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ROTUNDITY, THE C.S.R.P., AND THE λ-PROPERTY IN BANACH SPACES

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ABSTRACT. Two open questions stemming from the λ-property in Banach spaces are solved. The following are shown to be equivalent in a Banach space X: (a) X has the λ-property; (b) every vector in the closed unit ball of X is expressible as a convex series of extreme points of the unit ball of X. Also, by exhibiting a class of nonrotund Orlicz spaces for which the λ-function is identically 1 on the unit spheres, we answer negatively the question of whether the λ-function characterizes rotund Banach spaces.

Given a normed space X, B_X denotes its closed unit ball, $\mathrm{ext}(B_X)$ the set of extreme points of B_X, and S_X the closed unit sphere of X. If $x \in B_X$, a triple (e, y, λ) is said to be *amenable to* x if $e \in \mathrm{ext}(B_X)$, $y \in B_X$, $0 < \lambda \leq 1$, and $x = \lambda e + (1-\lambda)y$. In this case, we define

$$\lambda(x) = \sup\{\lambda : (e, y, \lambda) \text{ is amenable to } x\}. \tag{1}$$

X is said to have the *λ-property* if each $x \in B_X$ admits an amenable triple. If, in addition, $\inf\{\lambda(x) : x \in B_X\} > 0$, then X is said to have the *uniform λ-property*. Finally, X is said to have the *convex series representation property* if for each $x \in B_X$, there is a sequence (e_k) of extreme points of B_X and a sequence of nonnegative real numbers (λ_k) such that $\sum_{k=1}^{\infty} \lambda_k = 1$ and $x = \sum_{k=1}^{\infty} \lambda_k e_k$.

In this note, we settle two open questions which stem from the study of the λ-property. It was shown in [1] (see Theorem 3.1 and Remark 3.2) that X has the uniform λ-property if and only if X has a uniform version of the convex series representation property; that is, there exists a fixed sequence (λ_k) of nonnegative real numbers with $\sum_{k=1}^{\infty} \lambda_k = 1$ such that for each $x \in B_X$ there is a sequence $(e_k) \subset \mathrm{ext}(B_X)$ for which $x = \sum_{k=1}^{\infty} \lambda_k e_k$. It is clear that the convex series representation property implies the λ-property. In this note, we establish the final connection between these pairs of properties by showing that every Banach space with the λ-property has the convex series representation

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property. Our ideas borrow from the proof of Theorem 1.1 of [3], where this result was established under additional assumptions on the λ-function.

The second question we settle relates to rotundity. In [1], it was shown that, if $\dim X < \infty$ and $x \in B_X$, then $x \in \operatorname{ext}(B_X)$ if and only if $\lambda(x) = 1$. Hence when X is a finite-dimensional space, the λ-function characterizes individual extreme points in B_X. Consequently, in this case X is rotund if and only if $\lambda(x) = 1$ for all $x \in S_X$. It was shown in [3], however, that when X is an infinite-dimensional Banach space, the λ-function may fail to characterize individual extreme points in B_X. Thus, it is possible to have $\lambda(x) = 1$ for a point $x \in S_X$ that is not an extreme point of B_X. This left open the question raised in [3] of whether nonrotund spaces X exist for which $\lambda(x) = 1$ for all $x \in S_X$. We show that this is possible for a class of Orlicz spaces.

1. The convex series representation property

Theorem 1. *If X is a Banach space with the λ-property, then X has the convex series representation property.*

Proof. Let $x_0 \in B_X$. By Lemma 1.2 and Corollary 1.3 of [3], we may assume $x_0 \in S_X$. By repeated use of the λ-property, we obtain sequences $(e_k) \subset \operatorname{ext}(B_X)$, $(x_k) \subset S_X$, and (λ_k) such that for $k = 0, 1, 2, \ldots$

(a) $x_k = \lambda_{k+1}e_{k+1} + (1 - \lambda_{k+1})x_{k+1}$,
(b) $0 < \frac{1}{2}\lambda(x_k) < \lambda_{k+1} \leq 1$.

Condition (b) ensures that the sequence (λ_k) is chosen reasonably large. As we will see, the point of proof is to show that $(\lambda_k) \notin l_1$.

Let $P_0 = 1$ and for $n \geq 1$, write $P_n = \prod_{k=1}^n (1 - \lambda_k)$. For all n, we have

$$x_0 = \sum_{k=1}^n \lambda_k P_{k-1} e_k + P_n x_n, \tag{2}$$

$$\sum_{k=1}^n \lambda_k P_{k-1} + P_n = 1. \tag{3}$$

The sequence (P_n) is decreasing and we let $P = \lim_n P_n$. It suffices to show $P = 0$.

