THE EXTENSION OF THE THEOREMS OF Č. V. STANOJEVIĆ AND V. B. STANOJEVIĆ

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Abstract. The new necessary-sufficient conditions for $L^1$ convergence of Fourier series are obtained; Č. V. Stanojević and V. B. Stanojević's theorem [5] and Singh and Sharma's theorem [2] are modified; the convergence theorem for a function sequence in $L^1$ space is obtained; and the extensions are made for the Sheng and Yang theorem [6] and Singh and Sharma's results.

1. Introduction

Let $\{q_n\}$ be a monotone decreasing sequence for sufficiently large $n$. If the limit of $q_n$ is zero as $n \to \infty$, then it is denoted by $q_n \downarrow 0$. If there exists $\beta > 0$ such that $n^{-\beta} q_n \downarrow 0$, then the sequence $\{q_n\}$ is called a quasi-monotone sequence and is denoted by $q_n \Downarrow 0$. It is obvious that $q_n \downarrow 0$ implies $q_n \Downarrow 0$.

Assume $\{C_n\} = \{a_n + ib_n\}$ is a null sequence of complex numbers. A complex null sequence $\{C_n\}$ satisfying $\sum_{n=1}^{\infty} |\Delta(C_n - C_{n-1})| \ln n < \infty$, is called weakly even and is denoted by $C_n \in W$. It is obvious that if $\{C_n\}$ is an even sequence then it is weakly even. If $\{C_n\}$ is satisfying $\sum_{n=1}^{\infty} |\Delta(C_n - C_{n-1})| n^r \ln n < \infty$, $r = 0, 1, 2, \ldots$, then it is denoted by $C_n \in W_r$, where $W_0 = W$.

For convenience, the following notations are used:

$$M = \left\{ A_n | A_n \downarrow 0 \text{ and } \sum_{n=1}^{\infty} A_n < \infty \right\},$$

$$M_\alpha = \left\{ A_n | A_n \downarrow 0 \text{ and } \sum_{n=1}^{\infty} n^\alpha A_n < \infty \text{ for some } \alpha \geq 0 \right\},$$

$$S = \left\{ a_n | a_n \to 0, n \to \infty, |\Delta a_n| \leq A_n \forall n, \text{ and } A_n \in M \right\},$$

$$S_{p\alpha} = \left\{ a_n | a_n \to 0, n \to \infty, (1/n) \sum_{k=1}^{n} |\Delta a_k| A_k^p = O(1), 1 \leq p \leq 2, \text{ and, for some } \alpha \geq 0, A_n \in M_\alpha \right\},$$

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\[ S_p^* = \left\{ C_n | C_n \in W, (1/n) \sum_{k=1}^{n} |\Delta a_k| / A_k^p = O(1), 1 < p \leq 2, \text{ and } A_n \in M \right\}, \]

and

\[ S_{p,r}^* = \left\{ C_n | \text{for some } \alpha \geq 0, \ r \in \{ 0, 1, \ldots, [\alpha] \} \right\}, \]

\[ C_n \in W, (1/n^{\alpha(r)+1}) \sum_{k=1}^{n} |\Delta C_k| / A_k^p = O(1), 1 < p \leq 2, \text{ and } A_n \in M \alpha \} \].

When \( \alpha = 0 \), denote \( M_\alpha = M_0 \), \( S_\alpha = S_0 \), and when \( r = \alpha = 0 \), denote \( S_{p,r}^* = S_{p,0}^* \). It is obvious that \( M_\alpha \supset M_0 \supset M \), \( S_{p,r}^* \supset S_{p,0}^* \supset S_0^* \), and \( S_{p,r}^* \supset S_{p,0}^* \supset S_0^* \supset S \).

