A SHORT SOLUTION OF MARKOV'S PROBLEM ON CONNECTED GROUP TOPOLOGIES

DIETER REMUS

(Communicated by Dennis Burke)

Abstract. In 1945 A. A. Markov posed a problem concerning the existence of connected group topologies that was recently solved by V. G. Pestov. In this note, a short and independent solution is given.

All topological groups are assumed to be Hausdorff. For a Russian article, the cited translation is used.

In [4] A. A. Markov discussed the existence of connected group topologies. He called a subset $M$ of an abstract group $G$ unconditionally closed in $G$ if $M$ is closed with respect to any group topology on $G$. (Note that this property is called "absolutely closed" in [5].) The following fact can be shown easily [4, Theorem 2.9, p. 270]: If a group $G$ has an unconditionally closed proper subgroup of index less than $2^\aleph_0$, then $G$ does not admit a connected group topology.

Now Markov asked whether the inverse theorem is true [4, Problem 5, p. 271]. Recently V. G. Pestov [5] has answered this old question in the negative. For the construction of his counterexample, Pestov develops a complicated technique of independent interest (cf. [5, §§1 and 2]). In this note, quite different examples are given independently by a short proof (see theorem). In March 1989 the author reported on this result at Sofia (Bulgaria). Two weeks later, D. Dikranjan informed him of Pestov's paper.

Let $M$ be an infinite set. Then $S(M)$ denotes the group of all bijections from $M$ onto $M$. $S(M)$ is called the symmetric group of $M$.

Theorem. Let $M$ be a set with $|M| \geq 2^\aleph_0$. Then every unconditionally closed proper subgroup of $S(M)$ is of index not less than $2^\aleph_0$, but $S(M)$ admits only totally disconnected group topologies.

Proof. By [2] $S(M)$ has no proper subgroups of index less than $|M|$. Let $\tau_M$ be the topology of pointwise convergence on $S(M)$ with respect to the discrete topology on $M$. It is easy to see that $\tau_M$ is a group topology which has, as a subbasis for the neighbourhoods of the identity, the collection of all subgroups...
of the form \( E_x = \{ f \in S(M): f(x) = x \} \), where \( x \in M \) (cf. [1, §7.1]). Then [3, Theorem 2] implies that on \( S(M) \) every group topology \( \tau \) is finer than \( \tau_M \). Since \( \tau_M \) is totally disconnected, \( \tau \) has the same property.

It is natural to pose the following:

**Question.** Is there an abelian group \( G \) such that \( G \) does not admit a connected group topology, but every unconditionally closed proper subgroup of \( G \) is of index not less than \( 2^{\aleph_0} \) ?

**References**