Assume, to the contrary, that $P > 0$. Then $\sum_{k=1}^\infty \lambda_k < \infty$ and hence $\sum_{k=1}^\infty \lambda_k P_{k-1} e_k$ converges. From (2), it follows that (x_n) converges to a vector $y \in S_X$. Let (e, w, μ) be amenable to y. Choose K large enough so that $P/P_K > 1/2$ and $\lambda_{K+1} < \mu/4$. Apply (a) repeatedly, starting with $k = K$, to obtain

$$x_K = \frac{1}{P_K}\left[\sum_{j=1}^\infty \lambda_{K+j} P_{K+j-1} e_{K+j} + Py\right].$$

Since $y = \mu e + (1-\mu)w$, we have

$$\begin{aligned} x_K &= \left(\frac{P\mu}{P_K}\right) e + \frac{1}{P_K}\left[P(1-\mu)w + \sum_{j=1}^{\infty} \lambda_{K+j} P_{K+j-1} e_{K+j}\right] \\ &= \left(\frac{P\mu}{P_K}\right) e + \left(1 - \frac{P\mu}{P_K}\right) z, \end{aligned}$$

where

$$z = \frac{P(1-\mu)w + \sum_{j=1}^{\infty} \lambda_{K+j} P_{K+j-1} e_{K+j}}{P_K - P\mu}.$$

However,

$$\begin{aligned} \|z\| &\le \frac{P(1-\mu) + \sum_{j=1}^{\infty} \lambda_{K+j} P_{K+j-1}}{P_K - P\mu} \\ &= \frac{P + \sum_{k=K+1}^{\infty} \lambda_k P_{k-1} - P\mu}{P_K - P\mu}. \end{aligned}$$

By (3), we have $\sum_{k=1}^{K} \lambda_k P_{k-1} + P_K = 1$ and $\sum_{k=1}^{\infty} \lambda_k P_{k-1} + P = 1$. Consequently, $P + \sum_{k=K+1}^{\infty} \lambda_k P_{k-1} = P_K$. This implies that $\|z\| \le 1$ and hence $(e, z, P\mu/P_K)$ is amenable to x_K. It follows that

$$\lambda_{K+1} > \frac{1}{2}\lambda(x_K) \ge \frac{P\mu}{2P_K} > \frac{\mu}{4},$$

contradicting the choice of K and completing the proof.

Remark 2. (1) The preceding theorem was proved in [3] assuming that the λ-function is locally bounded away from zero on S_X at each of its points. In view of the general result just proved, we see that this additional hypothesis is not needed. In fact, it has been shown in [4] that the unit ball of Hilbert space can be modified very slightly to obtain a unit ball of a Banach space X such that B_X is the convex hull of $\text{ext}(B_X)$, yet the λ-function fails to be locally bounded away from zero on S_X at a member of $\text{ext}(B_X)$.

(2) In a sense, the proof of the preceding theorem is constructive. Namely, if condition (b) is satisfied at each step, then we are guaranteed to obtain $x_0 = \sum_{k=1}^{\infty} \lambda_k P_{k-1} e_k$. If the sequence (λ_k) is chosen in a more primitive manner, so that only condition (a) is satisfied, then a more tedious proof, using transfinite induction, can be given to obtain the same representation for x_0.

2. Nonrotund Orlicz spaces X for which $\lambda \equiv 1$ on S_X

Let (Ω, Σ, μ) be a measure space and let $\phi\colon \mathbf{R} \to \mathbf{R}^+ \cup \{0\}$ be a Young's function: that is, ϕ is nonnegative, not identically zero, even, convex, and $\phi(0) = 0$. The Orlicz space $L^{\phi}(\mu)$ is then defined as the set of equivalence classes of measurable functions $f\colon \Omega \to \mathbf{R}$ such that the functional $I_{\phi}(\lambda f) = \int_{\Omega} \phi(\lambda f)\, d\mu < \infty$ for some $\lambda < \infty$ equipped with Luxemburg norm $\|f\| = \inf\{\lambda > 0\colon I_{\phi}(f/\lambda) \le 1\}$. Recall that, for $f \in L^{\phi}(\mu)$, $I_{\phi}(f) = 1$ implies

$\|f\| = 1$, but from $\|f\| = 1$ we can only conclude $I_\phi(f) \le 1$. Also, from the Dominated Convergence Theorem, we obtain that, if $I_\phi(f) < 1$, the following are equivalent: (a) $\|f\| = 1$; (b) $I_\phi((1+\varepsilon)f) = +\infty$ for all $\varepsilon > 0$.

