ULTRASEPARATING FUNCTION SPACES
AND OPERATING FUNCTIONS
FOR THE REAL PART OF A FUNCTION ALGEBRA

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Abstract. We show how a local version of a result due to A. Bernard on ul-
traseparating Banach function spaces can be used to give a short proof of the
theorem stating that only affine functions operate on the real part of a function
algebra.

The paper [5] by K. de Leeuw and Y. Katznelson contains the following
generalization of the Stone-Weierstrass Theorem:

Let $X$ be a compact Hausdorff space and $B$ a subspace of $C_{\mathbb{R}}(X)$, which
separates the points of $X$ and contains the constant functions. If there is a
continuous non-affine function $h$ defined on a subinterval $I$ of $\mathbb{R}$ which operates
on $B$, i.e., $h \circ b \in B$ if $b \in B$ and $h \circ b$ is defined, then $B$ is dense in $C_{\mathbb{R}}(X)$.

At the same time J. Wermer [8] proved the following theorem:

Let $A$ be a function algebra on a compact Hausdorff space $X$. Then if $\text{Re } A$ is an algebra, $A = C(X)$.

Here, $\text{Re } A$ denotes the space of the real parts of functions in $A$. Another
formulation of Wermer's result is that if the function $t \mapsto t^2$ operates on $\text{Re } A$, then
$A = C(X)$. These two results lead to the following conjecture:

Let $A$ be a function algebra on the compact Hausdorff space $X$. If there is a
non-affine function defined on a subinterval of $\mathbb{R}$ which operates on $\text{Re } A$, then
$A = C(X)$.

(Here, it is not necessary to assume that the operating function is continuous
because a function which operates on the real part of a function algebra is
automatically continuous [6].)

If one tries to apply directly the theorem of de Leeuw and Katznelson one
only finds that $\text{Re } A$ is a dense subspace of $C_{\mathbb{R}}(X)$, i.e., that $A$ is a Dirichlet
algebra. Thus, more is needed to prove the conjecture.

In [1] A. Bernard introduced a powerful method to attack the conjecture
and related problems: Let $\beta(\mathbb{N} \times X)$ denote the Stone-Čech compactification
of $\mathbb{N} \times X$, and let $l^{\infty}(\mathbb{N}, C(X))$ denote the space of all bounded sequences

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of continuous functions on $X$. This space can be identified in a natural way with $C(\beta(N \times X))$, the space of all continuous complex-valued functions on $\beta(N \times X)$. Now, if $A$ is a function algebra on $X$, one can introduce a norm on $\text{Re} A$ in the following manner:

$$N(u) = \inf\{\|a\| : a \in A \text{ and } \text{Re}a = u\} \quad \text{for all } u \in \text{Re} A,$$

where $\| \cdot \|$ denotes the supremum norm. This norm dominates the supremum norm and turns $\text{Re} A$ into a Banach space. The space $l^\infty(N, \text{Re} A)$ of all $N$-bounded sequences of elements from $\text{Re} A$ satisfies the inclusion $l^\infty(N, \text{Re} A) \subseteq l^\infty(N, C_R(X)) = C_R(\beta(N \times X))$. It turns out that if $\text{Re} A$ is a Dirichlet algebra then $l^\infty(N, \text{Re} A)$ separates the points of $\beta(N \times X)$, (cf. [1]). From the theorem of de Leeuw and Katznelson it thus follows that if there is a non-affine function which operates on $l^\infty(N, \text{Re} A)$ then $l^\infty(N, \text{Re} A)$ is dense in $C_R(\beta(N \times X))$. This has some interesting consequences, as the next theorem, which is due to Bernard [1], shows.

**Theorem.** Let $E$ be a subspace of $C(X)$ which is a Banach space in a norm that dominates the supremum norm. Then, if $l^\infty(N, E)$ is dense in $C(\beta(N \times X))$, it follows that $E = C(X)$.

