RECURRENT HOMEOMORPHISMS ON $\mathbb{R}^2$ ARE PERIODIC

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Abstract. A homeomorphism $f: (X, d) \to (X, d)$ of a metric space $(X, d)$ onto $X$ is recurrent provided that for each $\varepsilon > 0$ there exists a positive integer $n$ such that $f^n$ is $\varepsilon$-close to the identity map on $X$. The notion of a recurrent homeomorphism is weaker than that of an almost periodic homeomorphism. The result announced in the title generalizes the theorem of Brechner for almost periodic homeomorphisms and answers a question of R. D. Edwards.

1. Introduction

Let $(X, d)$ be a locally compact metric space. Let $\text{id}_X$ denote the identity function on $X$, and let $Z$ (resp. $Z^+$) denote the set of integers (resp. non-negative integers). A homeomorphism $g: X \to X$ of $X$ onto $X$ is almost periodic if for each $\varepsilon > 0$ there exists a relatively dense set $A$ in $Z$ (i.e., there exists $N \in Z^+$ such that $[n, n + N] \cap A \neq \emptyset$ for each $n \in Z$) such that $d(g^n, \text{id}_X) < \varepsilon$ for each $m \in A$.

A homeomorphism $g: X \to X$ of $X$ onto $X$ is recurrent if for each $\varepsilon > 0$ there exists $n > 0$ such that $d(g^n, \text{id}_X) < \varepsilon$.

For $X$ compact the following are equivalent [Got]:

1. $g$ is almost periodic.
2. \{ $g^n | n \in Z$ \} is equicontinuous.
3. \{ $g^n | n \in Z$ \} has compact closure in the space of all homeomorphisms of $X$ onto $X$ with compact open topology.

Clearly, periodic homeomorphisms are almost periodic and almost periodic homeomorphisms are recurrent. None of these implications can be reversed. By [Bre], almost periodic homeomorphisms of the plane $\mathbb{R}^2$ with the usual metric $d$ are periodic. Hence, each almost periodic homeomorphism of the
plane is conjugate either to a rational rotation or to a reflection about a line [Got], [Eil].

The main purpose of this paper is to prove that for the plane \( \mathbb{R}^2 \) with its usual metric \( d \) recurrent homeomorphisms are periodic. This answers a recent question of R. D. Edwards. This result was claimed in [Hac-1], but the proof given there appears to be deficient [Hac-2]. Theorem 1 gives a positive solution to a problem raised by J. Hachigian for \( n = 2 \). The case \( n = 1 \) was done in [Coh-Hac]. We are indebted to Professor Morton Brown for references [Hac-1], [Hac-2], and [Coh-Hac] and for suggesting to us the term “recurrent homeomorphism.”

2. The main result

In this section we will prove the main result of the paper, but first we prove a special case of the main result under somewhat weaker hypotheses.

By a domain we will mean a nonempty, bounded, connected, simply connected, open subset of \( \mathbb{R}^2 \). We denote the closure (resp. boundary) of a set \( A \) by \( \text{Cl}(A) \) (resp. \( \text{Bd}(A) \)). We let \( B(A, \varepsilon) = \{ x \in X | d(x, A) < \varepsilon \} \).

Let \( h: X \rightarrow X \) be a homeomorphism. A set \( A \) in \( X \) is \( h \)-invariant if \( h(A) \subset A \), and \( A \) is completely \( h \)-invariant if \( h(A) = A \). A homeomorphism \( h: X \rightarrow X \) of \( X \) onto \( X \) is arc-recurrent (resp. point-recurrent) provided that for each arc \( A \) (resp. for each point \( p \) ) in \( X \) and for each \( \varepsilon > 0 \) there exists a positive integer \( n \) such that \( h^n(A) \subset B(A, \varepsilon) \) (resp. \( d(h^n(p), p) < \varepsilon \)). Clearly, each recurrent homeomorphism is arc-recurrent and each arc-recurrent homeomorphism is point-recurrent. Notice that each irrational rotation of \( \mathbb{R}^2 \) is arc-recurrent but not recurrent. For domains we can prove the following:

**Theorem 1.** Let \( U \) be a domain and let \( h: \text{Cl}(U) \rightarrow \text{Cl}(U) \) be an arc-recurrent homeomorphism such that \( h|\text{Bd}(U) = \text{id}_{\text{Bd}(U)} \). Then \( h = \text{id}_{\text{Cl}(U)} \).

