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SETS OF MINIMAL POINTS IN l_p

MIGUEL LACRUZ

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ABSTRACT. We show that the minimal hull of a convex set in a Banach space is not necessarily convex, even in l_p spaces (finite- or infinite-dimensional). This answers a question raised by B. Beauzamy and B. Maurey in their joint paper of 1977. We also carry out a careful study of the minimal hull and the saturation of the unit ball in $l_1^{(N)}$. Finally, we give a compactness theorem for the minimal hull in l_1.

INTRODUCTION

Let M be a subset of a metric space X. A point $x \in X$ is said to be *minimal with respect to* M if and only if the condition

$$d(y, m) \leq d(x, m) \quad \text{for all } m \in M$$

implies $y = x$. The set of all points minimal with respect to M is called the *minimal hull of* M and is denoted by $\min(M)$. The minimal hull is somehow a generalization of the closed convex hull, since in Hilbert spaces they are the same [1].

The main result of this paper is that in l_p and $l_p^{(N)}$, $N \geq 4$, the minimal hull of the unit ball fails to be convex provided that $1 \leq p \leq 1 + \varepsilon$ for some $\varepsilon > 0$ small enough. This is in contrast with the situation in $L_p[0, 1]$, $p > 1$. J.-O. Larsson proved that in these spaces the minimal hull of the unit ball is a closed ball of radius ρ_p, say, centered at the origin. He also gave estimates for ρ_p and studied the continuity of the function $p \to \rho_p$ [3].

Remarks. The following facts are easy to check and will be needed later on. The first two statements hold in general metric spaces, while the last one makes sense only in Banach spaces.

1. If $M_1 \subseteq M_2$, then $\min(M_1) \subseteq \min(M_2)$.

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2. If M is contained in a closed ball of center a and radius r, then $\min(M)$ is contained in an open ball of center a and radius $2r$.
3. If $M = \{m_1, m_2\}$ is a two-point set, then $\min(M) \subseteq \operatorname{conv}(M)$.

The minimal hull of the unit ball in l_p

We start by giving conditions for a point in $l_1^{(4)}$ to be minimal with respect to the unit ball.

1. **Lemma.** *Let $x \in l_1^{(4)}$ satisfy the following conditions:*
   (a) $|x_i| + |x_j| \leq 1$ *for all* $1 \leq i < j \leq 4$, *and*
   (b) $|x_i| + |x_j| + |x_k| \leq 1$ *for all* $1 \leq i < j < k \leq 4$, *with two possible exceptions.*

Then x is minimal with respect to $B_{l_1^{(4)}}$.

Proof. Suppose x satisfies (a) and (b), with the possible exceptions $1 < 2 < 3$ and $1 < 2 < 4$, say. Assume that $\|y - m\| \leq \|x - m\|$ $\forall m \in B_{l_1^{(4)}}$. Test with the following m's: $(0, x_2, x_3, x_4)$, $(x_1, 0, x_3, x_4)$, and $(x_1, x_2, 0, 0)$. We get

$$\begin{aligned} |y_1| + |y_2 - x_2| + |y_3 - x_3| + |y_4 - x_4| &\leq |x_1|, \\ |y_1 - x_1| + |y_2| + |y_3 - x_3| + |y_4 - x_4| &\leq |x_2|, \\ |y_1 - x_1| + |y_2 - x_2| + |y_3| + |y_4| &\leq |x_3| + |x_4|. \end{aligned}$$

Addition of these inequalities gives $\|y\| + 2\|y - x\| \leq \|x\|$. Consequently, $\|x\| \leq \|y\| + \|y - x\| \leq \|y\| + 2\|y - x\| \leq \|x\|$. Therefore, $\|y - x\| = 0$ and x is minimal with respect to $B_{l_1^{(4)}}$.

The next lemma exhibits some points which fail to be minimal with respect to the unit ball of $l_1^{(4)}$.

2. **Lemma.** *Let $u = (1, 1, 1, 1) \in l_1^{(4)}$. If $\lambda > \frac{1}{3}$ then λu is not minimal with respect to $B_{l_1^{(4)}}$.*

Proof. It will be enough to prove that $\|\frac{1}{3}u - m\| \leq \|\lambda u - m\|$ $\forall m \in B_{l_1^{(4)}}$. Let $m \in B_{l_1^{(4)}}$. We need to show that $\sum_{i=1}^4 |\frac{1}{3} - m_i| \leq \sum_{i=1}^4 |\lambda - m_i|$. First we restrict our attention to the case $m_i \geq 0$ for all $1 \leq i \leq 4$. By symmetry, one can assume without loss of generality that $m_1 \leq m_2 \leq m_3 \leq m_4$. Notice that $m_2 \leq \frac{1}{3}$, since $m_1 + m_2 + m_3 + m_4 \leq 1$. Thus, we have six different cases.

