PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 111, Number 1, January 1991

A COMPARISON THEOREM FOR SELFADJOINT OPERATORS

AMIN BOUMENIR

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. In this work we shall establish a result concerning the spectral theory of differential operators. Let L_1 and L_2 be two self-adjoint operators acting in two different Hilbert spaces. Then under some conditions we shall prove that

$$(d\Gamma_1/d\Gamma_2)(L_2) = \overline{V}V',$$

where $\Gamma_1(\lambda)$ and $\Gamma_2(\lambda)$ are the spectral functions associated with L_1 and L_2 respectively. V is the shift operator mapping the set of generalized eigenfunctions of L_1 into the set of generalized eigenfunctions of L_2, that is

$$y = V\varphi,$$

where $L_2 y = \lambda y$ and $L_1\varphi = \lambda\varphi$.

1. INTRODUCTION

As we shall be manipulating eigenfunctions we need to recall the theory of operators in rigged Hilbert spaces. Let Φ be a nuclear space, N-space, that is a countably normed space $\Phi = \bigcap_{n>1}\Phi_n$, such that, for any p, there exists $n > p$ so that the embedding $\Phi_n \hookrightarrow \Phi_p$ is a Hilbert–Schmidt operator, (see [7]). We recall that an N-space is a perfect space, and so each bounded set is relatively compact.

We now come to some interesting applications of the above idea. Suppose that an operator L is symmetric in a Hilbert space H. Assume that there exists an N-space Φ (perfect) invariant under the operator L and such that H can be obtained as a completion of Φ under the inner product of H. We shall assume that the embedding $\Phi \hookrightarrow H$ is the identity. Since

$$\Phi \xrightarrow{L} \Phi,$$

we have

$$\Phi' \xrightarrow{L^*} \Phi'.$$

Then, using the symmetry of L, i.e., $L \subset L^*$, and the fact that

$$\Phi \hookrightarrow H \hookrightarrow \Phi',$$

Received by the editors October 27, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46XXX.

we obtain that $L^* = L$ on Φ, and L^* is seen as an extension of L to Φ'. Thus we shall agree to denote by L the operator L^*, and so the definition:

Definition 1. We shall say that a linear functional $\varphi \in \Phi'$ is a generalized eigenfunction or eigenfunctional if

$$L\varphi = \lambda\varphi \quad \text{in } \Phi'.$$

Below we recall a well known result, see [5, vol. 3].

Result. *Let L be a symmetric linear operator which is defined on Φ and maps Φ into itself. Assume that Φ is an N-space and that L admits a self-adjoint extension to the Hilbert space H. Then L possesses a complete system of eigenfunctionals in the space Φ'.*

Let L be a self-adjoint operator with simple spectrum acting in a separable Hilbert space H. If $\varphi(\lambda)$ are the eigenfunctionals, that is $L\varphi(\lambda) = \lambda\varphi(\lambda)$ in Φ', then the associated isometry or φ-Fourier transform is given by

$$\hat{f}(\lambda) = \langle f, \varphi(\lambda)\rangle_{\Phi\times\Phi'} \qquad \forall f \in \Phi$$

and the inverse is

$$f = \int \overline{\hat{f}(\lambda)}\varphi(\lambda)\, d\Gamma(\lambda) \in H.$$

$\Gamma(\lambda)$ is a nondecreasing function and is called the spectral function. The Parseval equality reads

$$\forall f, \forall\psi \in \Phi \qquad (f(x), \psi(x))_H = \int \hat{f}(\lambda)\overline{\hat{\psi}(\lambda)}\, d\Gamma(\lambda).$$

Let us agree on some notations. Let L_1 and L_2 be two self-adjoint operators with simple spectrum and acting in two separable Hilbert spaces H_1 and H_2 respectively.

We assume the existence of two (perfect) N-spaces Φ_1 and Φ_2 such that

$$\Phi_i \xrightarrow{L_i} \Phi_i, \qquad i = 1, 2.$$

In all that follows, $\{\Phi_i, H_i, \Phi_i'\}$ and $\Gamma_i(\lambda)$ will denote, respectively, the rigged spaces and the spectral functions associated with the self-adjoint operator L_i, where $i = 1, 2$. Denote by $\varphi(\lambda)$ and $y(\lambda)$ the generalized eigenfunctionals defined by

$$\begin{aligned} L_1\varphi(\lambda) &= \lambda\varphi(\lambda) \quad \text{in } \Phi_1' \\ L_2 y(\lambda) &= \lambda y(\lambda) \quad \text{in } \Phi_2'. \end{aligned} \tag{1.1}$$

σ_i denotes the spectrum of L_i, for $i = 1, 2$. The Fourier transform in this case is given by

$$\begin{aligned} f \in \Phi_1 &\qquad \hat{f}^1(\lambda) \equiv \langle f, \varphi(\lambda)\rangle_{\Phi_1\times\Phi_1'}, \\ \psi \in \Phi_2 &\qquad \hat{\psi}^2(\lambda) \equiv \langle \psi, y(\lambda)\rangle_{\Phi_2\times\Phi_2'}. \end{aligned}$$

In order to compare operators, we shall need to establish a correspondence between the two sets of eigenfunctionals. Assume the existence of a one-to-one mapping between the real sets σ_1 and σ_2, namely,

$$T\colon \sigma_2 \to \sigma_1 .$$

Definition 2. Let $\varphi(\lambda)$ and $y(\lambda)$ be the eigenfunctionals defined by (1.1), and let $T\colon \sigma_2 \to \sigma_1$ be a one-to-one mapping. Then V is said to be a shift operator if

$$V\varphi(T(\lambda)) = y(\lambda) \qquad \forall \lambda \in \sigma_2 . \tag{1.2}$$