Telyakovskii [7] proved that if \( a_n \in S \) then the series \( a_0/2 + \sum_{k=1}^{\infty} a_k \cos kx \)
is the Fourier series of its sum \( f \), and \( \| S_n(f) - f \| = o(1), \ n \to \infty \), is equivalent to \( a_n \log n = o(1) \), where \( S_n(f) = S_n(f, x) = a_0/2 + \sum_{k=1}^{\infty} a_k \cos kx \) and \( \| \cdot \| \)denotes the \( L^1(0, \pi) \)-norm (a notation to be used throughout the paper). Singh and Sharm [2] showed that the condition \( a_n \in S \) can be reduced to be \( a_n \in S_0 \); Sheng and Yang [6] showed that the condition again can be reduced to be \( a_n \in S_{p,0} \); in this paper, we are going to show that the condition can be further reduced to be \( a_n \in S_{p,r} \).

The partial sums of the complex trigonometric series \( \sum_{|n|<\infty} C_n e^{int} \) will be denoted by \( S_n(c) = S_n(c, t) = \sum_{|k| \leq n} C_k e^{ikt}, t \in T = R/2\pi Z \). If a trigonometric series is the Fourier series of some \( f(t) \in L^1 \), we will write \( C_n = \hat{f}(n), \) for all \( n \), and \( S_n(c, t) = S_n(f, t) = S_n(f) \). Č. V. Stanojević and V. B. Stanojević [5] studied this problem from another aspect and improved Telyakovskii’s results. They proved that if \( C_n \in S_p^* \) then, for \( t \neq 0 \), \( \lim_{n \to \infty} S_n(C, t) = f(t) \)
epeat exists and \( f \in L^1(T) \) and that \( \| S_n(f) - f \| = o(1), \ n \to \infty \), is equivalent to \( \hat{f}(n) \log |n| = o(1), \ |n| \to \infty \). In this paper we will prove that the condition \( C_n \in S_p^* \) can be reduced to be \( C_n \in S_{p,r}^* \); also, we will improve the Singh and Sharma’s result [1] about \( L^1 \) convergence.

2. Lemmas

Lemma 2.1. Let \( A_n \in M_\alpha \), \( \alpha \geq 0 \). Then \( \sum_{n=1}^{\infty} n^{1+\alpha}|\Delta A_n| < \infty \).

Proof. It follows from [3] and [4] that \( A_n = 0 \ (n^{-1-\alpha}) \) and \( |\Delta A_n| = \Delta A_n + O(A_n/n) \). Then

\[ \sum_{k=1}^{n-1} k^{1+\alpha}|\Delta A_k| = \sum_{k=1}^{n-1} k^{1+\alpha}\Delta A_k + \sum_{k=1}^{n-1} O(k^\alpha A_k) \]

\[ = \sum_{k=1}^{n} [k^{1+\alpha} - (k-1)^{1+\alpha}] A_k - n^1 A_n + \sum_{k=1}^{n-1} O(k^\alpha A_k) \]

\[ = \sum_{k=1}^{n} O(k^\alpha A_k) + O(1). \]

This completes the proof. \( \square \)
Lemma 2.2. Let $r$ be a nonnegative integer, and $x \in [\pi/n, \pi]$ where $n \geq 1$. Then

\[
D_n^{(r)}(\alpha) = \sum_{k=0}^{r-1} \frac{(n+1/2)^k \sin((n+1/2)x + k\pi/2) \varphi}{(\sin(x/2))^{r+1-k}} + \frac{(n+1/2)^r \sin((n+1/2)x + r\pi/2)}{2\sin x/2},
\]

where the same $\varphi$ denotes various analytical functions of $x$ independent of $n$, and $D_n(x)$ is the Dirichlet kernel.

Proof. The proof is straightforward in the case of $r = 0$. Assuming that Equation (2.1) holds, and taking the derivatives of both sides in this equation, we have

\[
D_n^{(r+1)}(x) = \sum_{k=0}^{r-1} \left\{ (n+1/2)^k \sin((n+1/2)x + (k+1)\pi/2)(\sin(x/2))^{k-r-1} \varphi + (n+1/2)^k \sin((n+1/2)x + k\pi/2)(\sin(x/2))^{k-r-2} \varphi + \frac{(n+1/2)^{r+1} \sin((n+1/2)x + (r+1)\pi/2)}{2\sin(x/2)} \right\}
\]

\[
= \sum_{r=0}^{r} (n+1/2)^k \sin((n+1/2)x + k\pi/2) \cdot (\sin(x/2))^{k-r-1} \varphi
\]

The proof follows by induction. \(\square\)

Lemma 2.3. Let $n \geq 1$, and let $r$ be a nonnegative integer, $x \in [\epsilon, \pi]$. Then

\[|D_n^{(r)}(x)| \leq C_\epsilon n^r / x,\]

where $C_\epsilon$ is a positive constant depending on $\epsilon$, and $0 < \epsilon < \pi$.