(i) $m_1 \leq m_2 \leq \frac{1}{3} < \lambda \leq m_3 \leq m_4$. In this case we have $\sum_{i=1}^4 |\frac{1}{3} - m_i| = -m_1 - m_2 + m_3 + m_4 = \sum_{i=1}^4 |\lambda - m_i|$.

(ii) $m_1 \leq m_2 \leq \frac{1}{3} \leq m_3 \leq \lambda \leq m_4$. This gives $\sum_{i=1}^4 |\frac{1}{3} - m_i| = -m_1 - m_2 + m_3 + m_4 \leq -m_1 - m_2 - m_3 + m_4 + 2\lambda = \sum_{i=1}^4 |\lambda - m_i|$.

(iii) $m_1 \le m_2 \le \frac{1}{3} \le m_3 \le m_4 \le \lambda$. In this case we get $\sum_{i=1}^4 |\frac{1}{3} - m_i| = -m_1 - m_2 + m_3 + m_4 \le -m_1 - m_2 - m_3 - m_4 + 4\lambda = \sum_{i=1}^4 |\lambda - m_i|$.

(iv) $m_1 \le m_2 \le m_3 \le \frac{1}{3} < \lambda \le m_4$. Here, $\sum_{i=1}^4 |\frac{1}{3} - m_i| = -m_1 - m_2 - m_3 + m_4 + \frac{2}{3} \le -m_1 - m_2 - m_3 + m_4 + 2\lambda = \sum_{i=1}^4 |\lambda - m_i|$.

(v) $m_1 \le m_2 \le m_3 \le \frac{1}{3} \le m_4 \le \lambda$. This time, $\sum_{i=1}^4 |\frac{1}{3} - m_i| = -m_1 - m_2 - m_3 + m_4 + \frac{2}{3} \le -m_1 - m_2 - m_3 - m_4 + 4\lambda = \sum_{i=1}^4 |\lambda - m_i|$.

(vi) $m_1 \le m_2 \le m_3 \le m_4 \le \frac{1}{3} < \lambda$. In this case, $\sum_{i=1}^4 |\frac{1}{3} - m_i| = \frac{4}{3} - m_1 - m_2 - m_3 - m_4 \le 4\lambda - m_1 - m_2 - m_3 - m_4 = \sum_{i=1}^4 |\lambda - m_i|$.

Finally, let us remove the assumption that $m_i \ge 0$ for all $1 \le i \le 4$. For each $1 \le i \le 4$, let $m_i^+ = \max\{m_i, 0\}$. Let $I^+ = \{i\colon m_i > 0\}$ and $I^- = \{i\colon m_i \le 0\}$. Then,

$$\begin{aligned}\sum_{i=1}^4 \left|\frac{1}{3} - m_i\right| &= \sum_{i\in I^-}\left(\frac{1}{3} - m_i\right) + \sum_{i\in I^+}\left|\frac{1}{3} - m_i\right| = \sum_{i\in I^-}(-m_i) + \sum_{i=1}^4\left|\frac{1}{3} - m_i^+\right| \\ &\le \sum_{i\in I^-}(-m_i) + \sum_{i=1}^4 |\lambda - m_i^+| = \sum_{i\in I^-}(\lambda - m_i) + \sum_{i\in I^+}|\lambda - m_i| \\ &= \sum_{i=1}^4 |\lambda - m_i|.\end{aligned}$$

3. **Proposition.** *In $X = l_1^{(4)}$, the minimal hull of $M = B_{l_1^{(4)}}$ fails to be convex.*

Proof. Put $x = (\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ and $y = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2})$. By Lemma 2 we have $x, y \in \min(B_{l_1^{(4)}})$. On the other hand, Lemma 1 shows that the point $\frac{1}{2}(x+y) = \frac{3}{8}(1, 1, 1, 1)$ is not minimal with respect to $B_{l_1^{(4)}}$, since $\frac{3}{8} > \frac{1}{3}$.

Our next goal is to investigate this lack-of-convexity phenomenon in $l_p^{(4)}$ for $p > 1$. For the proof of the following lemma we refer to [2].