Remark. It is clear that the shift operator V is a one-to-one mapping between the sets $\{\varphi(\lambda)\}_{\sigma_1}$ and $\{y(\lambda)\}_{\sigma_2}$. Hence it is defined on $\{\varphi(\lambda)\}_{\sigma_1}$, a subset of Φ_1'. We next extend V to the algebraic span of $\{\varphi(\lambda)\}_{\sigma_1}$. For our immediate use, we shall only need the fact that V is densely defined. Indeed, from the reflexivity of Φ_1 and the completeness of $\{\varphi(\lambda)\}_{\sigma_1}$, the space spanned by $\{\varphi(\lambda)\}_{\sigma_1}$ is dense in Φ_1'. Therefore V is densely defined. This enables us to define the adjoint operator V'. By definition we have

$$\langle \psi , Vf\rangle_{\Phi_2''\times\Phi_2'} = \langle V'\psi , f\rangle_{\Phi_1''\times\Phi_1'} . \tag{1.3}$$

Since the spaces are reflexive,

$$\langle \psi , Vf\rangle_{\Phi_2\times\Phi_2'} = \langle V'\psi , f\rangle_{\Phi_1\times\Phi_1'} . \tag{1.4}$$

The domain of V' is defined by

$$D_{V'} \equiv \{\psi \in \Phi_2 | f \to \langle \psi , Vf\rangle \text{ is continuous}\}.$$

However there is a simple connection between $D_{V'}$ and Φ_2, indeed, we have the well-known result that, for example see [6, Chapter 2],

$$V \textit{ admits closure} \Leftrightarrow D_{V'} \textit{ is dense in } \Phi_2 . \tag{1.5}$$

2. The factorization theorem

Let us start with some notations. We know that V' acts between Φ_2 and Φ_1, which are imbedded in H_2 and H_1 respectively. Hence V' has a natural extension as an operator from H_2 into H_1, which we shall denote by $\widetilde{V}'$. Thus

$$H_2 \xrightarrow{\widetilde{V}'} H_1$$

and, for $f \in D_{V'}$, $\widetilde{V}'f = V'f$ in H_1.

Define an operator G in Φ_2 by

$$Gf = \int \overline{\hat{f}^2(\lambda)}y(\lambda)\, d\Gamma_1(T(\lambda)),$$

where $\Gamma_1(\lambda)$ is the spectral function of L_1, and $T(\cdot)$ is the mapping used in the definition of the shift operator. The domain of G is $D_G = \{f \in \Phi_2 | Gf \in \Phi_2'\}$. Clearly if $\hat{f}^2(\lambda)$ has a compact support then Gf is defined. It is possible to represent G if we knew the behaviour of $\Gamma_1(T(\lambda))$.

Theorem 3. *Let L_1 and L_2 be two self-adjoint operators with simple spectrum, acting in the rigged Hilbert spaces $\Phi_1 \hookrightarrow H_1 \hookrightarrow \Phi_1'$ and $\Phi_2 \hookrightarrow H_2 \hookrightarrow \Phi_2'$ respectively. If the shift operator V admits closure, then*

$$G = \overline{V}V',$$

where $\overline{V}$ denotes the closure of V.

Proof. Let us give the diagram of the operator V

$$\begin{array}{ccccc} \Phi_1 & \hookrightarrow & H_1 & \hookrightarrow & \Phi_1' \\ {\scriptstyle V'} \uparrow & & & & \downarrow {\scriptstyle V}, \\ \Phi_2 & \hookrightarrow & H_2 & \hookrightarrow & \Phi_2' \end{array}$$

where $V\phi(T(\lambda)) = y(\lambda)$.

Let f and ψ be two arbitrary elements of $D_{V'}$.

$$(2.1)\qquad \begin{aligned} \hat{f}^2(\lambda) &= \langle f, y(\lambda)\rangle_{\Phi_2\times\Phi_2'} \\ &= \langle f, V\varphi(T(\lambda))\rangle_{\Phi_2\times\Phi_2'} = \langle V'f, \varphi(T(\lambda))\rangle_{\Phi_1\times\Phi_1'} = \widehat{V'f}^1(T(\lambda)) \end{aligned}$$

and, similarly,

$$(2.2)\qquad \hat{\psi}^2(\lambda) = \widehat{V'\psi}^1(T(\lambda)).$$

Observing that the right-hand side of equations (2.1), (2.2) are the φ-Fourier transform of $V'f$ and $V'\psi$, respectively, we obtain by using the Parseval equality,

$$(2.3)\qquad \begin{aligned} &(V'f, V'\psi)_{H_1} = \int \widehat{V'f}^1\overline{\widehat{V'\psi}^1(\lambda)}\,d\Gamma_1(\lambda). \\ &\int \widehat{V'f}^1(T(\lambda))\overline{\widehat{V'\psi}^1(T(\lambda))}\,d\Gamma_1(T(\lambda)) = \int \hat{f}^2(\lambda)\overline{\hat{\psi}^2(\lambda)}\,d\Gamma_1(T(\lambda)). \end{aligned}$$

Clearly,

$$\begin{aligned} \langle f, G\psi\rangle_{\Phi_2\times\Phi_2'} &= \left\langle f, \int \overline{\hat{\psi}^2(\lambda)}y(\lambda)\,d\Gamma_1(T(\lambda))\right\rangle_{\Phi_2\times\Phi_2'} \\ &= \int \overline{\hat{\psi}^2(\lambda)}\hat{f}^2(\lambda)\,d\Gamma_1(T(\lambda)). \end{aligned}$$

