, one easily gets

$$\mu(D\setminus E) = 4\int_{D\setminus E} \frac{1}{(1+|f(z)|^2)^2} \sum_{j=1}^{n} \left| \frac{\partial f}{\partial z_j}(z) \right|^2 dm(z)$$

for each domain D relatively compact in Ω. Compare [HR2, p. 297], [Ri2, p. 549] and [Ya, p. 402]. Since $\mu(D\setminus E) < \infty$, the result follows then from Theorem 3.1.

3.3. *Remark.* Note that the above proof for Corollary 3.2 applies for all $n \geq 1$. Our previous, more complicated proof [HR1, pp. 477-479] for the case $n \geq 2$ was based on Parreau's result (the case $n = 1$) and on certain measure-theoretic results of Mattila and Sadullaev. For a third proof for the case $n = 1$ see [Hej, Theorem 3, pp. 105–108].

3.4. *Remark.* A proof of Lehto's and Virtanen's result, Theorem 1.1 above, can be based on a special case of Theorem 3.1. As a matter of fact, this is essentially Järvi's method in his proof [Jä, Lemma 1, p. 1172]: he considers the special case when $n = 1$, $\Omega = B^2(a, r) \subset\subset G$, $p = 2$, and $E = \{a\} \cup N_f$. Here N_f is the set of poles of f in $G \backslash \{a\}$.

Next we give a generalization to [Jä, Theorem 2, p. 1174].

3.5. **Theorem.** *Suppose that Ω is a domain in C^n, $n \geq 1$, that $E \subset \Omega$ is closed in Ω, and that $H^{2n-2}(E \cap K) < \infty$ for each compact set $K \subset \Omega$. Suppose that $f : \Omega \backslash E \to \mathbf{C}^*$ is a meromorphic function. If f is normal, then f has a meromorphic extension f^* to Ω.*

Proof. We may suppose that $f \not\equiv \infty$. Write $E_1 = E \cup N_f$, and note that f is holomorphic in $\Omega \backslash E_1$. The result follows then at once with the aid of Lemma 2.1 (b), Lemma 2.2, Theorem 1.1, and Lemma 2.3.

3.6. If one wants the extension f^* in Theorem 3.5 to be a meromorphic *function* from Ω to $\mathbf{C}^*$, then some additional restrictions must be imposed on the exceptional set E. This is seen from Järvi's example [Jä, Remark 2, p. 1174], which we recall here for the convenience of the reader.

Let $n \geq 3$. Set

$$V_j = \{z = (z_1, \dots, z_n) \in C^n : z_j = 0\},$$

and

$$W_1 = \{z = (z_1, \dots, z_n) \in C^n : z_1 - z_2 = 0\}.$$

for each j, $1 \leq j \leq n$. Then $V = V_1 \cup V_2 \cup W_1$ is an analytic subvariety in C^n of codimension 1. One sees at once that f, $f(z) = z_1/z_2$, is meromorphic in C^n and is holomorphic and normal in $C^n \backslash V$. However, $f | C^n \backslash V$ does not extend to a meromorphic *function* from C^n to $\mathbf{C}^*$.

In order to exclude such cases, Järvi considers the case in which the exceptional set $E = V$ is an analytic subvariety in Ω of codimension 1 whose singularities are *normal crossings*; that is, $\Omega \backslash V$ is locally biholomorphic to $[B^2(0, 1) \backslash \{0\}]^k \times B^2(0, 1)^{n-k}$ for some $k \in \{0, 1, \dots, n\}$. Below we propose one possibility for a less restrictive condition on the exceptional set. Since our method of proof is based on induction, our condition will be given on complex lines.

3.7. Suppose that Ω is an open set in C^n, $n \geq 1$. Let $\delta > 0$. We say that a set $A \subset C^n$ is *almost separately δ-sparse in Ω*, or briefly *δ-sparse in Ω*, if the following conditions are satisfied.

(i) If $n = 1$, then
$$|a - b| \geq \delta \cdot d(a, \partial\Omega)$$
for all $a, b \in A \cap \Omega$, $a \neq b$.

(ii) If $n \geq 2$, then for each k, $1 \leq k \leq n$, $A(Z_k)$ is δ-sparse in $\Omega(Z_k)$ for H^{2n-2}-almost all $Z_k \in C^{n-1}$.