4. **Lemma.** *Let $1 < p < \infty$. A point $x \in l_p^{(N)}$ is minimal with respect to a finite subset $M = \{m_1, \dots, m_r\}$ if and only if there are scalars $\lambda_1, \dots, \lambda_r \ge 0$ with $\sum_{i=1}^r \lambda_i = I$ such that the function $\varphi(z) = \sum_{i=1}^r \lambda_i \|z - m_i\|^p$ attains its minimum at $z = x$.*

Notice that the function $z \to \varphi(z)$ attains its minimum at a unique point. We will say that x is the *minimal point with respect to M associated with* $\lambda_1, \dots, \lambda_r$.

5. **Lemma.** *If $1 < p < \infty$ and $K_p = (1 + 2^{-1/(p-1)})^{-1}$, then the point*

$$K_p(2^{-1/p}, 2^{-1/p}, 4^{-1/p}, 4^{-1/p})$$

is minimal with respect to $B_{l_p^{(4)}}$.

Proof. Let $a = (2^{-1/p}, 2^{-1/p}, 4^{-1/p}, 4^{-1/p})$. We will apply Lemma 4 to find a minimal point with respect to the set $M = \{m_1, m_2, m_3\}$ where

$$\begin{aligned} m_1 &= (0, 2^{-1/p}, 4^{-1/p}, 4^{-1/p}), \\ m_2 &= (2^{-1/p}, 0, 4^{-1/p}, 4^{-1/p}), \\ m_3 &= (2^{-1/p}, 2^{-1/p}, 0, 0). \end{aligned}$$

By the remarks we made at the beginning, such a point must be minimal with respect to $B_{l_p^{(4)}}$.

Let x be the minimal point with respect to M associated with $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{3}$. Then x is the point that minimizes

$$\varphi(z) = \tfrac{1}{3}\|z - m_1\|^p + \tfrac{1}{3}\|z - m_2\|^p + \tfrac{1}{3}\|z - m_3\|^p = \tfrac{1}{3}\|z\|^p + \tfrac{2}{3}\|z - a\|^p.$$

Therefore, x is the minimal point with respect to $\{0, a\}$ associated with $\frac{1}{3}$, $\frac{2}{3}$. Again, by the remarks at the beginning, x is forced to lie on the line segment $\operatorname{conv}(\{0, a\})$.

In other words, there is a scalar t such that $x = ta$. Also, t has to minimize

$$\varphi(ta) = \left(\tfrac{1}{3}|t|^p + \tfrac{2}{3}|1 - t|^p\right)\|a\|^p.$$

A straightforward computation with derivatives shows that $t = K_p$. Hence the point $K_p a$ is minimal with respect to $B_{l_p^{(4)}}$.

If we switch the coordinates around, a similar argument shows that the point $K_p(4^{-1/p}, 4^{-1/p}, 2^{-1/p}, 2^{-1/p})$ is minimal with respect to $B_{l_p^{(4)}}$.

6. **Lemma.** *Let* $1 < p < \infty$, $u = (1, 1, 1, 1)$, $m \in B_{l_p^{(4)}}$, *and set*

$$f(t) = \|tu - m\|^p.$$

There is an $\varepsilon > 0$ *such that* $f'(3/8) > 0$, *provided that* $1 < p < 1 + \varepsilon$.

Proof. First choose $\varepsilon > 0$ so small that $3^{-1/p} < 3/8$ for all $1 < p < 1 + \varepsilon$. As in the proof of Lemma 2, we assume that $0 \le m_1 \le m_2 \le m_3 \le m_4$. There are two possible nontrivial cases.

Case I. $0 \le m_1 \le m_2 \le \frac{3}{8} < m_3 \le m_4$. We have $f(t) = (t - m_1)^p + (t - m_2)^p + (m_3 - t)^p + (m_4 - t)^p$. So

$$f'(\tfrac{3}{8}) = p\left[(\tfrac{3}{8} - m_1)^{p-1} + (\tfrac{3}{8} - m_2)^{p-1} - (m_3 - \tfrac{3}{8})^{p-1} - (m_4 - \tfrac{3}{8})^{p-1}\right].$$

Thus, it is enough to show that

$$(\tfrac{3}{8} - m_1)^{p-1} + (\tfrac{3}{8} - m_2)^{p-1} - (m_3 - \tfrac{3}{8})^{p-1} + (m_4 - \tfrac{3}{8})^{p-1} > 0.$$