Hence $(V'f, V'\psi)_{H_1} = \langle f, G\psi\rangle_{\Phi_2\times\Phi_2'}$ which implies that $D_{V'} \subset D_G$. Since the imbedding $\Phi_1 \hookrightarrow H_1 \hookrightarrow \Phi_1'$ is the identity,

$$(V'f, V'\psi)_{H_1} = \langle V'f, V'\psi\rangle_{\Phi_1\times\Phi_1'}.$$

Therefore

$$\langle f, G\psi\rangle_{\Phi_2\times\Phi_2'} = \langle V'f, V'\psi\rangle_{\Phi_1\times\Phi_1'},$$

and, since V admits closure, we have

$$\langle V'f, V'\psi\rangle_{\Phi_1\times\Phi_1'} = \langle f, \overline{V}V'\psi\rangle_{\Phi_2\times\Phi_2'}$$

and so $\langle f, G\psi\rangle_{\Phi_2\times\Phi_2'} = \langle f, \overline{V}V'\psi\rangle_{\Phi_2\times\Phi_2'}$. Therefore for $\psi \in D_{V'}$, $G\psi = \overline{V}V'\psi$ in Φ_2', and so $D_G = D_{\overline{V}V'}$. □

Remark. We had to use the fact that V admitted closure. We shall see that we do not need such an assumption if we took $\widetilde{V}'$ instead of V'. For that define

$$\widetilde{G}\colon H_2 \to H_2$$
$$\widetilde{G}f \equiv \int \overline{\hat{f}^2(\lambda)}y(\lambda)\, d\Gamma_1(T(\lambda)),$$

$D_{\widetilde{G}} = \{f \in H_2 | \widetilde{G}f \in H_2\}$. Here the Fourier transform is extended to H_2 by taking the closure of the Fourier transform to H_2. It is clear that $D_{\widetilde{G}}$ is dense in H_2. To see that, take the dense set of smooth compactly supported functions in $L^2_{\Gamma_2(\lambda)}$. Then, using the inverse y-Fourier transform, we shall obtain a dense set in H_2, which is also contained in $D_{\widetilde{G}}$. Hence $D_{\widetilde{G}}$ is dense in H_2, and so $\widetilde{G}$ is densely defined. From (2.3),

$$(V'f, V'\psi)_{H_1} = \int \hat{f}^2(\lambda)\overline{\hat{\psi}^2(\lambda)}\, d\Gamma_1(T(\lambda)).$$

Since f and ψ are also in H_2, $(V'f, V'\psi)_{H_1} = (\widetilde{V}'f, \widetilde{V}'\psi)_{H_1}$ and

$$\int \hat{f}^2(\lambda)\overline{\hat{\psi}^2(\lambda)}\, d\Gamma_1(T(\lambda)) = (f, \widetilde{G}\psi)_{H_2}.$$

Therefore

$$(2.4) \qquad (\widetilde{V}'f, \widetilde{V}'\psi)_{H_1} = (f, \widetilde{G}\psi)_{H_2}.$$

It is readily seen that $\widetilde{V}'$ is densely defined. Indeed, since

$$\|\widetilde{V}'f\|^2 = \int |\hat{f}^2(\lambda)|^2\, d\Gamma_1(T(\lambda)),$$

the argument used for $D_{\widetilde{G}}$ will go through. $\widetilde{V}'$ densely defined means that the adjoint operator is well defined and, by (2.4), we deduce that

$$(f, [\widetilde{V}']'\widetilde{V}'\psi)_{H_2} = (f, \widetilde{G}\psi)_{H_2}.$$

Therefore $\widetilde{G} = [\widetilde{V}']'\widetilde{V}'$, and we have just proved

Theorem 4. *Let L_1 and L_2 be two self-adjoint operators with simple spectrum, acting in the rigged Hilbert spaces $\Phi_1 \hookrightarrow H_1 \hookrightarrow \Phi_1'$ and $\Phi_2 \hookrightarrow H_2 \hookrightarrow \Phi_2'$, respectively. Then*

$$\widetilde{G} = [\widetilde{V}']'\widetilde{V}'.$$

There exists a particular case where it is possible to obtain a simple representation of the operator G. For that we need

Definition 5. $\Gamma_1(T(\lambda))$ is said to be absolutely continuous with respect to $\Gamma_2(\lambda)$ (denoted by $\mathrm{ABS}-d\Gamma_2(\lambda)$) if there exists a $d\Gamma_2(\lambda)$-summable function $g(\lambda)$ such that

$$\Gamma_1(T(\lambda)) = \int_{-\infty}^{\lambda} g(\lambda)\, d\Gamma_2(\lambda).$$

(Notation: $g(\lambda) = (d\Gamma_1(T)/d\Gamma_2)(\lambda)$.

Let $g(L_2)$ be the operator defined by

$$\begin{array}{ccc} \Phi_2 & \overset{g(L_2)}{\rightarrow} & H_2 \\ f(x) & \rightarrow & g(L_2)f(x) \\ \hat{2}\downarrow & & \uparrow \hat{2}^{-1} \\ \hat{f}^2(\lambda) & \rightarrow & g(\lambda)\hat{f}^2(\lambda) \end{array}$$

or $g(L_2)f(x) = \int g(\lambda)\overline{\hat{f}^2(\lambda)}y(\lambda)\, d\Gamma_2(\lambda)$. Its domain is given by

$$D_{g(L_2)} = \{f \in \Phi_2 | g(\lambda)\hat{f}^2(\lambda) \in L^2_{\Gamma_2}\}.$$

