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NONEXISTENCE OF 4-DIMENSIONAL ALMOST KAEHLER MANIFOLDS OF CONSTANT CURVATURE

DAVID E. BLAIR

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ABSTRACT. It is shown that in dimension 4 there are no almost Kaehler manifolds of constant curvature unless the constant is 0, in which case the manifold is Kaehlerian. This was previously shown in dimensions ≥ 8 by Z. Olszak and remains open in dimension 6.

1. INTRODUCTION

In 1978 Z. Olszak proved the following result.

Theorem (Olszak [7]). *In dimensions ≥ 8 there are no almost Kaehler manifolds of constant curvature unless the constant is 0, in which case the manifold is Kaehlerian.*

The theorem leaves open the question of whether or not there exist non-Kaehler, almost Kaehler manifolds of constant curvature in dimensions 4 and 6. Here our purpose is to show that none exist in dimension 4.

Theorem. *In dimension 4 there are no almost Kaehler manifolds of constant curvature unless the constant is 0, in which case the manifold is Kaehlerian.*

There are some reasons for trying to answer this question in both dimensions 4 and 6. In [5], O. T. Kassabov proved that if M is an almost Kaehler manifold of dimension ≥ 6, of pointwise constant anti-holomorphic sectional curvature and if its curvature tensor satisfies

$$(1.1)\qquad R(X, Y, Z, U) = R(X, Y, JZ, JU) + R(X, JY, Z, JU) + R(JX, Y, Z, JU),$$

then M is locally isometric to a complex space form or M is 6-dimensional and of constant negative curvature, but he did not give an example of the latter

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phenomenon. Similarly in [6] he showed that a conformally flat almost Kaehler manifold of dimension ≥ 4 satisfying (1.1) is either a flat Kaehler manifold, a 6-dimensional almost Kaehler manifold of constant negative curvature, or locally the product of a 4 or 6-dimensional almost Kaehler manifold of constant negative curvature $-c$ and a 2-dimensional sphere of constant curvature c.

On the other hand we have the following conjecture of S. I. Goldberg.

Conjecture (Goldberg [4]). *A compact almost Kaehler, Einstein space is Kaehlerian.*

For a compact symplectic manifold M, S. Ianus and the author studied [1] the functionals

$$I(g) = \int_M R\, dV_g \quad \text{and} \quad K(g) = \int_M R - R^*\, dV_g$$

on the set $\mathscr{A}$ of all Riemannian metrics associated to the symplectic structure, R being the scalar curvature and R^* the *-scalar curvature (see §2). These metrics are almost Kaehler and the main result of [1] is that a metric $g \in \mathscr{A}$ is a critical point of I, and respectively of K, if and only if the Ricci operator Q of g commutes with the almost complex structure J corresponding to g. It is well known and easy to see that on a Kaehler manifold $QJ = JQ$. Thus this result raises a stronger question than the Goldberg conjecture, viz. is a compact almost Kaehler manifold satisfying $QJ = JQ$, Kaehlerian? Thus we should not overlook a constant curvature example if it exists.

2. Preliminaries and a first observation

Let M^{2n} be an almost Hermitian manifold with structure tensors J and G, i.e., M^{2n} is a C^∞ manifold with almost complex structure J and Riemannian metric G such that $G(JX, JY) = G(X, Y)$. Define the *fundamental 2-form* Ω by $\Omega(X, Y) = G(X, JY)$. (M^{2n}, J, G) is said to be *almost Kaehler* if $d\Omega = 0$, and *Kaehler* if $\nabla J = 0$, ∇ being the Levi–Civita connection of G.

On an almost Hermitian manifold the **-Ricci tensor* is defined by

$$R^*_{ij} = R_{irst} J^{rs} J_j^{\ t}$$

and the **-scalar curvature* by

$$R^* = R^{*i}_i .$$

The relation of R^* to the scalar curvature R on an almost Kaehler manifold is

$$R - R^* = -\tfrac{1}{2}|\nabla J|^2, \tag{2.1}$$

[9, p. 196]. The following lemma is known and essentially given in [9, p. 197]; we also give the proof as it sets the stage for our problem.