The function $s \to s^{p-1}$ being concave, it will suffice to prove the above inequality for $m_3 = m_4$. In other words, we will be all set if we can show that

$$(\tfrac{3}{8} - m_1)^{p-1} + (\tfrac{3}{8} - m_2)^{p-1} - 2(m_3 - \tfrac{3}{8})^{p-1} > 0,$$

provided that $m_1^p + m_2^p + 2m_3^p \leq 1$. Now observe that $m_1 \leq 4^{-1/p}$, $m_2 \leq 3^{-1/p}$, $m_3 \leq 2^{-1/p}$. Hence the right-hand side of our inequality is at most $2(2^{-1/p-3/8})^{p-1}$. In order to get a lower estimate for the left-hand side, we note that

$$\left(\frac{1 - m_1^p - m_2^p}{2}\right)^{1/p} \geq m_3 \geq \tfrac{3}{8}$$

and so

$$m_2 \leq (1 - 2(\tfrac{3}{8})^p)^{1/p}, \qquad m_1 \leq [\tfrac{1}{2}(1 - 2(\tfrac{3}{8})^p)]^{1/p}.$$

Consequently, the left-hand side is at least

$$(\tfrac{3}{8} - [\tfrac{1}{2}(1 - 2(\tfrac{3}{8})^p)]^{1/p})^{p-1} + (\tfrac{3}{8} - (1 - 2(\tfrac{3}{8})^p)^{1/p})^{p-1}.$$

Hence, it suffices to show that

$$(\tfrac{3}{8} - [\tfrac{1}{2}(1 - 2(\tfrac{3}{8})^p)]^{1/p})^{p-1} + (\tfrac{3}{8} - (1 - 2(\tfrac{3}{8})^p)^{1/p})^{p-1} > 2(2^{-1/p} - \tfrac{3}{8})^{p-1}.$$

In order to do that, we consider the auxiliary function

$$\begin{aligned} g(p) = {} & (\tfrac{3}{8} - [\tfrac{1}{2}(1 - 2(\tfrac{3}{8})^p)]^{1/p})^{p-1} + (\tfrac{3}{8} - (1 - 2(\tfrac{3}{8})^p)^{1/p})^{p-1} \\ & - 2(2^{-1/p} - \tfrac{3}{8})^{p-1}. \end{aligned}$$

We have $g(1) = 0$ and $g'(1) = \log(2) > 0$. Therefore, there is an $\varepsilon_1 > 0$ such that $g(p) > 0$ for all $1 < p < 1 + \varepsilon_1$, as we wanted.

Case II. $0 \leq m_1 \leq m_2 \leq m_3 < \frac{3}{8} < m_4$. Then

$$f(t) = (t - m_1)^p + (t - m_2)^p + (t - m_3)^p + (m_4 - t)^p.$$

We get

$$f'(\tfrac{3}{8}) = p[(\tfrac{3}{8} - m_1)^{p-1} + (\tfrac{3}{8} - m_2)^{p-1} + (\tfrac{3}{8} - m_3)^{p-1} - (m_4 - \tfrac{3}{8})^{p-1}],$$

so we need to show that

$$(\tfrac{3}{8} - m_1)^{p-1} + (\tfrac{3}{8} - m_2)^{p-1} + (\tfrac{3}{8} - m_3)^{p-1} > (m_4 - \tfrac{3}{8})^{p-1}$$

under the conditions $m_1^p + m_2^p + m_3^p + m_4^p = 1$, $m_1 \leq m_2 \leq m_2 < \frac{3}{8} < m_4$. Thus, the left-hand side of our inequality must be at least

$$(\tfrac{3}{8} - 4^{-1/p})^{p-1} + (\tfrac{3}{8} - 3^{-1/p})^{p-1},$$

while the right-hand side is at most $(1 - \frac{3}{8})^{p-1}$. Now we consider the auxiliary function

$$h(p) = (\tfrac{3}{8} - 4^{-1/p})^{p-1} + (\tfrac{3}{8} - 3^{-1/p})^{p-1} - (1 - \tfrac{3}{8})^{p-1}.$$

Here $h(1) = 1$ and h is continuous at $p = 1$, so there is an $\varepsilon_2 > 0$ such that $h(p) > 0$ for all $1 < p < \varepsilon_2$. Finally we choose $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, so the inequalities in Cases I and II are both fulfilled. The proof of Lemma 6 is complete.