Therefore $Gf = I_{\Phi_2'} g(L_2)f$, for any $f \in D_{g(L_2)}$, and where $I_{\Phi_2'}$ is the imbedding from $H_2 \hookrightarrow \Phi_2'$, the identity. So we can write, for any $f \in D_{g(L_2)}$,

$$Gf = g(L_2)f \quad \text{in } \Phi_2'.$$

In this way G is an extension of $g(L_2)$ to Φ_2'. We shall agree to write $G \equiv g(L_2)$ in Φ_2'. Thus

Corollary 6. *Let L_1 and L_2 be two self-adjoint operators with simple spectrum, acting in the rigged Hilbert spaces $\Phi_1 \hookrightarrow H_1 \hookrightarrow \Phi_1'$ and $\Phi_2 \hookrightarrow H_2 \hookrightarrow \Phi_2'$, respectively. Assume that V admits closure. If the function $\Gamma_1(T(\lambda))$ is ABS-$d\Gamma_2(\lambda)$, i.e., $d\Gamma_1(T(\lambda)) = g(\lambda)\, d\Gamma_2(\lambda)$. Then, for any $f \in D_{g(L_2)}$,*

$$g(L_2)f = \overline{V}V'f \quad \textit{in } \Phi_2',$$

where $\overline{V}$ denotes the closure of V.

If $\tilde{g}(L_2)$ denotes the extension of $g(L_2)$ to $H_2 \to H_2$, then $\tilde{g}(L_2) = \widetilde{G}$ in H_2, and so from Theorem 4,

Corollary 7. *Let L_1 and L_2 be two self-adjoint operators with simple spectrum, acting in the rigged Hilbert spaces $\Phi_1 \hookrightarrow H_1 \hookrightarrow \Phi_1'$ and $\Phi_2 \hookrightarrow H_2 \hookrightarrow \Phi_2'$, respectively. If the function $\Gamma_1(T(\lambda))$ is ABS-$d\Gamma_2(\lambda)$, i.e., $d\Gamma_1(T(\lambda)) = g(\lambda)\, d\Gamma_2(\lambda)$. Then*

$$\tilde{g}(L_2) = [\widetilde{V}']'\widetilde{V}'.$$

We have defined $g(L_2)$ through the Fourier transform and claimed that it was the usual function of the operator L_2. Let us briefly show that the two definitions coincide. If E_λ is the spectral family associated with L_2, then since the operator L_2 has a simple spectrum,

$$dE_\lambda f = \overline{\hat{f}^2(\lambda)} y(x, \lambda)\, d\Gamma_2(\lambda).$$

Therefore

$$g(L_2)f \equiv \int g(\lambda)\, dE_\lambda f = \int g(\lambda) \overline{\hat{f}^2(\lambda)} y(x, \lambda)\, d\Gamma_2(\lambda),$$

and so the two definitions are in fact identical.

3. General results

We have shown that $g(L_2) = [\widetilde{V}']'\widetilde{V}'$ on $D_{V'}$. Clearly the boundedness of $g(L_2)$, which depends on the behaviour of $g(\lambda)$, must be related to the boundedness of $\widetilde{V}'$.

Theorem 8. *Assume that conditions in Corollary 7 hold; then $\widetilde{V}'$ is a bounded operator $H_2 \to H_1$ if and only if $\sqrt{g(\lambda)}$ is bounded $d\Gamma_2(\lambda)$ a.e.*

Proof. Assume that $\sqrt{g(\lambda)}$ is bounded $d\Gamma_2(\lambda)$ a.e. Then there exists M such that $|\sqrt{g(\lambda)}| \le M$ $d\Gamma_2(\lambda)$ a.e. From (2.4) we obtain

$$\|\widetilde{V}' f\|_{H_1} = \|\sqrt{g(\lambda)} \hat{f}^2(\lambda)\|_{d\Gamma_2} \quad \text{for } f \in \Phi_2 \hookrightarrow H_2,$$

but $\|\sqrt{g(\lambda)}\hat{f}^2(\lambda)\|_{d\Gamma_2} \le M\|\hat{f}^2(\lambda)\|_{d\Gamma_2} \le M\|f\|_{H_2}$. Hence $\|\widetilde{V' f}\|_{H_1} \le M\|f\|_{H_2}$, which shows that $\widetilde{V}'$ is a bounded operator from H_2 to H_1. Conversely, if $\widetilde{V}'$ is bounded then, for any $f \in \Phi_2$, we do have

$$\|\sqrt{g(\lambda)}\hat{f}^2(\lambda)\|_{d\Gamma_2} = \|\widetilde{V}' f\|_{H_1} \le M\|f\|_{H_2} \le M\|\hat{f}^2(\lambda)\|_{d\Gamma_2}.$$

From the above inequality it is readily seen that $\sqrt{g(\lambda)}$ is $d\Gamma_2$ bounded. □

Theorem 9. $\widetilde{V}'$: *$H_2 \to H_1$ is invertible if and only if*

$$\int_{k_q} d\Gamma_2(\lambda) = 0,$$

where $k_g \equiv \{\lambda | g(\lambda) = 0\}$ and

$$\|\widetilde{V}'\| = \operatorname*{ess\,sup}_{\lambda \in \sigma_2} \sqrt{g(\lambda)}.$$

From (2.4) we have that $\|\widetilde{V' f}\|_{H_1} = \|\sqrt{g(\lambda)}\hat{f}^2(\lambda)\|_{d\Gamma_2}$. So the operator $\widetilde{V}$: $H_2 \to H_1$ is invertible if and only if $\sqrt{g(L_2)}$ is invertible. Thus we should have

$$\sqrt{g(L_2)f} = 0 \Rightarrow f = 0.$$

Using the Fourier transform,

$$\sqrt{g(\lambda)}\hat{f}^2(\lambda) = 0 \Rightarrow \hat{f}^2(\lambda) = 0 \quad d\Gamma_2(\lambda) \text{ a.e.} \tag{3.1}$$