Lemma 2.1. *In dimensions $2n \geq 4$, there are no almost Kaehler manifolds of positive constant curvature and a flat almost Kaehler manifold is Kaehlerian*;

thus if a non-Kaehler almost Kaehler manifold of constant curvature exists, the curvature must be negative.

Proof. $R^*_{ij} = c(G_{it}G_{rs} - G_{is}G_{rt})J^{rs}J_j^{\ t} = cG_{ij}$ giving $R^* = 2nc$; but $R = 2n(2n-1)c$ and hence from (2.1) we have $8n(n-1)c = -|\nabla J|^2$ which gives the result.

3. Proof of the theorem

If there exist non-Kaehler almost Kaehler structures of constant curvature, then, since the problem is really a local one, we have, in view of Lemma 2.1, that locally hyperbolic space must carry such a structure. We will construct the most general almost Hermitian structure on 4-dimensional hyperbolic space and study the partial differential equations of the almost Kaehler condition $d\Omega = 0$.

Lemma 3.1. *A* 4×4 *skew-symmetric orthogonal matrix is of the form*

$$\begin{pmatrix} 0 & \alpha & \beta & \gamma \\ -\alpha & 0 & -\gamma & \beta \\ -\beta & \gamma & 0 & -\alpha \\ -\gamma & -\beta & \alpha & 0 \end{pmatrix} \quad or \quad \begin{pmatrix} 0 & \alpha & \beta & \gamma \\ -\alpha & 0 & \gamma & -\beta \\ -\beta & -\gamma & 0 & \alpha \\ -\gamma & \beta & -\alpha & 0 \end{pmatrix}.$$

Proof. Consider a 4×4 skew-symmetric orthogonal matrix

$$\begin{pmatrix} 0 & \alpha & \beta & \gamma \\ -\alpha & 0 & \zeta & \eta \\ -\beta & -\zeta & 0 & \kappa \\ -\gamma & -\eta & -\kappa & 0 \end{pmatrix}.$$

The conditions

$$\begin{aligned} \alpha^2 + \beta^2 + \gamma^2 &= 1, \\ \alpha^2 + \zeta^2 + \eta^2 &= 1, \\ \beta^2 + \zeta^2 + \kappa^2 &= 1, \\ \gamma^2 + \eta^2 + \kappa^2 &= 1 \end{aligned}$$

imply that $\alpha^2 = \kappa^2$, $\beta^2 = \eta^2$, $\gamma^2 = \zeta^2$. So if $\kappa = -\alpha \neq 0$, the orthogonality of the rows gives $\alpha\eta + \beta\kappa = 0$ and $\alpha\gamma - \zeta\kappa = 0$ and hence $\eta = \beta$ and $\zeta = -\gamma$. If $\kappa = \alpha \neq 0$, we similarly have the second matrix. If $\kappa = \alpha = 0$, it is again easy to see that the matrix is of one of the two types.

As a model of 4-dimensional hyperbolic space, we take the Poincaré model consisting of the unit ball B^4 in $\mathbb{R}^4$ and the metric G given by

$$ds^2 = \frac{4}{(1-r^2)^2}(dx^2 + dy^2 + dz^2 + dt^2).$$

Now if G is an almost Hermitian metric, the matrices of G, J, and Ω with respect to the coordinate basis, denoted by the same letters, satisfy

$$GJ = \Omega.$$

Since Ω is skew-symmetric, G proportional to the identity and $J^2 = -I$, J is a skew-symmetric orthogonal matrix.

If J is of the form of the second matrix in Lemma 3.1, the matrix of Ω is then of the form

$$\begin{pmatrix} 0 & h & -g & f \\ -h & 0 & f & g \\ g & -f & 0 & h \\ -f & -g & -h & 0 \end{pmatrix}$$

where f, g, h are functions on B^4 satisfying

$$f^2 + g^2 + h^2 = 1/(1 - r^2)^4.$$

If J is of the form of the first matrix in Lemma 3.1, the results are similar and we comment on what happens in due course.