7. **Theorem.** *In* $X = l_p^{(4)}$, *the minimal hull of* $M = B_{l_p^{(4)}}$ *fails to be convex whenever* $1 \le p < 1 + \varepsilon$, *for some* $\varepsilon > 0$ *small enough.*

Proof. We have already proved this in Proposition 3, for $p = 1$. Hence, we assume that $p > 1$. First, we choose $\varepsilon > 0$ as small as in Lemma 6, and for $1 < p < 1 + \varepsilon$ we set

$$x = K_p(2^{-1/p}, 2^{-1/p}, 4^{-1/p}, 4^{-1/p}), \qquad y = K_p(4^{-1/p}, 4^{-1/p}, 2^{-1/p}, 2^{-1/p}).$$

(Recall that $K_p = (1 + 2^{-1/(p-1)})^{-1}$.) By Lemma 5, we know these points are minimal with respect to $B_{l_p^{(4)}}$. Next, we consider the midpoint

$$\tfrac{1}{2}(x + y) = \tfrac{1}{2}K_p(2^{-1/p} + 4^{-1/p})u.$$

Let $\lambda_p = \frac{1}{2}(K_p(2^{-1/p} + 4^{-1/p}) = \frac{1}{2}(2^{-1/p} + 4^{-1/p})(1 + 2^{-1/(p-1)})^{-1}$. Notice that $\lambda_1 = \frac{3}{8}$ and that

$$\frac{d\lambda_p}{dp}\Big|_{p=1} = \frac{\log(4)}{4} + \frac{\log(2)}{2} > 0,$$

so we can choose $\varepsilon > 0$ so small that $\lambda_p > \frac{3}{8}$ for all $1 < p < 1 + \varepsilon$.

Notice that the function $t \to \|tu - m\|$ is convex. Thus, Lemma 6 gives that this function is increasing on $[\frac{3}{8}, \infty)$ for all $1 < p < 1 + \varepsilon$. Therefore

$$\|\tfrac{3}{8}u - m\| \le \|\lambda_p u - m\|$$

for all $m \in B_{l_p^{(4)}}$. Verdict: the midpoint $(x + y)/2 = \lambda_p u$ is not minimal with respect to $B_{l_p^{(4)}}$.

Remarks. A numerical computation gives that auxiliary functions $g(p)$ and $h(p)$ of Lemma 6 are both positive on the range $1 < p < 1.0385$, and that the function λ_p of Theorem 7 satisfies $\lambda_p > \frac{3}{8}$ for all $1 < p < 2$.

This indicates that Theorem 7 holds for all $1 < p < 1.0385$. However, this is just a heuristic estimate.

Now we want to extend Theorem 7 to the spaces l_p and $l_p^{(N)}$, $N \ge 4$. In order to do this we carefully localize $l_p^{(4)}$ inside them.

8. **Lemma.** *Let* n *be a positive integer and let* $1 \le p < \infty$. *Consider the canonical maps* $I\colon l_p^{(n)} \hookrightarrow l_p$ *and* $P\colon l_p \to l_p^{(n)}$ *given by* $Ix = x$ *and* $P(\sum_{i=1}^{\infty} x_i e_i) = \sum_{i=1}^{n} x_i e_i$.

(a) *If* $M \subseteq l_p^{(n)}$ *and* $0 \in M$ *then* $I(\min(M)) \subseteq \min(I(M))$.

(b) *If* $M \subseteq l_p$ *then* $P(\min(M)) \subseteq \min(P(M))$.

Proof. First let $M \subseteq l_p^{(n)}$ and $x \in \min(M)$. Assume that for some $y \in l_p$, we have $\|y - m\| \le \|x - m\|$ for all $m \in M$. Then $\|Py - m\| \le \|y - m\| \le \|x - m\|$ for all $m \in M$, and hence $Py = x$. Now, if we plug $m = 0$ and $x = Py$ into the last inequality, we get $\|y\| \le \|Py\|$, and therefore $y = Py = x$. This takes care of (a).

Next, let $M \subseteq l_p$ and $x \in \min(M)$. Assume that for some $y \in l_p^{(n)}$ we have $\|y - Pm\| \leq \|Px - Pm\|$ for all $m \in M$. Then

$$\begin{aligned}\|y - Px + x - m\|^p &= \|y - Px + x - Pm + Pm - m\|^p \\ &= |y - Pm\|^p + \|(\mathrm{id} - P)(x - m)\|^p \\ &\leq \|Px - Pm\|^p + \|(\mathrm{id} - P)(x - m)\|^p = \|x - m\|^p .\end{aligned}$$

Hence $y - Px + x = x$, and we are all set.