As $\|\hat{f}^2\|$ depends on the support of Γ_2, (3.1) will have to be verified only on the support of Γ_2, or, in other words, on σ_2. Thus (3.1) means that there is no set of nonzero measure, where $g(\lambda)$ vanishes. □

Suppose that we need to find L_1 from its spectral function Γ_1, i.e., the inverse spectral problem. Let L_2 be given with its spectral function Γ_2, and form the operator G. If we can solve $G = \overline{V}V'$, then we claim that the inverse spectral problem is solved. Indeed, if we regard the eigenfunctions as a basis for the differential operator then the result is immediate,

$$L_2 y = \lambda y \qquad \forall \lambda \in \sigma_2.$$

So, by using the shift operator,

$$\begin{gathered} y = V\varphi \quad \text{or} \quad \varphi = V^{-1} y \\ L_2 V\varphi = \lambda V\varphi, \end{gathered}$$

and clearly

$$V^{-1} L_2 V\varphi = \lambda\varphi$$

so that

$$L_1 = V^{-1} L_2 V \quad \text{in } \Phi_1', \tag{3.2}$$

which is exactly the formula for the change of basis. From (3.2) we can see that we can recover L_1 if V^{-1} exists.

4. The second factorization

Notice that the function $g(\lambda)$ in Definition 5 might not exist. In this section we shall give another way of relating the spectral functions. Let T be a non-decreasing one-to-one one mapping between σ_1 and σ_2. As usual the shift operator is defined by $y(\lambda) = V\varphi(T(\lambda))$. Since $\Gamma_1(T(\lambda))$ and $\Gamma_2(\lambda)$ are non-decreasing functions we can assume the existence of an increasing function $s(\lambda)$ such that

$$\Gamma_1(T(s(\lambda))) = \Gamma_2(\lambda).$$

With the help of $s(\lambda)$ we can define the following operator:

$$\begin{array}{ccc} \Phi_2 & \xrightarrow{A_s} & H_2 \\ f & \rightarrow & A_s(f) \\ \hat{}\,2 \downarrow & & \uparrow \hat{}\,2^{-1} \\ \hat{f}^2(\lambda) & \rightarrow & \hat{f}^2(s(\lambda)) \end{array}$$

The domain of A_s is $D_{A_s} = \{f \in \Phi_2 | \hat{f}^2(s(\lambda)) \in L^2_{\Gamma_2}\}$. Clearly $A_s = \hat{}\,2^{-1} s\circ \hat{}\,2$, where $s\circ$ denotes the composition with the function $s(\lambda)$.

Denote by $\widetilde{A}_s$ the closure of A_s in H_2.

Theorem 10. *Let L_1 and L_2 be two self-adjoint operators having their spectral functions such that $\Gamma_1(T(s(\lambda))) = \Gamma_2(\lambda)$. Then*

$$[\widetilde{V}']'\widetilde{V}' = \widetilde{A}'_{Ts}\widetilde{A}_{Ts}. \tag{4.1}$$

Proof. Let f and ψ be two elements of $D_{\widetilde{V}'}$. As usual we shall work with the Fourier transform. By (2.3),

$$\begin{aligned}(\widetilde{V}'f, \widetilde{V}'\psi)_{H_1} &= \int \hat{f}^2(\lambda)\overline{\hat{\psi}^2(\lambda)}\, d\Gamma_1(T(\lambda))\\ &= \int \hat{f}^2(T(s(\lambda)))\overline{\hat{\psi}^2(T(s(\lambda)))}\, d\Gamma_1(T(s(\lambda)))\\ &= \int \widehat{A_{Ts}f}^2(\lambda)\overline{\widehat{A_{Ts}\psi}^2(\lambda)}\, d\Gamma_2(\lambda) = (\widetilde{A_{Ts}}f, \widetilde{A_{Ts}}\psi)_{H_2}.\end{aligned} \tag{4.2}$$

From (4.2), we deduce that $D_{\widetilde{A}_{Ts}} = D_{V'}$ and $\widetilde{A}_{Ts}$ is densely defined in H_2, so

$$(f, [\widetilde{V}']'\widetilde{V}'\psi)_{H_2} = (f, \widetilde{A}'_{Ts}\widetilde{A}_{Ts}\psi(x))_{H_2}.$$

Hence

$$[\widetilde{V}']'\widetilde{V}' = \widetilde{A}'_{Ts}\widetilde{A}_{Ts}. \quad \square$$

Let us illustrate the next idea by an example. Let $s(\lambda)$ be an increasing function and define

$$L_1 \equiv s(L_2). \tag{4.3}$$

It is clear that $L_1\varphi(\lambda) = \lambda\varphi(\lambda)$, where $\varphi(\lambda) = y(s^{-1}(\lambda))$. Indeed

$$s(L_2)y(s^{-1}(\lambda)) = s(s^{-1}(\lambda))y(s^{-1}(\lambda)) = \lambda y(s^{-1}(\lambda)).$$