Proof. This is a direct result from Lemma 2.2. \(\square\)

Lemma 2.4. $\|D_n^{(r)}(x)\| = (4/\pi)n^r \log n + O(n^r)$, $r \in \{0, 1, \ldots\}$.

Proof. It follows from $D_n^{(r)}(x) = O(n^{r+1})$ and Lemma 2.2 that

\[
\|D_n^{(r)}(x)\| = 2 \left\{ \int_0^{\pi/n} + \int_0^{\pi/n} \right\} |D_n^{(r)}(x)| \, dx
\]

\[
= 2n^r \int_0^{\pi/n} \frac{\left| \sin(nx + r\pi/2) \right|}{x} \, dx + \sum_{k=0}^{n-1} \int_{\pi/k}^{\pi/n} n^k x^{k-1-r} \, dx + O(n^r)
\]

\[
= \frac{4}{\pi} n^r \log n + O(n^r). \quad \square\]
Lemma 2.5. \[ \|D^{(r)}_n(x)\| = O(n^{r/2n} \log n), \quad r \in \{0, 1, \ldots\}. \]

Proof. With the Bernstein inequality in $L$ space, we have
\[ \int_0^\pi |D^{(r)}_n(x)| \, dx \leq n^{r} \int_0^\pi |D_n(x)| \, dx. \tag{2.2} \]
On the other hand,
\[ \int_0^\pi |D_n(x)| \, dx \leq \int_0^\pi \sin^2(nx/2) \, dx + O(1) \nonumber = \log(1 + n\pi) + O(1) = O(n \log n). \]
By combining Equations (2.2) and (2.3) the proof is completed. \qed

Lemma 2.6. For each nonnegative integer $n$, there holds
\[ \|C_n E_n^{(r)}(t) + C_{-n} E_{-n}^{(r)}(t)\| = o(1) \iff n^{r} C_n \log |n| = o(1), \]
where \{C_n\} is a complex sequence, $E_n(t) = \sum_{k=0}^n e^{ikt}$ and $t \in R/2\pi Z$.

Proof. Assuming $r \geq 1$ and denoting $J_n = \|C_n E_n^{(r)}(t) + C_{-n} E_{-n}^{(r)}(t)\|$, from Lemma 2.4 we have
\[ J_n = \int_0^\pi \left| C_n E_n^{(r)}(t) + C_{-n} E_{-n}^{(r)}(t) \right| \, dt \tag{2.4} \geq |C_n + C_{-n}| \int_0^\pi |E_n^{(r)}(t) + E_{-n}^{(r)}(t)| \, dt \nonumber \]
\[ = |C_n + C_{-n}| \int_0^\pi 2|D_n^{(r)}(t)| \, dt \geq \frac{4}{\pi} |C_n + C_{-n}| n^{r} \log n + O(1). \]
On the other hand, using
\[ J_n = \int_{-\pi}^\pi \left| (C_n + C_{-n}) E_n^{(r)}(t) + C_{-n}(E_n^{(r)}(t) - E_{-n}^{(r)}(t)) \right| \, dt \tag{2.5} \leq |C_n + C_{-n}| \int_{-\pi}^\pi |E_n^{(r)}(t)| \, dt + |C_{-n}| \int_{-\pi}^\pi |E_{-n}^{(r)}(t) - E_{n}^{(r)}(t)| \, dt \]
with Lemma 2.4 and Lemma 2.5, the right-hand side of Equation (2.5) can be written as
\[ O\{ |C_n + C_{-n}| n^{r} \log n \} + O\{ |C_{-n}| n^{r} \log n \} = O\{ |C_n + C_{-n}| n^{r} \log n \}. \tag{2.6} \]
Combining Equations (2.4), (2.5), and (2.6) completes the proof. \qed