A set A is *almost separately sparse in* Ω, or briefly *sparse in* Ω, if A is δ-sparse in Ω for some $\delta > 0$.

When $n = 1$ and $\Omega = B^2(0, 1)$ our condition is equivalent to the following well-known sparseness condition on the pseudohyperb lic distance for the set A: there is a $\delta' > 0$ such that
$$\frac{|a - b|}{|1 - a\bar{b}|} \geq \delta'$$
for all $a, b \in A \cap B^2(0, 1)$, $a \neq b$. See [CG2, p. 691, 694] and also [VH, p. 147], [Sa, p. 22].

It is easy to see that measurable sparse sets (in Ω) are of Lebesgue measure zero. Sparse sets (in Ω) can, of course, by very large. In fact, using a suitable sequence of Cartesian products of appropriate Cantor sets one easily gets a compact sparse set in C^n whose Hausdorff dimension is $2n$. There are also more surface-like examples: in C^2 there exists a compact set K whose Hausdorff dimension is 4 such that for each $z = (z_1, z_2) \in C^2$ the sections $K(z_1)$ and $K(z_2)$ contain at most one point. The existence of such a set is based on [DF, Theorem 2, p. 118], see [HR2, p. 298]. For more natural examples, see 3.12 and 3.13 below.

3.8. **Lemma.** *Suppose that Ω is a domain in C^n, $n \geq 2$, and that $A \subset \Omega$ is δ-sparse in Ω. Then for each k, $1 \leq k \leq n$, $A(z_k)$ is δ-sparse in $\Omega(z_k)$ for H^2-almost all $z_k \in \mathbf{C}$.*

Proof. Since the assertion clearly holds when $n = 2$, we may suppose that $n \geq 3$. Suppose that $1 \leq k \leq n$. Write, for each $j \neq k$, $1 \leq j \leq n$,
$$A_j = \{Z_j \in C^{n-1} \colon A(Z_j) \text{ is not } \delta\text{-sparse in } \Omega(Z_j)\}.$$
Since $H^{2n-2}(A_j) = 0$, we see with the aid of Fubini's theorem that $H^2(A_{jk}) = 0$, where
$$A_{jk} = \{z_k \in \mathbf{C} \colon H^{2n-4}(A_j(z_k)) > 0\}.$$
Write
$$A_k^* = \bigcup_{j=1,\, j\neq k}^{n} A_{jk}.$$
Then $H^2(A_k^*) = 0$. It is easy to see that $A(z_k)$ is δ-sparse in $\Omega(z_k)$ for all $z_k \in \mathbf{C}\setminus A_k^*$.

3.9. **Theorem.** *Suppose that Ω is a domain in C^n, $n \geq 1$, and that $E \subset \Omega$ is closed in Ω and sparse in Ω. Suppose that $f \colon \Omega\setminus E \to \mathbf{C}^*$ is a meromorphic* *function. If f is normal, then f extends to a meromorphic function $f^*: \Omega \to \mathbf{C}^*$.*

Proof. It is sufficient to show that f has a spherically continuous extension to a neighborhood of each point $z_0 \in E$. Because of Theorem 1.1, we may suppose that $n \geq 2$. We give an induction proof. One sees at once that the proof of Theorem 3.5 works also in the present case, i.e., when the exceptional set is sparse in Ω. Thus f has a meromorphic extension f^* to Ω. We may suppose that $f^* \not\equiv \infty$. Moreover, we may suppose that E is δ-sparse in Ω, where $0 < \delta < 1$.

Take $z_0 \in E$ and $R > 0$ such that $\overline{D} \subset \Omega$, where $D = D(z_0, R) = B^2(z_1^0, R) \times \cdots \times B^2(z_n^0, R)$. Let $C \geq 0$ be the order of normality of f. Choose $R_1 > 0$ and $M > 1$ such that

$$R_1 < \frac{\delta R}{5} e^{-300C^2} \tag{1}$$

and

$$\frac{4\pi C^2}{M^2 - 1} < \frac{\pi}{150}. \tag{2}$$

Write $A = A_1 \cap A_2 \cap A_3$, where

$$A_1 = \{Z_n \in C^{n-1} : E(Z_n) \text{ is } \delta\text{-sparse in } \Omega(Z_n)\},$$