Therefore

$$Vy(s^{-1}(\lambda)) = y(\lambda).$$

For any $f \in D_{V'}$ we have

$$\begin{aligned}\langle f, y(\lambda)\rangle_{\Phi_2\times\Phi_2'} &= \langle f, Vy(s^{-1}(\lambda))\rangle_{\Phi_2\times\Phi_2'}\\ &= \langle V'f, y(s^{-1}(\lambda))\rangle_{\Phi_1\times\Phi_1'},\end{aligned}$$

$$\hat{f}^2(\lambda) = \widehat{V'f}^2(s^{-1}(\lambda)),$$

or

$$\hat{f}^2(s(\lambda)) = \widehat{V'f}^2(\lambda),$$

and so, taking the inverse y-Fourier transform, for any $f \in D_{V'}$, $V'f = \widetilde{A}_s f$ holds in H_2. In other words

$$\widetilde{V}'f = \widetilde{A}_s f.$$

The spectral functions are

$$f = \int \overline{\hat{f}^1(\lambda)}\varphi(\lambda)\, d\Gamma_1(\lambda) = \int \overline{\hat{f}^2(s^{-1}(\lambda))} y(s^{-1}(\lambda))\, d\Gamma_1(\lambda)$$
$$= \int \overline{\hat{f}^2(\lambda)} y(\lambda)\, d\Gamma_1(s(\lambda)).$$

On the other hand,

$$f = \int \overline{\hat{f}^2(\lambda)} y(\lambda)\, d\Gamma_2(\lambda).$$

Therefore

$$d\Gamma_1(s(\lambda)) = d\Gamma_2(\lambda). \quad \square$$

Theorem 11. *Let L_2 be a self-adjoint operator acting in the rigged Hilbert space $\Phi_2 \hookrightarrow H_2 \hookrightarrow \Phi_2'$. If $L_1 = s(L_2)$, where s is an increasing function, then*

$$\Gamma_1(s(\lambda)) = \Gamma_2(\lambda) \quad \textit{for any } f \in D_{\widetilde{V}}, \qquad \widetilde{V}' f = \widetilde{A}_s f.$$

We now discuss the case where we are given two spectral functions, $\Gamma_1(\lambda)$ and $\Gamma_2(\lambda)$ such that $\Gamma_1(s(\lambda)) = \Gamma_2(\lambda)$, where $s(\lambda)$ is a one-to-one mapping $\sigma_2 \to \sigma_1$. The operators are not supposed to commute, and so they are not functions of each other.

$\Gamma_1(s(\lambda))$ is ABS-$d\Gamma_2(\lambda)$ since $g(\lambda) = (d\Gamma_1(s)/d\Gamma_2)(\lambda) = 1$, and so $\tilde{g}(L_2) =$ Id on $D_{g(L_2)} \subset H_2 \to H_2$. The shift operator is given by

$$V\varphi(s(\lambda)) = y(\lambda) \quad \text{for } \lambda \in \sigma_2.$$

From Corollary 4 we deduce that $\tilde{g}(L_2) = [\widetilde{V}']'\widetilde{V}'$ or that $\mathrm{Id} = [\widetilde{V}']'\widetilde{V}'$. Hence $\widetilde{V}'$ is a unitary operator, in fact

$$(\widetilde{V}' f, \widetilde{V}'\psi)_{H_1} = (f, \psi)_{H_2}.$$

We can decompose V into two shifts:

$$\varphi(s(\lambda)) \xrightarrow{R} \varphi(\lambda) \xrightarrow{W} y(\lambda),$$

so $V = W \cdot R$. By definition $R\varphi(s(\lambda)) = \varphi(\lambda)$ or $R\varphi(\lambda) = \varphi(s^{-1}(\lambda))$. So, by Theorem 11, if $\widetilde{R}'$ is the closure of R' in H_1, $\hat{f}^1(s^{-1}(\lambda)) = \widehat{R'f}^1(\lambda) \Rightarrow \widetilde{A}_{s-1} = \widetilde{R}'$. Therefore

$$\mathrm{Id} = [\widetilde{V}']'\widetilde{V}' = \widetilde{W}''\widetilde{R}''\widetilde{R}'\widetilde{W}' = [\widetilde{W}']'(\widetilde{A}_s - 1)'(\widetilde{A}_s - 1)\widetilde{W}'.$$

5. Examples

Example. As an example we shall show how to apply the above ideas and obtain an extension of the Gelfand Levitan theory to the generalized second order differential operator $L_2 \equiv -d^2/w(x)\,dx^2$, where $w(x) \geq 0$. Suppose we are given two second order differential operators such that Theorem 4 holds:

$$L_1 = L_2 + q(x)$$

where

$$L_2 = -\frac{d^2}{w(x)\,dx^2},$$

$w(x) \geq 0$, and the boundary conditions are included in the definition of the operators. Notice that the operators L_1 and L_2 act in the same space $L^2_{w(x)\,dx}(0, \infty)$. In what concerns the rigged Hilbert space structure we refer to the construction done by Aleksandrjian. Or, since the Fourier transform is defined, one can simply take $\Phi \equiv \hat{S}^{-1}$ where S is a space of rapidly decreasing functions which is nuclear and invariant by the multiplication by λ. In this way Φ is also an N-space. The shift operator is given by

$$V = 1 + H, \tag{5.1}$$

where

$$1(f) = f$$

and

$$H(f) \equiv \int_0^x H(x, t) f(t) w(t)\,dt.$$

The relation between the functions $H(x, t)$ and $q(x)$ is as follows: we have shown in (3.2) that

$$L_2 V = V L_1 \quad \text{in } \Phi_2'$$

or

$$L_2(1 + H) = (1 + H)(L_2 + q).$$

So $L_2 H - H L_2 = q + Hq$, which is a hyperbolic equation, and $G = VV'$ or, in other words,

$$G = 1 + H + H' + HH',$$

which means that

$$\{G - 1\} f(x) = \{H + H' + HH'\} f(x) \tag{5.2}$$

where $f(x)$ is a smooth function with compact support. Now notice that the term on the left-hand side of (5.2) is nothing other than

$$[G - 1] f(x) = \int \hat{f}^2(\lambda) y(x, \lambda)\, d[\Gamma_1(\lambda) - \Gamma_2(\lambda)]. \tag{5.3}$$