Notice that Lemma 8 is still valid when we replace l_p by $l_p^{(N)}$, with $N \geq n$. Now the main result follows from Theorem 7 and Lemma 8.

9. **Theorem.** *There is an $\varepsilon > 0$ such that for all $1 < p < 1 + \varepsilon$ the minimal hull of the unit ball in l_p or $l_p^{(N)}$ $(N \geq 4)$ fails to be convex.*

Proof. We give the proof only for the case of l_p, since the same argument works for the case of $l_p^{(N)}$. Let $I: l_p^{(4)} \hookrightarrow l_p$ and $P: l_p \to l_p^{(4)}$ be as in Lemma 8.

Use Theorem 7 to get x, $y \in \min(B_{l_p^{(4)}})$ with $(x + y)/2 \notin \min(B_{l_p^{(4)}})$ and look at Ix, Iy. Then Lemma 8(a) gives $Ix, Iy \in \min(B_{l_p})$. However, the midpoint $(Ix+Iy)/2$ is not in $\min(B_{l_p})$, since in that case the point $(x+y)/2 = P((Ix + Iy)/2)$ would be, by Lemma 10(b), in $\min(B_{l_p^{(4)}})$.

Some special features of the minimal hull and the saturation of the unit ball in $l_1^{(N)}$

Let X be a general Banach space. We always have the inclusion

$$\min(B_X) \subseteq 2B_X^0 .$$

In the space $L_1[0, 1]$, the minimal hull of the unit ball is as large as it can be; in other words, the above inclusion is actually an equality. The first result of this section shows that in l_1 we can find points which are minimal with respect to the unit ball and have norm arbitrarily close to 2. This is not so in $L_p[0, 1]$, $p > 1$ (see the comments in the Introduction).

10. **Theorem.** *Let $x \in l_1^{(2N+1)}$ satisfy $\sum_{k=1}^{N+1} |x_{i_k}| \leq 1$ for all possible choices $1 \leq i_1 < \cdots < i_{N+1} \leq 2N + 1$. Then x is minimal with respect to $B_{l_1^{(2N+1)}}$.*

Proof. Assume that $\|y - m\| \leq \|x - m\|$ for all $m \in B_{l_1^{(2N+1)}}$. We want to show that $y = x$. For any $1 \leq i_1 < \cdots < i_{N+1} \leq 2N + 1$, put $m = \sum_{k=1}^{N+1} x_{i_k} e_{i_k}$. By assumption, m is in $B_{l_1^{(2N+1)}}$, so $\|y - m\| \leq \|x - m\|$; in other words,

$$\sum_{k=1}^{N+1} |y_{i_k} - x_{i_k}| + \sum_{i \neq i_k} |y_i| \leq \sum_{i \neq i_k} |x_i| .$$

If we add up these $\binom{2N+1}{N+1}$ inequalities, then we get

$$\binom{2N}{N} \|y - x\| + \binom{2N}{N+1} \|y\| \leq \binom{2N}{N+1} \|x\| .$$

Hence,

$$\begin{aligned}\binom{2N}{N+1}\|x\| &\le \binom{2N}{N+1}\|y-x\| + \binom{2N}{N+1}\|y\| \\ &\le \binom{2N}{N}\|y-x\| + \binom{2N}{N+1}\|y\| \\ &\le \binom{2N}{N+1}\|x\|.\end{aligned}$$

Since $\binom{2N}{N+1} < \binom{2N}{N}$, we conclude that $y = x$.

Remarks. This theorem ensures that the point $x_N = (e_1 + \cdots + e_{2N+1})/(N+1)$ is minimal with respect to the unit ball of $l_1^{(2N+1)}$ (and, thanks to Lemma 8(a), with respect to the unit ball of l_1). Also, $\|x_N\| = (2N+1)/(N+1)$, and this quantity converges to 2 as $N \to \infty$.

Notice that it follows from the proof that a point satisfying the conditions of Theorem 10 is minimal not only with respect to the unit ball of $l_1^{(2N+1)}$, but also with respect to a finite subset. It was proven in [2] that, in a uniformly convex Banach space, the following approximation theorem holds: any point x minimal with respect to a given subset M can be approximated by a point x' which is arbitrarily close to x and which is minimal with respect to a finite subset of M. More precisely, *if* $x \in \min(M)$, *then for all* $\varepsilon > 0$ *there is a finite subset* $M' \subseteq M$ *and a point* $x' \in \min(M')$ *with* $\|x - x'\| < \varepsilon$.