3. $L^1$ CONVERGENCE OF FOURIER SERIES AND FUNCTION SEQUENCES IN $L^1$ SPACE

Let $S_n(C, t) = \sum_{|k| \leq n} C_k e^{ikt}, \quad t \in T = R/2\pi Z$. The limit of $S_n(C, t)$ is denoted by $f(t)$. If $f(t) \in L$, then $\hat{f}(k) = C_k$ is called the Fourier coefficient of $f(t)$. Let
\[ \lim_{n \to \infty} S^{(r)}_n(C, t) = f^{(r)}(t), \quad r \in \{1, 2, \ldots\}. \]
If $f^{(r)}(t) \in L$, then it is denoted by $f^{(r)}(t)$. 

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**Theorem 3.1.** (main theorem). Let \( C_n \in S_{\text{par}}^* \), \( \alpha \geq 0 \), and \( r \in \{0, 1, \ldots \lceil \alpha \rceil \} \). Then, for \( t \neq 0 \),

(i) \( \lim_{n \to \infty} S_n^{(r)}(C, t) = f_r(t) \),

(ii) \( f_r(t) \in L(T) \),

(iii) \( \|S_n^{(r)}(f, t) - f^{(r)}(t)\| = o(1) \iff n^r \hat{f}(n) \log |n| = o(1) \).

**Proof.** It can be shown that

\[
\sum_{k=1}^{n} |\Delta[(ik)^r C_k]| = \sum_{k=1}^{n} |(ik)^r C_k + C_{k+1} \Delta[(ik)^r]| \\
\leq \sum_{k=1}^{n} k^r |\Delta C_k| + \sum_{k=1}^{n} k^{r-1} |C_{k+1}|.
\]

Notice that

\[
\sum_{k=1}^{n} k^{r-1} |C_{k+1}| = O \left( \sum_{k=1}^{n-1} |\Delta C_{k+1}| k^r + \sum_{k=n+1}^{\infty} k^r |\Delta C_k| \right)
\]

and

\[
\sum_{k=1}^{n} k^r |\Delta C_k| = \sum_{k=1}^{n-1} \Delta A_k \sum_{j=1}^{k} \frac{\Delta C_j}{A_j} j^r + A_n \sum_{j=1}^{n} \frac{|\Delta C_j|}{A_j} j^r \\
\leq \sum_{k=1}^{n-1} \Delta A_k k^{1+\alpha} \left( k^{p(r-\alpha)-1} \sum_{j=1}^{k} \frac{|\Delta C_j|^p}{A_j^p} \right)^{1/p} + n^{1+\alpha} A_n \left( n^{p(r-\alpha)-1} \sum_{j=1}^{n} \frac{|\Delta C_j|^p}{A_j^p} \right)^{1/p} \\
= O(1)
\]

Therefore \( \sum_{k=1}^{\infty} |\Delta ik C_k| < \infty \), and (i) is proved.

Now the proof of (ii): it is obvious for the case of \( r = 0 \). Assuming \( r \geq 1 \), it can be shown that

\[
S_n^{(r)}(C, t) - (C_n E_n^{(r)}(t) + C_{-n} E_{-n}^{(r)}(t)) \\
= \sum_{k=0}^{n} C_k (ik)^r e^{ikt} + \sum_{k=1}^{n} (C_{-k} - C_k) (-ik)^r e^{-ikt} \\
+ \sum_{k=1}^{n} C_k (-ik)^r e^{-ikt} - C_n E_n^{(r)}(t) - C_{-n} E_{-n}^{(r)}(t) \\
= 2 \left( \sum_{k=0}^{n-1} \Delta C_k D_k^{(r)}(t) \right) + \sum_{k=1}^{n-1} \Delta(C_{-k} - C_k) E_{-k}^{(r)}(t) \equiv g_n(t).
\]
For \( t \neq 0 \), it follows from (i) that
\[
\begin{align*}
  f_r(t) - g_{n,r}(t) &= \sum_{k=0}^{\infty} C_k (e^{ikt})^r + \sum_{k=1}^{\infty} C_{-k} (e^{-ikt})^r - g_{n,r}(t) \\
  &= 2 \sum_{k=0}^{\infty} \Delta C_k D_k^r(t) + \sum_{k=1}^{\infty} \Delta (C_{-k} - C_k)(E_k^r(t)) \\
  &\quad - 2 \sum_{k=0}^{n-1} \Delta C_k D_k^r(t) - \sum_{k=0}^{n-1} \Delta (C_{-k} - C_k)(E_k^r(t)) \\
  &= 2 \sum_{k=r}^{\infty} \Delta C_k D_k^r(t) + \sum_{k=n}^{\infty} \Delta (C_{-k} - C_k)(E_k^r(t)).
\end{align*}
\]