Now, using the expression of the Fourier transform, we have

$$\hat{f}^2(\lambda) = \int f(t) y(t, \lambda) w(t)\,dt.$$

Assuming that

$$P(x, t) \equiv \int y(t, \lambda) y(x, \lambda) d[\Gamma_1 - \Gamma_2](\lambda)$$

is a continuous function of t and x and, using the fact that $f(x)$ is of compact support, we obtain, by applying Fubini's theorem,

$$(5.4)\qquad \begin{aligned}[G-1]f(x) &= \int f(t)\left[\int y(t,\lambda)y(x,\lambda)\,d[\Gamma_1-\Gamma_2](\lambda)\right] w(t)\,dt \\ &= \int f(t)P(t,x)w(t)\,dt.\end{aligned}$$

We can now express the right-hand side of (5.2):

(5.5)

$$\begin{aligned}[H+H'+HH']f(x) = &\int_0^x H(x,t)f(t)w(t)\,dt + \int_x^\infty H(t,x)f(t)w(t)\,dt \\ &+ \int_0^x H(x,s)\int_s^\infty H(s,t)f(t)w(t)\,dt\,w(s)\,ds.\end{aligned}$$

The last term can be written as

$$\begin{aligned}&\int_0^x\int_0^x H(x,s)H(s,t)w(s)\,ds\,f(t)w(t)\,dt \\ &\quad+ \int_x^\infty\int_0^x H(x,s)H(s,t)w(s)\,ds\,f(t)w(t)\,dt,\end{aligned}$$

and so in the weak sense we do have the result

$$(5.6)\qquad P(t,x) = H(x,t) + \int_0^x H(x,s)H(s,t)w(t)\,dt, \qquad t<x.$$

Suppose that V^{-1} exists and $V^{-1}\equiv 1+K$. Then we do have

$$\begin{gathered}V^{-1}G = V' \\ (1+K)G = 1+H^* \\ (1+K)Gf(x) = (1+H^*)f(x)\end{gathered}$$

$$(5.7)\qquad Gf(x) + \int_0^x K(x,t)Gf(t)w(t)\,dt = f(x) + \int_x^\infty H(t,x)f(t)w(t)\,dt$$

or

$$\begin{aligned}&[G-1]f(x) + \int_0^x K(x,t)f(t)w(t)\,dt + \int_0^x K(x,t)[G-1]f(t)w(t)\,dt \\ &\quad= \int_x^\infty H(t,x)f(t)w(t)\,dt.\end{aligned}$$

But $[G-1]f(x) = \int P(t,x)f(t)w(t)\,dt$, so we have

(5.8)

$$\begin{aligned}&\int P(t,x)f(t)w(t)\,dt + \int_0^x K(x,t)f(t)w(t)\,dt \\ &\qquad+ \int\int_0^x K(x,s)P(s,t)w(s)\,ds\,f(t)w(t)\,dt = \int_x^\infty H(t,x)f(t)w(t)\,dt\end{aligned}$$

Hence,

$$(5.9)\qquad P(t,x) + K(x,t) + \int_0^x K(x,s)P(s,t)w(s)\,ds = H(x,t).$$

Observe that $H(t, x) = 0$ if $t < x$; hence

$$P(t, x) + K(x, t) + \int_0^x K(x, s)P(s, t)w(s)\,ds = 0 \tag{5.10}$$

where $0 \le t < x$.

Example 2. Everitt and Zettl computed the Weyl–Titchmarsh function, associated with the operator

$$L_2 f \equiv \frac{-1}{x^\alpha} f''(x), \qquad x \in [0, \infty).$$

They proved that the spectral function associated with

$$L_2 f \equiv \frac{-1}{x^\alpha}\frac{d^2}{dx^2} f(x), \qquad x \ge 0,$$
$$f'(0) = 0$$

is of the form

$$\Gamma_2(\lambda) = \begin{cases} c \cdot \lambda^{(\alpha+1)/(\alpha+2)} & \text{for } \lambda \ge 0 \\ 0 & \text{for } \lambda < 0. \end{cases}$$

For further details see [2].

Let us prove the same result but using our method. We shall define another operator L_1 and then obtain the shift operator V. Having V we shall use $(d\Gamma_1/d\Gamma_2)(L_2) = VV'$ to obtain an equation for $\Gamma_1(\lambda)$.

Denote by $y(x, \lambda)$ the eigenfunctions of $L_2 y(x, \lambda) = \lambda y(x, \lambda)$

$$\begin{aligned} &y''(x, \lambda) + \lambda x^\alpha y(x, \lambda) = 0 \\ &y(x, \lambda) = 1 \quad \text{and} \quad y'(x, \lambda) = 0. \end{aligned} \tag{5.11}$$

Solutions of (5.11) can be written in terms of the Bessel functions $(\lambda = \mu^2)$:

$$y(x, \mu^2) = Ar(x)\sqrt{t(x)\mu}\,J_\nu(t(x)\mu) + Br(x)\sqrt{t(x)\mu}\,J_{-\nu}(t(x)\mu),$$

where $r(x) = x^{-\alpha/4}$, $t(x) = 2\nu x^{1/2\nu}$ and $\nu = 1/(\alpha+2)$. A and B are determined from the boundary conditions

$$y(0, \lambda) = 1 \quad \text{and} \quad y'(0, \lambda) = 0.$$

So

$$A = 0 \text{ and } B = \mu^{\nu - 1/2} \cdot C(\nu),$$

where $C(\nu)$ is a function of ν only.