As we mentioned in the Introduction, in Hilbert spaces the minimal hull is the closed convex hull, so this approximation result is immediate. We do not know whether this is true in l_1. However, there are Banach spaces in which such a result does not hold, as the following example shows.

Example. In the space $X = c_0$, we consider the set $M = \{\pm e_k : k = 1, 2, \dots\}$. Then we have $0 \in \min(M)$. Indeed, assume $\|y - m\| \le \|m\|$ for all $m \in M$. Then we get $|y_k - 1| \le 1$ and $|y_k + 1| \le 1$ for all $k = 1, 2, \dots$. Hence $y_k = 0$ for all $k = 1, 2, \dots$ and 0 is minimal with respect to M. However, given any $x' \in c_0$ with $\|x'\| < 1/2$, x' cannot be minimal with respect to a finite subset of M. Indeed, if that were the case, we would have $x' \in \min(\{\pm e_1, \dots, \pm e_N\})$ for some positive integer N. But then we could move the point x' to a different point y without decreasing its distance to any point in $\{\pm e_1, \dots, \pm e_N\}$; in order to do that, it suffices to set $y_k = x'_k$ if $k \ne N+1$, $y_{N+1} = 0$ if $x_{N+1} \ne 0$, and $y_{N+1} = 1/2$ if $x_{N+1} = 0$. Therefore the space $X = c_0$ fails the approximation theorem.

Let M be a subset of a metric space X. We set $M_0 = M$ and, for $k \ge 0$, we write

$$M_{k+1} = \min(M_k).$$

The *saturation* of M is then defined by

$$\operatorname{sat}(M) = \overline{\bigcup_{k \ge 0} M_k}.$$

The pathologies of the minimal hull disappear when one replaces it by the saturation. For instance, in a strictly convex space, the saturation of a set is always a closed convex set [1, p. 121]. The saturation of the unit ball in l_p, $1 < p < \infty$, $p \neq 2$, is the set

$$\{x \in l_p : |x_i|^p + |x_j|^p \leq 1 \ \forall i < j\}$$

[1, p. 136]. In the next theorem, we prove that this result still holds when $p = 1$. Also, we show that, when computing the saturation of the unit ball in $l_1^{(N)}$, this infinite process reduces to a finite number of steps.

11. **Theorem.**

(a) *In l_1, the saturation of the unit ball is the set*

$$S = \{x \in l_1 : |x_i| + |x_j| \leq 1 \ \forall 1 \leq i < j < \infty\}.$$

(b) *In $l_1^{(N)}$, the saturation of the unit ball is the set $S \cap l_1^{(N)}$, and the infinite process reduces to $N - 2$ steps.*

Proof. By the argument used in [1], we have the inclusions $\operatorname{sat}(B_{l_1}) \subseteq S$ and $\operatorname{sat}(B_{l_1^{(N)}}) \subseteq S \cap l_1^{(N)}$, so it will suffice to prove the reverse inclusion.

We claim that $S \cap l_1^{(N)} \subseteq \min^{N-2}(B_{l_1^{(N)}})$ and we prove this by induction on N. The case $N = 3$ is a particular instance of Theorem 10. Now let us assume that our claim holds for N. We want to prove that it also holds for $N + 1$. Given $1 \leq i \leq N + 1$ we consider the projection $P_i : l_1^{(N+1)} \to l_1^{(N)}$ defined by $P_i x = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{N+1})$. Our objective is to show that $S \cap l_1^{(N+1)} \subseteq \min^{N-1}(B_{l_1^{(N+1)}})$. Pick $x \in S \cap l_1^{(N+1)}$ and observe that $P_i x \in S \cap l_1^{(N)}$. By our inductive hypothesis, $P_i x \in \min^{N-2}(B_{l_1^{(N)}})$, and so $P_i x \in \min^{N-2}(B_{l_1^{(N+1)}})$. Finally, assume that $\|y - m\| \leq \|x - m\|$ for all $m \in \min^{N-2}(B_{l_1^{(N+1)}})$. Test with the $P_i x$'s and add up the $N + 1$ resulting inequalities to obtain $\|y\| + N\|y - x\| \leq \|x\|$. Hence $\|x\| \leq \|y\| + \|y - x\| \leq \|y\| + N\|y - x\| \leq \|x\|$. Therefore $y = x$, and x is minimal with respect to $\min^{N-2}(B_{l_1^{(N+1)}})$; in other words, x is in $\min^{N-1}(B_{l_1^{(N+1)}})$.