$$A_2 = \{Z_n \in C^{n-1} : N_{f^*}(Z_n) \text{ is countable}\},$$

$$A_3 = \{Z_n \in C^{n-1} : I_{f^*}(Z_n) = \varnothing\}.$$

With the aid of Lemma 2.1 (b), one sees that $H^{2n-2}(C^{n-1} \backslash A) = 0$. Take $Z_n \in A$ arbitrarily, and write $g = f_{Z_n}$. By Lemma 2.2, g is normal in $(\Omega \backslash E)(Z_n)$ and has therefore by Theorem 1.1 a meromorphic extension g^* to $\Omega(Z_n)$. Clearly $g^* = f^*_{Z_n}$, where $f^*_{Z_n} : \Omega(Z_n) \to \mathbf{C}^*$,

$$f^*_{Z_n}(z_n) = f^*(z_n, Z_n) = f^*(Z_n, z_n).$$

Consider first the case in which $E(Z_n) \cap B^2(z_n^0, R_1) \neq \varnothing$. Choose $z_n^* \in E(Z_n) \cap B^2(z_n^0, R_1)$ arbitrarily; then $B^2(z_n^*, \delta(R - R_1)) \backslash \{z_n^*\} \subset (\Omega \backslash E)(Z_n)$. Because g is normal also in $B^2(z_n^*, (\delta/2)(R - R_1)) \backslash \{z_n^*\}$, because the Kobayashi metric and the Poincaré metric of $B^2(z_n^*, (\delta/2)(R - R_1)) \backslash \{z_n^*\}$ are the same, and because $C_{g|B^2(z_n^*, (\delta/2)(R-R_1)) \backslash \{z_n^*\}} \leq C_g \leq C$ by Lemma 2.2, one has

$$\frac{|g^{*\prime}(z_n)|}{1 + |g^*(z_n)|^2} \leq C \frac{1}{|z_n - z_n^*| \log \frac{\delta(R - R_1)}{2|z_n - z_n^*|}} \tag{3}$$

for all $z_n \in B^2(z_n^*, (\delta/2)(R - R_1)) \backslash (\{z_n^*\} \cup N_{g^*})$. See [LV, proof of Theorem 9,

p. 63]. [Ah, p. 17] and [Jä, proof of Lemma 1, p. 1172]. Since $B^2(z_n^*, 2R_1) \subset B^2(z_n^*, (\delta/2)(R-R_1))$, it follows from (3) that

$$\int_{B^2(z_n^*, 2R_1)} \frac{|g^{*'}(z_n)|^2}{(1+|g^*(z_n)|^2)^2}\, dm(z_n) \le \frac{2\pi C^2}{\log \frac{\delta(R-R_1)}{4R_1}}.$$

Proceeding then as in the proof of Theorem 3.1, we see with the aid of (1) that

$$m_{sp}(g^*(B^2(z_n^0, R_1))) \le m_{sp}(g^*(B^2(z_n^*, 2R_1))) \le \frac{2\pi C^2}{\log \frac{\delta(R-R_1)}{4R_1}} < \frac{\pi}{150}.$$

By m_{sp} (the spherical measure in $\mathbf{C}^*$) we mean the following: If $H \subset \mathbf{C}^*$ is such that $H\backslash\{\infty\}$ is measurable in $\mathbf{C}$, then we write

$$m_{sp}(H) = \int_{H\backslash\{\infty\}} \frac{1}{(1+|w|^2)^2}\, dm(w).$$

On the other hand,

$$m_{sp}(B^2(0, 1)) = \tfrac{\pi}{2},$$
$$m_{sp}(B^2(0, 3)\backslash\overline{B}^2(0, 2)) = \tfrac{\pi}{10},$$
$$m_{sp}(B^2(0, 7)\backslash\overline{B}^2(0, 6)) > \tfrac{\pi}{150}.$$

Thus there is a number $\varepsilon > 0$ independent of Z_n such that g^* omits in $B^2(z_n^0, R_1)$ three points $a_{g^*}, b_{g^*}, c_{g^*} \in \mathbf{C}$ such that

$$q(a_{g^*}, b_{g^*})q(b_{g^*}, c_{g^*})q(c_{g^*}, a_{g^*}) \ge \varepsilon. \tag{4}$$

(For example, $\varepsilon = 1/125,000$ will do.)

