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THE DE BRANGES–ROVNYAK MODEL

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ABSTRACT. A characterization, extending results of A. Beurling, L. de Branges and J. Rovnyak, of those Hilbert spaces of formal power series, which are isometrically equal to a de Branges-Rovnyak (scalar-valued) function space $\mathscr{H}(b)$, is obtained.

Let $\mathscr{C}(z)$ denote the Hilbert space of square-summable power series $f(z) = \sum a_n z^n$ with complex coefficients such that $\|f(z)\|^2 = \sum |a_n|^2$. For a power series $b(z)$ which converges to a function which is bounded by one in the unit disk, the space $\mathscr{H}(b)$ of L. de Branges and J. Rovnyak [4] is the Hilbert space of series $f(z)$ in $\mathscr{C}(z)$ with the property that

$$\|f(z)\|_b^2 = \sup\{\|f(z) + b(z)g(z)\|^2 - \|g(z)\|^2\}$$

is finite, where the supremum is taken over all elements $g(z)$ of $\mathscr{C}(z)$. If $f(z)$ is in $\mathscr{H}(b)$, then so is the series $[f(z) - f(0)]/z$ and, moreover,

$$\|[f(z) - f(0)]/z\|_b^2 \leq \|f(z)\|_b^2 - |f(0)|^2. \tag{1}$$

The difference-quotient transformation on $\mathscr{H}(b)$ is the contraction $R(0)$ which maps $f(z)$ into $[f(z)-f(0)]/z$. There have been several recent developments [1, 6, 7, 8] with the space $\mathscr{H}(b)$ especially as a canonical model space for contractions on Hilbert spaces.

Let $\mathscr{H}$ be a Hilbert space of formal power series such that the difference-quotient inequality

$$\|[f(z) - f(0)]/z\|_{\mathscr{H}}^2 \leq \|f(z)\|_{\mathscr{H}}^2 - |f(0)|^2 \tag{2}$$

holds for every element $f(z)$, and let T denote the contraction on $\mathscr{H}$ which sends $f(z)$ into $[f(z)-f(0)]/z$. By the well-known invariant subspace theorem of A. Beurling [2], if $\mathscr{H}$ is contained isometrically in $\mathscr{C}(z)$, then $\mathscr{H}$ is a space $\mathscr{H}(b)$ for some inner function b. (Conversely, if b is inner, then $\mathscr{H}(b)$ is contained isometrically in $\mathscr{C}(z)$). More generally, de Branges and Rovnyak showed in [4, Theorem 15] that if equality holds in (2), for every f, and the rank of $1 - TT^*$ is one, then $\mathscr{H} = \mathscr{H}(b)$ isometrically, where b is an extreme

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point of the unit ball in the space of bounded analytic functions of the unit disk. (Conversely, if b is an extreme point, then equality holds in (1) for every f, and the ranges of $1 - R(0)R(0)^*$ and $1 - R(0)^*R(0)$ are spanned by $[b(z) - b(0)]/z$ and $1 - b(z)\overline{b(0)}$ respectively). In this note, we obtain a similar result for spaces $\mathscr{H}(b)$ where b is not an extreme point, which answers a question raised in [4, p. 39].

Suppose that b is nonconstant and not an extreme point. Then the range of $1 - R(0)R(0)^*$ is spanned by $[b(z) - b(0)]/z$, and the range of $1 - R(0)^*R(0)$ is spanned by 1 and b. Moreover, it follows directly from [8, Lemmas 1-3] that if $f(z) = [1 - b(z)\overline{b(0)}]\|b(z)\|_b^2 - b(z)\overline{b(0)}$, then $f(0) \neq 0$ and equality holds in (1) for f. These properties are sufficient to characterize $\mathscr{H}(b)$.

Theorem. *Let $\mathscr{H}$ be a Hilbert space of formal power series which satisfies* (2), *and suppose that equality holds in* (2) *for some element with nonzero constant coefficient. If the ranks of* $1 - TT^*$ *and* $1 - T^*T$ *are* 1 *and* 2 *respectively, where T is the difference-quotient transformation on $\mathscr{H}$, then $\mathscr{H}$ is isometrically equal to a space $\mathscr{H}(b)$ where b is not an extreme point.*

The following example shows that the above hypothesis on equality in (2) may not be omitted.

Example. Let ϕ be a nonconstant bounded analytic function, and, for $r \geq 1$, let $\mathscr{H} = \mathscr{C}(z)$ with the inner product given by

$$\langle f, g\rangle_{\mathscr{H}} = \langle (r + T_\phi T_\phi^*)f, g\rangle,$$

where T_ϕ denotes multiplication by ϕ on $\mathscr{C}(z)$. It is easily checked that $\mathscr{H}$ is a Hilbert space and, for every f, we have that

$$\begin{aligned}\|[f(z) - f(0)]/z\|_{\mathscr{H}}^2 &= \|f\|_{\mathscr{H}}^2 - |f(0)|^2 - (r-1)|f(0)|^2 - |(T_\phi^* f)(0)|^2 \\ &\leq \|f\|_{\mathscr{H}}^2 - |f(0)|^2.\end{aligned}$$

Furthermore, the range of $1 - TT^*$ is spanned by $[\phi(z) - \phi(0)]/z$, and the range of $1 - T^*T$ is spanned by 1 and ϕ. However, equality holds in (2) for f if and only if $(r-1)\langle f, 1\rangle = 0 = \langle f, \phi\rangle$. Hence, by the theorem, $\mathscr{H} = \mathscr{H}(b)$ isometrically if and only if $r = 1$.