Now the theorem is a consequence of our claim. We have

$$\operatorname{sat}(B_{l_1^{(N)}}) \subseteq S \cap l_1^{(N)} \subseteq \min^{N-2}(B_{l_1^{(N+1)}}),$$

and this gives (b). On the other hand,

$$\operatorname{sat}(B_{l_1}) \subseteq S = \overline{\bigcup_{N \geq 1} (S \cap l_1^{(N)})} = \overline{\bigcup_{N \geq 1} \operatorname{sat}(B_{l_1^{(N)}})} \subseteq \operatorname{sat} \overline{\bigcup_{N \geq 1} B_{l_1^{(N)}}} = \operatorname{sat}(B_{l_1}),$$

and (a) follows.

THE MINIMAL HULL OF COMPACT SETS IN l_1

Recall a classical theorem of Mazur: in a general Banach space, the closed convex hull of a compact set is compact. A natural question arises: do we have

an analogous result for the minimal hull? As we mentioned in the Introduction, Beauzamy and Maurey proved that result for locally uniformly convex, reflexive Banach spaces. Here we prove the corresponding result for l_1. Finally, we give a counterexample to show that the result does not hold in general metric spaces: even for a two-point set, the minimal hull might fail to be relatively compact.

In order to prove the following lemma, one has just to mimic the proof of Lemma 8.

12. **Lemma.** *For any* $n \geq 0$, *consider the projection* $P_n : l_1 \to l_1$ *given by* $P_n x = \sum_{i=n+1}^{\infty} x_i e_i$. *If* $M \subseteq l_1$ *then* $P_n(\min(M)) \subseteq \min(P_n(M))$.

13. **Theorem.** *If* M *is a compact subset of* l_1, *then* $\min(M)$ *is relatively compact.*

Proof. Recall that a subset K of l_1 is relatively compact if and only if K is bounded and

$$\lim_{n\to\infty} \sup_{x\in K} \left\| \sum_{i=n}^{\infty} x_i e_i \right\| = 0.$$

The second condition can be restated as follows: for all $\varepsilon > 0$ there is an n_0 such that if $n \geq n_0$ then $P_n(K) \subseteq \varepsilon B_{l_1}$.

Now let us see how $\min(M)$ satisfies this set of conditions. The boundedness of $\min(M)$ follows from the boundedness of M and the remarks at the end of the Introduction. Therefore, it is enough to check the second condition. Let $\varepsilon > 0$ be given. Since M is compact, there is an n_0 such that $n \geq n_0$ implies $P_n(M) \subseteq (\varepsilon/2)B_{l_1}$. But then, for $n \geq n_0$ we have

$$P_n(\min(M)) \subseteq \min(P_n(M)) \subseteq \min\left(\frac{\varepsilon}{2} B_{l_1}\right) \subseteq \varepsilon B_{l_1}.$$

Hence $\min(M)$ satisfies the second condition.

Example. Let $X = \{m_1, m_2, x_1, \dots, x_N, \dots\}$ be any countable set. We provide X with a metric d as follows:

$$d(m_1, m_2) = 2, \tag{1}$$

$$d(x_i, x_j) = 1 \qquad (i \neq j), \tag{2}$$

$$d(m_1, x_N) = 1 + \sum_{k=1}^{N} \frac{1}{2^k}, \tag{3}$$

$$d(m_2, x_N) = 1 - \sum_{k=1}^{N} \frac{1}{2^k}. \tag{4}$$

It is easily seen that this actually defines a metric on X (we only have to worry about the triangle inequality). Now look at the compact set $M = \{m_1, m_2\}$. We claim that $\min(M) = X$ (and therefore $\min(M)$ fails to be relatively compact, since the infinite subset $\{x_1, \dots, x_N, \dots\}$ has the discrete topology). Indeed, if we take an x_i in X and move it to m_1 (resp. m_2) then we increase its

distance to m_2 (resp. m_1). Also, if we move x_i to x_j, where $j > i$ (resp. $j < i$), then we increase its distance from m_1 (resp. m_2). Therefore, every x_i is minimal with respect to M.

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Department of Mathematics, Kent State University, Kent, Ohio 44242