Consider then the case in which $E(Z_n) \cap B^2(z_n^0, R_1) = \emptyset$. As above, g is normal in $B^2(z_n^0, R_1)$ and $C_{g|B^2(z_n^0, R_1)} \le C_g \le C$. Thus,

$$\frac{|g'(z_n)|}{1+|g(z_n)|^2} \le C \frac{2R_1}{R_1^2 - |z_n - z_n^0|^2}$$

for all $z_n \in B^2(z_n^0, R_1)\backslash N_g$. (Recall that the Kobayashi metric and the Poincaré metric of $B^2(z_n^0, R_1)$ are the same.) Using this inequality and (2), one gets

$$m_{sp}\left(g\left(B^2\left(z_n^0, \frac{R_1}{M}\right)\right)\right) \le \frac{4\pi C^2}{M^2-1} < \frac{\pi}{150}.$$

Proceeding again as above, one sees that g omits in $B^2(z_n^0, R_1/M)$ three points $a_g, b_g, c_g \in \mathbf{C}$ such that

$$q(a_g, b_g)q(b_g, c_g)q(c_g, a_g) \ge \varepsilon, \tag{5}$$

where $\varepsilon > 0$ is the same number as above. Because of (4) and (5), we see by Lemma 2.4 that the family

$$\left\{f_{Z_n}^* : B^2\left(z_n^0, \frac{R_1}{M}\right) \to \mathbf{C}^* : Z_n \in A \cap D(z_n^0)\right\}$$

of meromorphic functions is spherically equicontinuous. (As for the notation, recall that $D(z_n^0)$ is the section of D.)

To conclude the proof we show that f has a spherically continuous extension to $D_1 = B^2(z_1^0, R) \times \cdots \times B^2(z_{n-1}^0, R) \times B^2(z_n^0, R_1/M)$. To prove this, it is clearly sufficient to show that the following condition is fulfilled: for each $Z_n' \in D_1(z_n^0)$ and for each sequence $Z_n^j \to Z_n'$, $Z_n^j \in A \cap D_1(z_n^0)$, $j = 1, 2, \ldots$, the sequence

$$(6) \qquad f^*_{Z_n^j} : B^2\left(z_n^0, \frac{R_1}{M}\right) \to \mathbf{C}^*, \; j = 1, 2, \ldots,$$

of meromorphic functions converges spherically c-uniformly. (See, e.g., [Ri1, Lemma 2.6, p. 48]; in this elementary lemma one can of course equally well use a spherical metric instead of a Euclidean metric.) By [Vä, Theorem 20.3, p. 68], it is thus sufficient to show that the sequence (6) converges in a dense subset of $B^2(z_n^0, R_1/M)$.

We consider first the case when $n = 2$. Since $H^{2n-4}(I_{f^*} \cap K) < \infty$ for each compact set $K \subset \Omega$, $I_{f^*}(z_n) = \emptyset$ for H^2-almost all $z_n \in B^2(z_n^0, R_1/M)$. Thus the desired convergence follows.

Suppose then that $n \geq 3$. With the aid of Lemma 3.8 and Lemma 2.1 (b), we find a set $B \subset B^2(z_n^0, R_1/M)$ such that $H^2(B^2(z_n^0, R_1/M) \backslash B) = 0$ and such that for each $z_n \in B$ the set $E(z_n)$ is δ-sparse in $\Omega(z_n)$ and $H^{2n-4}(N_{f^*}(z_n) \cap K) < \infty$ for each compact set $K \subset \Omega(z_n)$. Choose $z_n \in B$ arbitrarily. Then $h = f_{z_n} : (\Omega \backslash E)(z_n) \to \mathbf{C}^*$ is clearly a meromorphic *function*. Since h is normal by Lemma 2.2, we see by the induction hypothesis that h extends to a meromorphic *function* $h^* : \Omega(z_n) \to \mathbf{C}^*$. Clearly $h^*(Z_n) = f^*(z_n, Z_n) = f^*(Z_n, z_n) = f^*_{Z_n}(z_n)$ for each $Z_n \in \Omega(z_n) \backslash I_{f^*}(z_n)$. Recalling then that $Z_n^j \in A$, $j = 1, 2, \ldots$, and the definition of A, we see that $Z_n^j \notin I_{f^*}(z_n)$. Thus $f^*_{Z_n^j}(z_n) \to h^*(Z_n')$ as $j \to \infty$. Hence we have shown that the sequence (6) converges for each $z_n \in B$. Therefore the induction step is done and the proof is complete.