Define an operator

$$\begin{array}{ccc} L^2 & \xrightarrow{T} & L^2_{x^\alpha dx} \\ f & \to & Tf(x) = r(x) f(t(x)). \end{array}$$

We do have $TT' = \mathrm{Id}$. Since $T'g(t) = x(t)^{\alpha/4} g(x(t))$, where $x(t)$ is the inverse function of $t(x)$. Thus $y(x, \mu^2) = \mu^{\nu-1/2} \cdot C(\nu) \cdot T(\sqrt{t\mu}\,J_{-\nu}(t\mu))$.

Now it is time to find the operator L_1. If

$$\varphi(t, \mu^2) = \sqrt{t\mu} J_{-\nu}(t\mu)$$

are the eigenfunctionals of L_1, then, from the inversion formula of the Bessel functions,

$$F(\lambda) = \int_0^\infty f(t)\sqrt{t\sqrt{\lambda}} J_{-\nu}(t\sqrt{\lambda})\, dt$$
$$f(t) = \int_0^\infty F(\lambda)\sqrt{t\sqrt{\lambda}} J_{-\nu}(t\sqrt{\lambda})\, d\sqrt{\lambda}.$$

We deduce that $d\Gamma_1(\lambda) = d\sqrt{\lambda}$.

The shift operator is given by

$$y(x, \mu^2) = \mu^{\nu-1/2} \cdot C(\nu) \cdot T[\varphi(x, \mu^2)],$$

since

$$y(x, \lambda) = \lambda^{(\nu-1/2)/2} \cdot C(\nu) \cdot T[\varphi(x, \lambda)].$$

Using the fact that $L_2 y = \lambda y$ we have

$$\begin{aligned} L_2^{(\nu-1/2)/2} y &= \lambda^{(\nu-1/2)/2} y \\ &= \lambda^{(\nu-1/2)/2} \cdot C(\nu) \cdot L_2^{(\nu-1/2)/2} = T[\varphi] \end{aligned}$$

Simplifying by $\lambda^{(\nu-1/2)/2}$,

$$y(x, \lambda) = C(\nu) \cdot L_2^{(\nu-1/2)/2} \cdot T[\varphi].$$

Recall that

$$y \equiv V\varphi,$$

hence

$$V[\cdot] = C(\nu) \cdot L_2^{(\nu-1/2)/2} \cdot T[\cdot].$$

From Theorem 4 we obtain that

$$\frac{d\Gamma_1}{d\Gamma_2}(L_2) = VV' = C(\nu)^2 \cdot L_2^{(\nu-1/2)/2} \cdot T \cdot T' L_2^{(\nu-1/2)/2}.$$

Since $TT' = 1$

$$\frac{d\Gamma_1}{d\Gamma_2}(L_2) = C(\nu)^2 \cdot L_2^{(\nu-1/2)}.$$

Hence

$$\frac{d\Gamma_1}{d\Gamma_2}(\lambda) = C(\nu)^2 \cdot \lambda^{(\nu-1/2)}.$$

Let us solve the above differential equation:

$$\begin{aligned} d\Gamma_2(\lambda) &= \frac{1}{C(\nu)^2}\lambda^{1/2-\nu}\, d\Gamma_1(\lambda) \\ &= \frac{1}{C(\nu)^2}\lambda^{1/2-\nu}\, d\sqrt{\lambda} \\ \Gamma_2(\lambda) &= \frac{1}{C(\nu)^2}\int_0^{\sqrt{\lambda}} s^{2(1/2-\nu)}\, ds \\ &= \frac{1}{2-2\nu}\cdot\frac{1}{C(\nu)^2}\int_0^{\sqrt{\lambda}} ds^{2-2\nu} \\ \Gamma_2(\lambda) &= \frac{1}{2-2\nu}\cdot\frac{1}{C(\nu)^2}\lambda^{1-\nu}. \end{aligned}$$

So

$$\Gamma_2(\lambda) = c\lambda^{1-1/(\alpha+2)},$$

where c is a constant. That is what Everitt and Zettl have shown.

References

1. Aleksandrjian, *Spectral decomposition of arbitrary self-adjoint operators into eigenfunctionals*, Soviet Mat. **5** (1985), 607–611.
2. W. N. Everitt and A. Zettl, *A class of integral inequalities*, J. London Math. Soc (2) **17** (1978), 291–303.
3. I. M. Gelfand and A. G. Kostyuchenko, *Eigenfunction expansions of differential and other operators*, Dokl. Akad. Nauk SSSR **103** (1955), 349–352.
4. I. M. Gelfand and B. M. Levitan, *On the determination of a differential equation from its spectral function*, Amer. Math. Soc. Transl. (2) **1** (1951), 239–253.
5. I.M. Gelfand and G. E. Shilov, *Generalized functions*, vols. 2–4, Academic Press, New York, 1961. (English transl.)
6. S. Goldberg, *Unbounded operators; theory and applications*, McGraw Hill, New York, 1966.
7. K. Maurin, *General eigenfunctions expansions*, Polska. Akad. Nauk. **48** (1969).
9. N. Naimark, *Linear differential operators*, Part 2, Ungar, New York, 1968. (English transl.)
10. A. I. Plesner and V. A. Rohlin, *Spectral theory of linear operators*, Amer. Math. Soc. Transl. (2) **62** (1946), 24–101.
11. J. Weidman, *Spectral theory of differential operators*, Lecture Notes in Math., vol. 1258, Springer-Verlag.

DEPARTMENT OF MATHEMATICS, KING FAHD UNIVERSITY, DHAHRAN, SAUDI ARABIA