This theorem holds for the real as well as the complex case. Thus one concludes that if $l^\infty(N, \text{Re} A)$ is dense in $C_R(\beta(N \times X))$, then $\text{Re} A = C_R(X)$, and hence $A = C(X)$ by a well known theorem of Bishop, Hoffman and Wermer.

In general it is not clear that a function which operates on $\text{Re} A$ also operates on $l^\infty(N, \text{Re} A)$. This led Bernard to introduce the following concept:

A real-valued function $h$, defined on an open subinterval of $R$ containing 0, operates boundedly on $\text{Re} A$ if there exist numbers $\alpha > 0$ and $M > 0$ such that $N(h \circ u) < M$ whenever $u \in \text{Re} A$ and $N(u) < \alpha$.

As an example, take the function $t \mapsto t^2$. If this function operates on $\text{Re} A$ then it operates boundedly on $\text{Re} A$ (cf. [1]). If $h$ operates boundedly on $\text{Re} A$ then $h \circ \{u_n\} \in l^\infty(N, \text{Re} A)$ whenever $\{u_n\} \in l^\infty(N, \text{Re} A)$ and $N(u_n) < \alpha$ for all $n$. The proof of the theorem of de Leeuw and Katznelson shows that if $h$ is also non-affine, then it follows that $l^\infty(N, \text{Re} A)$ is dense in $C_R(\beta(N \times X))$.

Using the Baire Category Theorem in a clever way S. Sidney [6] succeeded in proving the conjecture without the condition that the operating function should operate boundedly. However, he had to add another condition; the operating function must not be affine on any subinterval of the one on which it is defined. The finishing touch is due to O. Hatori [3], who proved the conjecture for the remaining case.

Recently O. Hatori [4] has given a local version of the theorem of Bernard stated above. In this note we show how a variant of Hatori’s local version can be used to give a simple proof of the above conjecture.
RESULTS AND APPLICATIONS

Let $X$ be a compact Hausdorff space. A subspace $B$ of $C(X)$ which separates the points of $X$, contains the constant functions and is complete in a norm which dominates the supremum norm is called a Banach function space on $X$. Bernard calls such a space ultraseparating on $X$ if $l^\infty(N, B)$ separates the points of the Stone-Čech compactification $\beta(N \times X)$ of $N \times X$. As mentioned earlier, if $A$ is a Dirichlet algebra on $X$, then $\text{Re} A$ is ultraseparating (see [1]).

If $f \in C(X)$ we let $\{f\}$ denote the element $\{f_n\} \in l^\infty(N, C(X))$ where $f_n = f$ for all $n$. For $x \in X$ and $f \in C(X)$ we let $\mathcal{F}_x = \mathcal{F}_x(f)$ denote the set of constancy for $\{f\}$ given by

$$\mathcal{F}_x = \{\xi \in \beta(N \times X): \{f\}(\xi) = f(x)\}.$$

The following theorem is a variant of Hatori’s local version of Bernard’s Theorem:

**Theorem 1.** Let $B$ be a Banach function space on a compact Hausdorff space $X$, and let $x \in X$. If $l^\infty(N, B)|_{\mathcal{F}_x}$ is dense in $C(\mathcal{F}_x)$ then there exists a compact neighbourhood $K_x$ of $x$ such that $B|_{K_x} = C(\mathcal{K}_x)$.

**Proof.** Let $g \in C(X)$ be a function such that $\mathcal{F}_x = \mathcal{F}_x(g)$ and let $K_n = \{x' \in X: |g(x') - g(x)| \leq 1/n\}$ for each $n \in N$. We assert that there exists a natural number $n_0$ having the following property:

For all $f \in C(X)$ for which $\|f\|_\infty \leq 1$ there is a function $u \in B$ such that $N(u) \leq n_0$ and $|f - u| < 1/2$ on $K_{n_0}$.