3.10. **Corollary.** *Suppose that Ω is a domain in C^n, $n \geq 1$, that E is closed in Ω, and that for each k, $1 \leq k \leq n$, $\operatorname{card} E(Z_k) \leq 1$ for H^{2n-2}-almost all $Z_k \in C^{n-1}$. Suppose that $f : \Omega \backslash E \to \mathbf{C}^*$ is a meromorphic function. If f is normal, then f extends to a meromorphic function $f^* : \Omega \to \mathbf{C}^*$.*

3.11. In concluding we give two examples. These give natural exceptional sets in the considered setting of meromorphic *functions* with values in C^*, which satisfy our condition in Corollary 3.10, but which are such that the extension result [Jä, Theorem 1, p. 1173] cannot be applied.

3.12. **Example.** Set $n = 3$; $V = V_1 \cup V_2 \cup V_3 \cup W_2$, where V_j, $j = 1, 2, 3$, are as in 3.6 above and

$$W_2 = \{z = (z_1, z_2, z_3) \in C^3 : z_1 = z_2 = z_3\}.$$

Then V is an analytic subvariety in C^3. However, V is not contained in any analytic subvariety V' in C^3 of codimension 1 whose singularities are normal crossings. On the other hand, V is sparse in C^3.

Note further that V is a natural exceptional set for meromorphic functions in our setting: for example, the meromorphic function $f\colon C^3\backslash V \to \mathbf{C}^*$,

$$f(z_1, z_2, z_3) = e^{-1/z_1} + e^{-1/z_2} + e^{-1/z_3} + \frac{z_1 - z_3}{z_2 - z_3},$$

cannot be extended to any larger set as a meromorphic function with values in $\mathbf{C}^*$.

3.13. **Example.** Set $n = 2$ and $\Omega = B^2(0, 1) \times B^2(0, 1)$. Choose a sequence $b_k = (b_1^k, b_2^k) \in \Omega$, $k = 1, 2, \ldots$, in such a way that $b_k \to (0, 0)$ as $k \to \infty$ and that the set

$$E = \{b_k = (b_1^k, b_2^k) \in \Omega\colon k = 1, 2, \ldots\} \cup [B^2(0, 1) \times \{0\}] \cup [\{0\} \times B^2(0, 1)]$$

is not contained in any analytic subvariety in Ω. As a matter of fact, using an antithesis and, for example, [Her, Corollary 3, p. 36; Remark, p. 33; Remark, p. 44; Theorem 6, p. 33; and part of the proof of Theorem 1 (ii), p. 11] one sees that one possibility for this is the following: choose a sequence of distinct points $a_j \in B^2(0, 1)$, $j = 1, 2, \ldots$, such that $a_j \to 0$ as $j \to \infty$ and write

$$b_1^k = a_j, \text{ when } \frac{(j-1)j}{2} < k \le \frac{j(j+1)}{2} \text{ for some } j \in \mathbf{N},$$
$$b_2^k = a_{j+k-\frac{(j-1)j}{2}}, \text{ when } \frac{(j-1)j}{2} < k \le \frac{j(j+1)}{2} \text{ for some } j \in \mathbf{N},$$

for $k = 1, 2, \ldots$. Nevertheless, E is clearly sparse in Ω.

Again, our set E is a natural exceptional set in the situation considered. In fact, suppose that

$$\sum_{j=1}^{\infty} |c_j| < \infty.$$

Then it is routine to check that f,

$$f(z_1, z_2) = e^{-1/z_1} + e^{-1/z_2} + \sum_{j=1}^{\infty} c_j \frac{z_1 - b_1^j}{z_2 - b_2^j},$$

is a meromorphic function from $\Omega\backslash E$ to $\mathbf{C}^*$, which, however, cannot be extended to any larger set as a meromorphic function with values in $\mathbf{C}^*$.

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