Recall the coboundary formula for $d\Omega$,

$$d\Omega(X, Y, Z) = \tfrac{1}{3}\{X\Omega(Y, Z) + Y\Omega(Z, X) + Z\Omega(X, Y) - \Omega([X, Y], Z) \\ - \Omega([Z, X], Y) - \Omega([Y, Z], X)\}.$$

For simplicity we write ∂_x for $\partial/\partial x$, etc. and f_x for $\partial f/\partial x$, etc. The almost Kaehler condition $d\Omega = 0$ then becomes

$$\begin{aligned} 0 &= 3d\Omega(\partial_x, \partial_y, \partial_z) = f_x + g_y + h_z, \\ 0 &= 3d\Omega(\partial_x, \partial_y, \partial_t) = g_x - f_y + h_t, \\ 0 &= 3d\Omega(\partial_x, \partial_z, \partial_t) = h_x - f_z - g_t, \\ 0 &= 3d\Omega(\partial_y, \partial_z, \partial_t) = h_y - g_z + f_t. \end{aligned}$$

Thus the proof of the theorem in the four dimensional case is to show that the system of first order partial differential equations

$$\begin{aligned} f_x + g_y + h_z &= 0 \\ g_x - f_y + h_t &= 0 \\ h_x - f_z - g_t &= 0 \\ h_y - g_z + f_t &= 0 \end{aligned}$$

has no solution on B^4 subject to $f^2 + g^2 + h^2 = 1/(1 - r^2)^4$. The system itself, although overdetermined, does have solutions and our method will be to interpret the problem as one in quaternionic analysis and show that there is no solution satisfying the growth condition.

We denote the set of quaternions by $\mathbb{H}$. $\mathbb{H}$ is a 4-dimensional division algebra over $\mathbb{R}$ with basis $1, i, j, k$ where 1 is the identity, $i^2 = j^2 = k^2 = -1$, and $ij = -ji = k$. In terms of this basis we write a quaternion q as $t + ix + jy + kz$. The conjugate $\overline{q}$ of q is defined by $t - ix - jy - kz$ and the modulus $|q|$ of q is defined by $\sqrt{t^2 + x^2 + y^2 + z^2}$ or equivalently $\sqrt{q\overline{q}}$. Conjugation and multiplication of quaternions are related by $\overline{q_1 q_2} = \overline{q}_2 \overline{q}_1$. For

a function F of the quaternionic variable q, we write $F = f_0 + if_1 + jf_2 + kf_3$, f_0 is called the *real part* of F and $if_1 + jf_2 + kf_3$ is called the *pure* or *imaginary part* of F. In [3] R. Feuter introduced left and right $\overline{\partial}$-operators, analogous to $\partial/\partial\overline{z} = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$ in the complex case, to generalize the Cauchy–Riemann equations. In particular let

$$\overline{\partial}_l = \frac{1}{2}\left(\frac{\partial}{\partial t} + i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right),$$
$$\overline{\partial}_r = \frac{1}{2}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right).$$

We then say that a quaternionic function F is *left regular* (resp. *right regular*) if it is differentiable in the real variable sense and $\overline{\partial}_l F = 0$ (resp. $\overline{\partial}_r F = 0$). A quaternionic function F is left regular in q if and only if its conjugate, $\overline{F}$, is right regular in $\overline{q}$. There are many left and right regular functions and much work has been done studying these functions, developing analogues of Cauchy's Theorem, the Cauchy integral formula, Taylor series in terms of special polynomials, etc. (cf. [8] or [2] for a recent treatment of these kinds of questions for functions with values in a Clifford algebra).

A basic property of left regular functions that is easy to prove is that the component functions are harmonic. Given a harmonic function f_0 one can construct $if_1 + jf_2 + kf_3$ such that $f_0 + if_1 + jf_2 + kf_3$ is left regular but the choice of "harmonic conjugate" $if_1 + jf_2 + kf_3$ is not unique to within a constant.