**Proof.** Note first that \( h \) is orientation preserving. Let \( \text{Fix}(h) = \{ x \in \text{Cl}(U) | h(x) = x \} \). Suppose \( \text{Fix}(h) \neq \text{Cl}(U) \), or there is nothing to prove. Then \( \text{Bd}(U) \subset \text{Fix}(h) \) and \( \text{Fix}(h) \) is closed. We prove first that \( \text{Fix}(h) \) is not connected. If \( \text{Fix}(h) \) were connected, let \( W \) be a component of \( U \setminus \text{Fix}(h) \). Then \( W \) would be homeomorphic to \( \mathbb{R}^2 \) and \( h|W \) would be an orientation preserving fixed-point free homeomorphism onto \( W \) [Bro-Kis]. By a theorem of Brouwer [And], \( \lim_{n \to \infty} \sup h^n(x) \in \text{Bd}(W) \) for \( x \in W \). This would contradict point-recurrence. Thus, there exists a component \( F' \) of \( \text{Fix}(h) \) such that \( U \setminus \text{Fix}(h) \) separates \( \text{Bd}(U) \) from \( F' \). Let \( F \) be the topological hull of \( F' \) (i.e., \( F \) is the union of \( F' \) together with all of the bounded components of \( \mathbb{R}^2 \setminus F' \)).

By [Bro-Kis], \( h(F) = F \). Since \( h \) is one to one, \( h(U \setminus F) = U \setminus F \).

Let \( T_1 \) be a simple closed curve in \( U \setminus \text{Fix}(h) \) which separates \( F \) from \( \text{Bd}(U) \). Let \( V_1 \) be the component of \( \text{Cl}(U) \setminus T_1 \) which meets \( \text{Bd}(U) \). Let \( A = \{ x \in \text{Cl}(U) | h^n(x) \notin V_1 \text{ for } n \in \mathbb{Z}^+ \} \). Then \( A \) is a closed, \( h \)-invariant set. Since \( h \) is point-recurrent, \( A \) is completely
$h$-invariant. So $A \subset \text{Cl}(U) \setminus V_1$ is a compact set and $F \subset A$. Let $C$ be the component of $F$ in $A$. Since $C \cap \text{Fix}(h) \neq \emptyset$, $C$ is invariant. Also, by the same argument as was used in the construction of $F$, $C$ does not separate the plane.

Claim 1. $F \neq C$.

Proof of Claim 1. Suppose $C = F$. Since the components of a compact Hausdorff space are quasi-components [Eng, p. 438], there exists a simple closed curve $T_2$ in $U \setminus A$ which separates $T_1$ and $F$. Let $V_2$ be the component of $\text{Cl}(U) \setminus T_2$ which contains $V_1$. Let

$$H = \bigcup_{n=0}^{\infty} h^n(\text{Cl}(V_2)) \subset \text{Cl}(U) \setminus F.$$ 

Now, $H$ is $h$-invariant and $H \cap C = 0$. If $H$ is compact, then $H$ is closed, and hence $H$ is completely $h$-invariant, since $h$ is point-recurrent. Hence, $\text{Cl}(U) \setminus H \subset \text{Cl}(U) \setminus V_2$ is $h$-invariant. So $\text{Cl}(U) \setminus H \subset A$. The Boundary Bumping Theorem [Eng, p. 439] states that if $G$ is a proper open subset of a continuum $M$, and $N$ is a component of $G$, then $\text{Cl}(G)$ meets $\text{Bd}(G)$. Since $C$ is a component of $\text{Cl}(U) \setminus H$, $C$ meets $\text{Bd}(\text{Cl}(U) \setminus H)$, which is a contradiction. Hence, $H$ is not compact.

By [Hom-Kin], $\{h^n(\text{Cl}(V_2))\}_{n=0}^{\infty}$ is a bulging sequence (i.e., $h^n(\text{Cl}(V_2)) \setminus \bigcup_{i=0}^{n-1} (\text{Cl}(V_2)) \neq \emptyset$ for $n \in \mathbb{Z}^+$), and there exists a point $x \in \text{Cl}(V_2)$ such that $h^n(x) \notin V_2$ for each $n \in \mathbb{Z}^+$. Since $h$ is point-recurrent, $x \in \text{Cl}(V_2) \cap (\text{Cl}(U) \setminus V_2) = T_2$. Clearly, $x \in A$. This contradicts the fact that $T_2 \cap A = \emptyset$. The claim is proved.