If not, then there is, for each $n \in N$, a function $f_n \in C(X)$ such that $\|f_n\|_\infty \leq 1$ and such that $N(u) > n$ if $u \in B$ and $|f_n - u| < 1/2$ on $K_n$. Since $\{f_n\} \in l^\infty(N, C(X))$ there is a sequence $\{u_n\} \in l^\infty(N, B)$ such that $\|\{f_n\} - \{u_n\}\| < 1/2$ on $\mathcal{F}_x$ and hence on some neighbourhood $\mathcal{U}$ of $\mathcal{F}_x$. Since $\mathcal{U}$ is open the definition of $\mathcal{F}_x$ shows that there is a number $n_0 \in N$ such that

$$\{\xi \in \beta(N \times X): |\{g\}(\xi) - g(x)| \leq 1/n_0\} \subseteq \mathcal{U}.$$

Now, $N \times K_{n_0}$ is contained in the set on the left hand side and hence in $\mathcal{U}$. We conclude that $|\{f_n\} - \{u_n\}| < 1/2$ on $N \times K_{n_0}$, which means that $|f_n - u_n| < 1/2$ on $K_{n_0}$ for all $n$. Since $\{u_n\} \in l^\infty(N, B)$ there is a number $n_1 > n_0$ such that $N(u_n) < n_1$ for all $n$. Since $K_{n_1} \subseteq K_{n_0}$ we have reached a contradiction, and the assertion has been proved.

Let $f \in C(X)_1$, the unit ball of $C(X)$. By induction we can construct elements $v_0, v_1, \ldots, v_n, \ldots$ in $B$ such that $N(v_n) < n_0$ for all $n$ and such that

$$\left| f - \left( v_0 + \frac{1}{2} v_1 + \cdots + \frac{1}{2^n} v_n \right) \right| < \frac{1}{2^{n+1}} \text{ on } K_{n_0}.$$

Let $v = \sum v_n/2^n$. Then $v \in B$ and $v = f$ on $K_{n_0}$. □
Using this theorem, the Baire Category Theorem in the same way as Sidney in [6] and a few results from the theory of function algebras, we obtain a simple proof of the conjecture.

**Theorem 2.** ([1, 3, 6]). Let $A$ be a function algebra on a compact Hausdorff space $X$ and suppose there is a non-affine function defined on a subinterval of $\mathbb{R}$ which operates on $\text{Re } A$. Then $A = C(X)$.

**Proof.** We have already noted that $\text{Re } A$ is ultraseparating on $X$. Suppose $A \neq C(X)$, let $\mu \in \text{ext}(\langle A^\perp \rangle_1)$, and let $E$ be the smallest $p$-set for $A$ which contains the support of $\mu$. Then $u(E)$ is an interval for any $u \in \text{Re } A$. To see this let $u \in \text{Re } A$ and pick $a \in A$ such that $u = \text{Re } a$. If $u(E)$ is not an interval, then we can find a real number $t$ such that $a(E) = C_1 \cup C_2$, where $C_1$ and $C_2$ are nonempty disjoint compact subsets of the complex plane such that $\text{Re } z < t$ for all $z \in C_1$ and $\text{Re } z > t$ for all $z \in C_2$. By Runge's Theorem there is a sequence of polynomials converging uniformly to $0$ on $C_1$ and to $1$ on $C_2$. It follows that $E = a^{-1}(C_1) \cup a^{-1}(C_2)$, a union of two disjoint $p$-sets. Since $\mu$ is an extreme point of $\langle A^\perp \rangle_1$, the support of $\mu$ is contained in $a^{-1}(C_1)$ or in $a^{-1}(C_2)$, contradicting the fact that $E$ is the smallest $p$-set for $A$ with this property.