From the above results and Lemmas 2.4, 2.5 and \( C_n \in S_{par}^* \), we can get
\[
(2.7) \quad \|f_r(t) - g_{n,r}(t)\| \leq 2 \int_0^\pi \left| \sum_{k=n}^{\infty} \Delta C_k D_k^r(t) \right| dt \\
  + O\left\{ \sum_{k=n}^{\infty} \left| \Delta (C_{-k} - C_k) \right| \int_0^\pi |E_k^r(t)| dt \right\} \\
  + O(1) = O\left\{ \int_0^\pi \left| \sum_{k=n}^{\infty} \Delta C_k D_k^r(t) \right| dt + O(1) \right\}
\]

and
\[
\int_0^\pi \left| \sum_{k=n}^{\infty} \Delta C_k D_k^r(t) \right| dt \\
\leq \sum_{k=n}^{\infty} |\Delta A_k| \int_0^\pi \left| \sum_{j=1}^{k} \frac{\Delta C_j D_j^r(t)}{A_j} \right| dt + A_n \int_0^\pi \left| \sum_{j=1}^{n-1} \frac{\Delta C_j D_j^r(t)}{A_j} \right| dt \\
= I_1 + I_2,
\]

where
\[
(2.9) \quad I_1 = \sum_{k=n}^{\infty} |\Delta A_k| \int_0^{\pi/k} \left| \sum_{j=1}^{k} \frac{\Delta C_j D_j^r(t)}{A_j} \right| dt + \sum_{k=n}^{\infty} |\Delta A_k| \int_0^\pi \left| \sum_{j=1}^{k} \frac{\Delta C_j D_j^r(t)}{A_j} \right| dt \\
= I_{11} + I_{12}.
\]

It follows from the hypotheses of Theorem 3.1 and Lemma 2.1 that
\[
(2.10) \quad I_{11} = O\left\{ \sum_{k=n}^{\infty} |\Delta A_k| k^{1+\alpha} \left( k^{p(r-\alpha)-1} \sum_{j=1}^{k} \frac{|\Delta C_j|^p}{A_j^p} \right)^{1/p} \right\} \\
= O\left\{ \sum_{k=n}^{\infty} k^{1+\alpha} |\Delta A_k| \right\} = O(1).
\]
Now we estimate $I_{12}$:

\[(2.11) \quad I_{12} = O \left\{ \sum_{k=n}^{\infty} |\Delta A_k| \left\| \sum_{j=1}^{k} \frac{\Delta C_j}{A_j} D_j^{(r)}(t) \right\| dt \right\} \triangleq O \left\{ \sum_{k=n}^{\infty} |\Delta A_k| |\varphi_k(t)| \right\} .\]

From Lemma 2.2, we have

\[(2.12) \quad \varphi_k(t) = \int_{\pi/k}^{\pi} \left\| \sum_{j=1}^{k} \frac{\Delta C_j}{A_j} D_j^{(r)}(t) \right\| dt
\]

\[= \int_{\pi/k}^{\pi} \left\| \sum_{j=1}^{k} \frac{\Delta C_j}{A_j} \left( \sum_{\nu=0}^{r-1} \frac{(j + 1/2)^\nu \sin[(j + 1/2)t + \nu \pi/2]}{(\sin(t/2))^{r+1-\nu}} \phi \right) \right\| dt
\]