Now, $U \setminus C$ is homeomorphic to the open annulus

$$Y = \{x \in \mathbb{R}^2 | 1 < |x| < 2\},$$

since $C$ is a continuum in the domain $U$ which does not separate the plane. Hence [Eps] there is a uniformization $\varphi : U \setminus C \to Y$, i.e., $\varphi$ is a homeomorphism of $U \setminus C$ onto $Y$ which maps crosscuts onto crosscuts. (A crosscut $K$ of $U \setminus C$ is an arc in $\text{Cl}(U \setminus C)$ such that $K \cap \text{Bd}(U \setminus C)$ is the set of endpoints of $K$ and these endpoints lie in one component of $\text{Bd}(U \setminus C)$.)

We may suppose $\varphi$ maps the points of $U \setminus C$ near $C$ to points of $Y$ near $S_1 = \{x \in \mathbb{R}^2 | |x| = 1\}$ and $\varphi$ maps the points of $U \setminus C$ near $\text{Bd}(U)$ to points of $Y$ near $S_2 = \{x \in \mathbb{R}^2 | |x| = 2\}$.

Let $g = \varphi \circ h \circ \varphi^{-1} : Y \to Y$. Then $g$ is a homeomorphism of $Y$ onto $Y$, and since $h$ is orientation preserving, $g$ is also orientation preserving. By [Eps], $g$ extends to an orientation preserving homeomorphism $G : \text{Cl}(Y) \to \text{Cl}(Y)$. Since $h|\text{Bd}(U) = \text{id}_{\text{Bd}(U)}$ and $\text{Bd}(C) \setminus \text{Fix}(h) \neq \emptyset$, we have $G|S_2 = \text{id}_{S_2}$ and $G|S_1 \neq \text{id}_{S_1}$.

Choose an arc $I$ in $\text{Cl}(U)$ such that $I$ irreducibly joins $\text{Bd}(U)$ to $C$, $I \cap \text{Bd}(U) = \{a\}$, $I \cap C = \{b\}$, and $h(b) \neq b$. Then there exist points $\alpha \in S_2$,
and $\beta \in S_1$ such that $J = \{\alpha, \beta\} \cup \varphi(I\setminus\{a, b\})$ is an arc in $\text{Cl}(Y)$ which is irreducible from $S_1$ to $S_2$. The points $\alpha$ and $\beta$ are the endpoints of $J$, $G(\alpha) = \alpha$ and $G(\beta) \neq \beta$. Let $\varepsilon > 0$.

**Claim 2.** There is a positive integer $n$ such that $G^n(J) \subseteq B(J, \varepsilon)$.

**Proof of Claim 2.** Choose sequences $\{L_i\}_i$ and $\{R_i\}_i$ of arcs in $\text{Cl}(U)\setminus I$ converging to $I$ such that

1. for each $i$ and $j$, $I$ separates $L_i$ from $R_j$ in a connected neighborhood $W$ of $I$ in $(U\setminus C) \cup I$,
2. each $L_i$ and each $R_j$ meets each of $C$ and $\text{Bd}(U)$ in exactly one point, and
3. in $W$, $L_{i+1}$ (resp. $R_{i+1}$) separates $L_i$ (resp. $R_i$) from $I$ for each $i \in \mathbb{Z}^+$.

For each $i \in \mathbb{Z}^+$, let $L_i^0$ (resp. $R_i^0$) be the arc $L_i$ (resp. $R_i$) minus its endpoints. Then $\tilde{L}_i = \text{Cl}(\varphi(L_i^0))$ (resp. $\tilde{R}_i = \text{Cl}(\varphi(R_i^0))$) are arcs in $\text{Cl}(Y)$ converging to $J$ such that each $\tilde{L}_i$ and each $\tilde{R}_i$ intersects each of $S_1$ and $S_2$ in exactly one point.

Suppose that, for each positive integer $n$, $G^n(J) \setminus B(J, \varepsilon) \neq \emptyset$. Choose $i \in \mathbb{Z}^+$ such that the component $O$ of $\text{Cl}(Y) \setminus (\tilde{L}_i \cup \tilde{R}_i)$ which contains $J$ is in $B(J, \varepsilon)$, and $G(\beta) \notin O$. Then for each $n > 0$, $G^n(J) \cap (\tilde{L}_i^0 \cup \tilde{R}_i^0) \neq \emptyset$. Hence, $h^n(I) \cap (\tilde{L}_i \cup \tilde{R}_i) \neq \emptyset$ for each $n$. This contradicts the fact that there exists a positive integer $n$ such that $h^n(I) \subseteq \text{Cl}(U)\setminus (R_i \cup L_i)$. The claim is proved.