Rotundity of the closed unit ball $B_{L^\phi(\mu)}$ has been studied by several authors:

(A) If μ is not purely atomic, B. Turett proved in [6] that $B_{L^\phi(\mu)}$ is rotund if and only if ϕ is strictly convex (i.e., $\phi((x+y)/2) < \frac{1}{2}[\phi(x)+\phi(y)]$, $x \ne y$) and satisfies a Δ_2 condition for large values of x (i.e., there exist positive constants M and x_0 such that $\phi(2x) \le M\phi(x)$ if $x \ge x_0$).

(B) If μ is purely atomic, the rotundity of $B_{L^\phi(\mu)}$ was characterized by A. Kaminska in [2]. In particular, if $L^\phi(\mu)$ is the "usual" Orlicz sequence space l^ϕ, then B_{l^ϕ} is rotund if and only if the following conditions are fulfilled: (a) there is an $x \in \mathbf{R}$ such that $\phi(x) = 1$; (b) ϕ satisfies the δ_2 condition (i.e., there exist positive constants M and x_0 such that $\phi(2x) \le M\phi(x)$ if $0 \le x \le x_0$); (c) ϕ is strictly convex on the set $A = \{x \in \mathbf{R} \colon \phi(x) \le \frac{1}{2}\}$.

As we need a nonrotund unit ball, we assume that (Ω, Σ, μ) is not purely atomic and that ϕ is strictly convex and fails Δ_2 conditions. Now let $e \in L^\phi(\mu)$. Since we are assuming ϕ is strictly convex, it follows from [5, Proposition 1] that $e \in \operatorname{ext}(B_{L^\phi(\mu)})$ if and only if $I_\phi(e) = 1$.

We claim that $\lambda(f) = 1$ for each $f \in L^\phi(\mu)$ with $\|f\| = 1$. First, if $I_\phi(f) = 1$, then, by [5, 1. Proposition] $f \in \operatorname{ext}(B_{L^\phi(\mu)})$ and so $\lambda(f) = 1$. Next, suppose that $I_\phi(f) < 1$ (note that $0 < I_\phi(f)$ because $\|f\| = 1$). Then $I_\phi((1+\varepsilon)f) = +\infty$ if $\varepsilon > 0$. Fix $\delta > 0$. We want to prove that $\lambda(f) \ge 1/(1+\delta)$. Put $A_n = \{w \in \Omega \colon \frac{1}{n} < |f(w)| < n\}$. Then for all $\alpha > 0$ and all $n \in \mathbf{N}$, $\int_{A_n} \phi(\alpha f)\, d\mu < \infty$ and $\mu(A_n) < \infty$. Also,

$$\lim_{n\to\infty} \int_{A_n} \phi((1+\delta)f)\, d\mu = \int_\Omega \phi((1+\delta)f)\, d\mu = \infty.$$

Therefore, there exists $k \in \mathbf{N}$ such that $1 \le \int_{A_k} \phi((1+\delta)f)\, d\mu < \infty$, and we can choose $0 < t \le \delta$ such that

$$\int_{A_k} \phi((1+t)f)\, d\mu + \int_{\Omega\setminus A_k} \phi(f)\, d\mu = 1.$$

Define $g, h \colon \Omega \to \mathbf{R}$ as follows:

$$g(w) = \begin{cases} (1+t)f(w) & \text{if } w \in A_k, \\ f(w) & \text{if } w \in \Omega\setminus A_k; \end{cases}$$

$$h(w) = \begin{cases} 0 & \text{if } w \in A_k, \\ f(w) & \text{if } w \in \Omega\setminus A_k. \end{cases}$$

Then we have:

1. $g \in \operatorname{ext}(B_{L^{\phi}(\mu)})$ because $I_{\phi}(g) = 1$.
2. $\|h\| = 1$ because $I_{\phi}(h) \leq I_{\phi}(f) < 1$ and, for each $\varepsilon > 0$, $I_{\phi}((1+\varepsilon)h) = \infty$, since $I_{\phi}((1+\varepsilon)f) = \infty$ and $\int_{A_k} \phi((1+\varepsilon)f)\, d\mu < \infty$.
3. $f = g/(1+t) + th/(1+t)$. Therefore, $\lambda(f) \geq 1/(1+t) \geq 1/(1+\delta)$.

As $\delta > 0$ is arbitrary, we conclude that $\lambda(f) = 1$.

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