We may assume that the operating function $h$ is defined on an interval containing $0$ as an interior point and is not affine on any open interval which contains $0$. We choose a positive number $\delta$ such that $h \circ u$ is defined if $N(u) \leq \delta$, put $H = \{u \in \text{Re } A: 0 \in u(E) \text{ and } N(u) \leq \delta\}$ and write $H = \cup H_n$, where $H_n = \{u \in H: N(h \circ u) \leq n\}$. By the Baire Category Theorem there is a number $n_0 \in \mathbb{N}$ such that $\overline{H_{n_0}}$ contains a set which is open in $H$. (The bar denotes the closure w.r.t. the norm $N$.) Thus there is a function $u_0 \in H$ and a positive number $\epsilon$ such that $\{u \in H: N(u - u_0) < \epsilon\} \subseteq \overline{H_{n_0}}$. Adding to or subtracting from $u_0$ a small multiple of the constant function $1$, one obtains a function $u'_0$ such that $0$ is an interior point of $u'_0(E)$. This assumes that $u_0$ is not constant on $E$. In that case we replace $u_0$ by another function $u$ in $H$ for which $N(u - u_0) < \epsilon$ and $\epsilon$ by a smaller number. Unless $E$ is a singleton, in which case we already have a contradiction, we can thus assume that $0$ is an interior point of $u_0(E)$. Since $u(E)$ is an interval for all $u \in \text{Re } A$ we conclude that $u \in H$ if $u$ is sufficiently close to $u_0$, and hence we can assume that

$$\{u \in \text{Re } A: N(u - u_0) < \epsilon\} \subseteq \overline{H_{n_0}}.$$

It follows that $N(h \circ (u_0 + u)) < n_0$ if $u$ belongs to a dense subset of $\text{Re } A_\epsilon := \{u \in \text{Re } A: N(u) < \epsilon\}$ and thus $h \circ \{u_0 + u_n\} \in \ell^\infty(N, \text{ Re } A)$ if the functions $u_n$ belong to a dense subset of $\text{Re } A_\epsilon$. We conclude that $h \circ \{u_0 + u_n\} \in \ell^\infty(N, \text{ Re } A)$ if the functions $u_n$ belong to $\text{Re } A_\epsilon$. (The bar denotes closure w.r.t. the sup-norm on $\beta(N \times X)$.)

We now pick a point $x \in E$ such that $u_0(x) = 0$. Let $\mathcal{F}_x = \{\xi \in \beta(N \times X): \{u_0\}(\xi) = u_0(x)\}$ and look at the space $\ell^\infty(N, \text{ Re } A)|_{\mathcal{F}_x}$. Since $\{u_0 + u_n\}|_{\mathcal{F}_x} = \{u_n\}|_{\mathcal{F}_x}$, the proof of the theorem of de Leeuw and Katznel-
son shows that \( \{u_n\}^2_{n=1} \subset l^\infty(N, \text{Re } A)\|\mathcal{F}_x \) \( \) (where the closure is taken w.r.t. the supremum norm on \( \mathcal{F}_x \)) if the functions \( u_n \) belong to \( \text{Re } A \), and hence \( l^\infty(N, \text{Re } A)\|\mathcal{F}_x = \mathbb{C}(\mathcal{F}_x) \) by the Stone-Weierstrass Theorem.

By Theorem 1 there is a compact neighbourhood \( K_x \) of \( x \) such that \( \text{Re } A|K_x = \mathbb{C}(K_x) \). Using a theorem of S. Sidney and E. Stout [7] we infer that \( A|K_x = \mathbb{C}(K_x) \). Since \( A \) is a Dirichlet algebra, a theorem of I. Glicksberg [2] shows that \( \nu|K_x = 0 \) for all \( \nu \in A^\perp \), and thus \( \nu|(X\setminus U_x) \in A^\perp \) for all \( \nu \in A^\perp \), where \( U_x \) is the interior of \( K_x \). Referring again to a theorem of I. Glicksberg [2], the set \( X\setminus U_x \) is a \( p \)-set for \( A \) supporting \( \mu \), hence \( E\setminus U_x \) is a \( p \)-set which supports \( \mu \), contradicting the fact that \( E \) is the smallest \( p \)-set with this property. This concludes the proof.

References


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