\[+ \int_{\pi/k}^{\pi} \left\| \sum_{j=1}^{k} \frac{\Delta C_j}{A_j} \frac{(j + 1/2)^\nu \sin[(j + 1/2)t + r \pi/2]}{2 \sin(t/2)} \right\| dt \triangleq \varphi'_k(t) + \varphi''_k(t),\]

where

\[\varphi'_k(t) = \varphi'_{k,1}(t) + \cdots + \varphi'_{k,r}(t)\]

\[\varphi'_{k,\nu}(t) = \int_{\pi/k}^{\pi} \left\| \sum_{j=1}^{k} \frac{\Delta C_j}{A_j} (j + 1/2)^\nu \sin[(j + 1/2)t + \nu \pi/2]}{(\sin(t/2))^{r+1-\nu}} \phi \right\| dt .\]

Since $\varphi$ is bounded, it can be shown by Hölder's inequality and Hausdorff-Young's inequality that

\[(2.13) \quad \varphi'_{k,\nu}(t) \leq C \int_{\pi/k}^{\pi} \left\| \sum_{j=1}^{k} \frac{\Delta C_j}{A_j} (j + 1/2)^\nu \sin[(j + 1/2)t + \nu \pi/2]}{(\sin(t/2))^{r+1-\nu}} \right\| dt
\]

\[= O \left\{ k^{(r+1-\nu)p-1/p} \left( \sum_{j=1}^{k} \frac{|\Delta C_j|^p}{A_j^p} j^{-\nu p} \right)^{1/p} \right\}
\]

\[= O \left\{ k^{p(r-\alpha)-1} \left( \sum_{j=1}^{k} \frac{|\Delta C_j|^p}{A_j^p} \right)^{1/p} \right\} \left[ k^{1+\alpha} \right].\]

Since $r$ is a finite value, we have

\[(2.14) \quad \varphi'_k(t) = O_r \left\{ k^{1+\alpha} \left[ k^{p(r-\alpha)-1} \left( \sum_{j=1}^{k} \frac{|\Delta C_j|^p}{A_j^p} \right)^{1/p} \right] \right\} .\]

where $O_r$ dependents only on $r$.

Similarly, we can get

\[(2.15) \quad \varphi''_k(t) = O_r \left\{ k^{1+\alpha} \left[ k^{p(r-\alpha)-1} \left( \sum_{j=1}^{k} \frac{|\Delta C_j|^p}{A_j^p} \right)^{1/p} \right] \right\} .\]
Combining Equations (2.11)–(2.15), it follows from Lemma 2.1 and the hypotheses of Theorem 3.1 that $I_{12} = o(1)$. From Equations (2.9)–(2.11) we have $I_1 = o(1)$. Similarly, $I_2 = o(1)$ holds. From Equations (2.7) and (2.8), we get $\|f(t) - g_{n,r}(t)\| = o(1)$, and $f(t) \in L$, since $g_{n,r}(t)$ is a polynomial. Therefore, (ii) is proved.

At last, the proof of (iii). Because

$$f^{(r)}(t) - S_n^{(r)}(f, t) - [\hat{f}(n)E_n^{(r)}(t) + \hat{f}(-n)E_{-n}^{(r)}(t)] = f^{(r)}(t) - g_{n,r}(t),$$

we have

$$\|f^{(r)}(t) - g_{n,r}(t)\| \geq \|f^{(r)}(t) - S_n^{(r)}(f, t)\| - \|\hat{f}(n)E_n^{(r)}(t) + \hat{f}(-n)E_{-n}^{(r)}(t)\|. $$

It follows from Lemma 2.6 and condition $\|f^{(r)}(t) - g_{n,r}(t)\| = o(1)$ that

$$\|f^{(r)}(t) - S_n^{(r)}(f, t)\| = O(1) \Rightarrow n^r \hat{f}(n) \log |n| = O(1).$$

This completes the proof of Theorem 3.1. □

**Corollary 3.1.** In the case of $\alpha = 0$ and $r = 0$, Theorem 3.1 is an extension of Theorem 2.1 in [5].

**Corollary 3.2.** Assume that $\alpha \geq 0$, $r \in \{0, 1, \ldots, \lfloor\alpha\rfloor\}$, and $C_n$ is a complex zero sequence that satisfies $\sum_{n=1}^{\infty} |\Delta(C_n - C_{n+1})|^n < \infty$. Let $\rho_n > 0$ such that $\sum_{n=1}^{\infty} n^\alpha / n \rho_n < \infty$, $1/(n^{1-\beta} \rho_n) \downarrow 0$, for some $\beta > 0$, and $n^{\rho(r-\alpha)-1} \sum_{k=1}^{n}(k \rho_k)^p |\Delta C_k|^p = O(1)$. Then the result of Theorem 3.1 is valid.