We now describe one of the main results of [8]. Let ν be an unordered set of n integers $\{i_1, \dots, i_n\}$ with $1 \le i_r \le 3$; ν is determined by three integers n_1, n_2, n_3 with $n_1 + n_2 + n_3 = n$, where n_1 is the number of 1's in ν, n_2 the number of 2's and n_3 the number of 3's. There are $\frac{1}{2}(n+1)(n+2)$ such sets ν and we denote the set of all of them by σ_n. Let e_{i_r} and x_{i_r} denote i, j, k and x, y, z according as i_r is $1, 2$, or 3 respectively and

$$P_\nu(q) = \frac{1}{n!}\sum(te_{i_1} - x_{i_1})\cdots(te_{i_n} - x_{i_n})$$

where the sum is over all $n!\,n_1!\,n_2!\,n_3!$ different orderings of n_1 1's, n_2 2's, and n_3 3's; when $n = 0$, so that $\nu = \varnothing$, we take $P_0(q) = 1$. It will be useful to exhibit the second degree P_ν explicitly.

$$\begin{aligned}
P_{11} &= \tfrac{1}{2}(x^2 - t^2) - xti,\\
P_{12} &= xy - tyi - txj,\\
P_{13} &= xz - tzi - txk,\\
P_{22} &= \tfrac{1}{2}(y^2 - t^2) - ytj,\\
P_{23} &= yz - tzj - tyk,\\
P_{33} &= \tfrac{1}{2}(z^2 - t^2) - ztk.
\end{aligned}$$

Theorem (Sudbery [8]). *Suppose F is left regular in a neighborhood of the origin 0. Then there is a ball B with center 0 in which $F(q)$ is represented by a*

uniformly convergent series

$$F(q) = \sum_{n=0}^{\infty} \sum_{\nu \in \sigma_n} P_\nu(q) a_\nu .$$

Turning to our problem, let $F(q) = fi + gj + hk$, where we have identified $q \in \mathbb{H}$ with $(x, y, z, t) \in \mathbb{R}^4$. Then our first order system of partial differential equations is the condition that F be left regular. If J is of the form of the first matrix in Lemma 3.1, the almost Kaehler condition $d\Omega = 0$ leads to an imaginary quaternionic function which is right regular and satisfies the same growth condition, $|F| = 1/(1-r^2)^2$. Thus to complete the proof that there is no 4-dimensional almost Kaehler manifold of constant negative curvature, it suffices to show that there is no left regular quaternionic function defined on a neighborhood of the origin whose modulus is $1/(1-r^2)^2$.

Let F be a left regular quaternionic function defined on a neighborhood of the origin, then by the above theorem of Sudbery

$$F(q) = a_0 + \sum_{i=1}^{3} P_i a_i + \sum_{i \le j} P_{ij} a_{ij} + \cdots .$$

For the conjugate we have

$$\overline{F(q)} = \overline{a}_0 + \sum_{i=1}^{3} \overline{a}_i \overline{P}_i + \sum_{i \le j} \overline{a}_{ij} \overline{P}_{ij} + \cdots .$$

Multiplying these and expanding $|F|^2 = 1/(1-r^2)^4$ we have

$$\begin{aligned} 1 + 4r^2 + \cdots = |a_0|^2 &+ \sum_{i=1}^{3} (P_i a_i \overline{a}_0 + a_0 \overline{a}_i \overline{P}_i) + \sum_{i \le j} (P_{ij} a_{ij} \overline{a}_0 + a_0 \overline{a}_{ij} \overline{P}_{ij}) \\ &+ \sum\nolimits_{i,j} P_i a_i \overline{a}_j \overline{P}_j + \cdots . \end{aligned}$$

Comparing the constant terms we see that $a_0 \neq 0$; then comparing the first degree terms we see that each $a_i = 0$. Now set $b_{ij} = a_{ij}\overline{a}_0 = b_{ij}^0 + b_{ij}^1 i + b_{ij}^2 j + b_{ij}^3 k$. Comparing the second degree terms we must have $2x^2 + 2y^2 + 2z^2 + 2t^2 = \sum_{i \le j} \mathrm{Re}(P_{ij} b_{ij})$, but now comparing the terms in x^2, y^2, and z^2, we see that $b_{11}^0 = b_{22}^0 = b_{33}^0 = 4$ but then $-6t^2 = 2t^2$, a contradiction.

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Department of Mathematics, Michigan State University, East Lansing, Michigan 48824