Let $Z$ be the universal covering space of the closed annulus $\text{Cl}(Y)$, let $\rho: Z \to \text{Cl}(Y)$ be the covering projection, and let $\tilde{G}: Z \to Z$ be a lifting of $G$. Then $Z$ is the product of the line with an arc and $\text{Bd}(Z) = Z_1 \cup Z_2$, where $Z_1$ and $Z_2$ are lines such that $\rho(Z_i) = S_i$ for $i = 1, 2$, $\tilde{G}|Z_2 = \text{id}_{Z_2}$ and $\tilde{G}|Z_1 \neq \text{id}_{Z_1}$. Assign a natural linear order to the line $Z_1$. Let $\tilde{L}_i$, $\tilde{J}$, and $\tilde{R}_i$ be lifts of $L_i$, $J$ and $R_i$, respectively, such that $\tilde{O}$, the component of $\tilde{J}$ in $Z\setminus (\tilde{L}_i \cup \tilde{R}_i)$, maps homeomorphically onto the set $O$ defined in the proof of Claim 2. Let $\tilde{\beta} \in \tilde{J} \cap \rho^{-1}(\beta)$. Since $G(\beta) \notin O$, $\tilde{G}(\tilde{\beta}) \notin \tilde{O}$. Without loss of generality, $\tilde{\beta} < \tilde{G}(\tilde{\beta})$ in $Z_1$. Since $G$, hence $\tilde{G}$, are orientation preserving homeomorphisms,

$$\tilde{\beta} < \tilde{G}(\tilde{\beta}) < \cdots < \tilde{G}^n(\tilde{\beta})$$

for each $n \in \mathbb{Z}^+$. Hence, for each $n \in \mathbb{Z}^+\setminus\{0\}$, $\tilde{G}^n(\tilde{J}) \cap (\tilde{L}_i \cup \tilde{R}_i) \neq \emptyset$. This implies that $G^n(J) \cap (L_i \cup R_i) \neq \emptyset$ for each $n > 0$, which contradicts Claim 2. This completes the proof of the theorem.

**Remark.** The hypothesis that $\rho|\text{Bd}(U) = \text{id}_{\text{Bd}(U)}$ in Theorem 1 can be replaced by the assumption that $h$ is orientation preserving and $h$ has at least one accessible fixed point on $\text{Bd}(U)$.
Lemma 2. Let \( h : X \to X \) be a recurrent homeomorphism of the metric space \((X, d)\) onto \(X\), and let \( n \) be a positive integer. Then \( h^n \) is recurrent.

Proof. Let \( \varepsilon > 0 \) be given. There is a positive integer \( k \) such that \( d(h^k, \text{id}_X) < \varepsilon/n \). Then, \( d(h^{n_k}, \text{id}_X) < n \cdot \frac{\varepsilon}{n} = \varepsilon \).

Theorem 3. If \( h : \mathbb{R}^2 \to \mathbb{R}^2 \) is a recurrent homeomorphism of the plane with its usual metric \( d \) onto \( \mathbb{R}^2 \), then \( h \) is periodic.

Proof. Let \( n \) be a positive integer such that \( d(h^n, \text{id}_{\mathbb{R}^2}) < 1 \). By [Bro-1], \( h^n \) is orientation preserving, and by Lemma 2, \( h^n \) is recurrent. Let \( D^0 \) be the open and \( D \) the closed unit ball centered at the origin. Define \( \psi : \mathbb{R}^2 \to D^0 \) by \( \psi(re^{i\theta}) = \frac{r}{1+r}e^{i\theta} \) and \( \varphi' = \psi \circ h^n \circ \psi^{-1} : D^0 \to D^0 \). Since \( d(h^n, \text{id}_{\mathbb{R}^2}) < 1 \), \( \varphi' \) extends to a homeomorphism \( \varphi : D \to D \) such that \( \varphi|_{\partial(D)} = \text{id}_{\partial(D)} \). Since \( d(\varphi^m, \text{id}_D) \leq d(h^{nm}, \text{id}_{\mathbb{R}^2}) \) and \( h^n \) is recurrent, \( \varphi \) is also recurrent. By Theorem 1, \( \varphi = \text{id}_D \). Hence, \( h^n = \text{id}_{\mathbb{R}^2} \).

Remark. Note that the hypothesis of Theorem 3 (i.e., \( h \) is recurrent) is used only to ensure that the induced map \( \varphi \) on the closed unit ball is arc-recurrent and the identity on its boundary. Hence the hypothesis of Theorem 3 can be weakened accordingly.

Added in proof. R. D. Edwards communicated to us that he proved Theorem 3 independently.

References