Let

$$f_n(x) = (a_0 - a_{n+1})/2 + \sum_{k=1}^{n}(a_k - a_{n+1}) \cos kx, S_n(x)$$

$$= a_0/2 + \sum_{k=1}^{n} a_k \cos kx, a_n$$

$$= O(1).$$

If $S_n(x)$ is convergent, then its limit is denoted by $f(x)$. It is obvious that $\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} S_n(x) = f(x)$.

**Theorem 3.2.** Let $a_n \in S_{p\alpha}$, $\alpha \geq 0$, $r \in \{0, 1, \ldots, \lfloor\alpha\rfloor\}$. Then

(i) $\|f^{(r)}(x) - f^{(r)}(x)\| = O(n^{r-\alpha})$,

(ii) $\|S^{(r)}(x) - f^{(r)}(x)\| = O(n^{r-\alpha}) \iff a_n \log n = O(n^{\alpha}).$

**Proof.** It can be shown from $a_n \in S_{p\alpha}$ and from $A_k k^\alpha \leq 1/k$ that $\sum_{k=1}^{\infty} k^\alpha |\Delta a_k| < \infty$. Indeed

$$\sum_{k=1}^{n} A_k k^\alpha |\Delta a_k| / A_k \leq \left(\sum_{k=1}^{n} k^{-q}\right)^{1/q} \left(\sum_{k=1}^{n} |A_k|^p / A_k^p\right)^{1/p}$$

$$= O(n^{-(1-q)/p + 1/p}) = O(1).$$

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From Lemma 2.3 we know that $\sum_{k=0}^{\infty} D_k^{(r)}(x)\Delta a_k$ is uniformly convergent on any compact subset of $(0, \pi)$; $f(x) = \sum_{k=0}^{\infty} D_k(x)\Delta a_k$ implies $f^{(r)}(x) = \sum_{k=0}^{\infty} D_k^{(r)}(x)\Delta a_k$. So

$$\|f_n^{(r)}(x) - f^{(r)}(x)\| = \left\| \sum_{k=n+1}^{\infty} D_k^{(r)}(x)\Delta a_k \right\|$$

$$\leq 2 \left( \sum_{k=n}^{\infty} |\Delta A_k| \int_0^\pi \left| \sum_{j=1}^k \frac{\Delta a_k}{A_j} D_j^{(r)}(x) \right| dx + A_n \int_0^\pi \left| \sum_{j=1}^n \frac{\Delta a_j}{A_j} D_j^{(r)}(x) \right| dx \right)$$

$$= 2(J_1 + J_2).$$

It follows from Lemma 2.4 and the estimates of $I_{12}$ in Theorem 3.1 that $J_1 = o(n^{-\alpha})$, $a_k \in S_{p\alpha}$. Similarly, $J_2 = o(n^{-\alpha})$ holds.

From

$$S_n^{(r)}(x) - f^{(r)}(x) - a_{n+1}D_n^{(r)}(x) = f_n^{(r)}(x) - f^{(r)}(x),$$

$$\|S_n^{(r)}(x) - f^{(r)}(x)\| - a_{n+1}\|D_n^{(r)}(x)\| \leq \|f_n^{(r)}(x) - f^{(r)}(x)\|,$$

and Lemma 2.4, (ii) can be proved.

**Corollary 3.3.** When $r = 0$, $\alpha = 0$, Theorem 3.2 becomes the extension of corresponding theorems in [2].

**Corollary 3.4.** If $a_n \in S_{\alpha} \subset S_{p\alpha}(\alpha \geq 0)$, Theorem 3.2 reduces to Theorem 1 in [6].

**References**


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