The key to the verification of the theorem is the theory of complementation [3] of de Branges: A Hilbert space $\mathscr{P}$ is contained contractively in a Hilbert space $\mathscr{K}$ if $\mathscr{P}$ is a vector subspace of $\mathscr{K}$ and if the inclusion map $i_{\mathscr{P}}$ of $\mathscr{P}$ into $\mathscr{K}$ is a contraction. If $\mathscr{P}$ is contained contractively in $\mathscr{K}$, then the space complementary to $\mathscr{P}$ in $\mathscr{K}$ is the Hilbert space $\mathscr{Q}$ of elements g of $\mathscr{K}$ with the property that

$$\|g\|_{\mathscr{Q}}^2 = \sup\{\|g + f\|_{\mathscr{K}}^2 - \|f\|_{\mathscr{P}}^2\}$$

is finite, where the supremum is taken over all f in $\mathscr{P}$. The space $\mathscr{Q}$ is contained contractively in $\mathscr{K}$ and is the unique Hilbert space such that the

inequality $\|h\|^2_{\mathscr{K}} \leq \|f\|^2_{\mathscr{P}} + \|g\|^2_{\mathscr{Q}}$ holds whenever $h = f + g$ is a decomposition of h in $\mathscr{K}$ into f in $\mathscr{P}$ and g in $\mathscr{Q}$, and such that every h in $\mathscr{K}$ admits the unique decomposition $h = (i^*_{\mathscr{P}} h) + (i^*_{\mathscr{Q}} h)$, where $\|h\|^2_{\mathscr{K}} = \|i^*_{\mathscr{P}} h\|^2_{\mathscr{P}} + \|i^*_{\mathscr{Q}} h\|^2_{\mathscr{Q}}$.

By (2), $\mathscr{H}$ is contained contractively in $\mathscr{C}(z)$. Moreover, if $\mathscr{M}$ is the complementary space to $\mathscr{H}$ in $\mathscr{C}(z)$, then multiplication by z is defined and bounded by one on $\mathscr{M}$. The essential result for us will be the following corollary of de Branges's extension [3, Theorem 15] of Beurling's theorem: if multiplication by z is an isometry on $\mathscr{M}$, then $\mathscr{H}$ is isometrically equal to a space $\mathscr{H}(b)$.

To begin the proof of the theorem, let us first observe that T is onto: If the kernel of T^* is nontrivial, then it must coincide with the range of $1 - TT^*$ since the rank of $1 - TT^*$ is 1. However, in this case, T^* and hence T are partial isometries so that $\ker T = \operatorname{range}(1 - T^*T)$ is two-dimensional, which is impossible. Thus the range of T is dense in $\mathscr{H}$. But T has closed range since T^*T has closed range (T^*T is the orthogonal direct sum of the identity operator on the kernel of $1 - T^*T$ and an operator on the finite-dimensional range of $1 - T^*T$). Therefore, $\mathscr{H} = T\mathscr{H}$.

Note that constants belong to $\mathscr{H}$ since $T(1 - T^*T) = (1 - TT^*)T$ and $\operatorname{rank}(1 - TT^*) < \operatorname{rank}(1 - T^*T)$. Hence, since T is onto, $\mathscr{H}$ has the property that $zf(z)$ is in $\mathscr{H}$ whenever $f(z)$ is in $\mathscr{H}$.

Consider the set $\mathscr{D}$ of those elements of $\mathscr{H}$ for which equality holds in (2). By (2) and the parallelogram law, $\mathscr{D}$ is a vector subspace of $\mathscr{H}$; since T is continuous and $\mathscr{H}$ is contained contractively in $\mathscr{C}(z)$, $\mathscr{D}$ is closed. Also by (2), $\mathscr{D}$ contains the kernel of $1 - T^*T$; by assumption, $\mathscr{D}$ contains an element with nonvanishing constant coefficient which we may assume belongs to the range of $1 - T^*T$.

Let us suppose first that 1 does not belong to $\mathscr{D}$. Since $\mathscr{H}$ is thus the orthogonal direct sum of the kernel of $1 - T^*T$ and the span of 1 and some element of $\mathscr{D}$, we have that $\mathscr{H} = T\mathscr{H} = T\mathscr{D}$. Therefore, for g in the space $\mathscr{M}$, complementary to $\mathscr{H}$ in $\mathscr{C}(z)$,

$$\begin{aligned}
\|g\|^2_{\mathscr{M}} &= \sup\{\|g + Tf\|^2 - \|Tf\|^2_{\mathscr{H}} : f \in \mathscr{D}\} \\
&= \sup\{(\|zg + f\|^2 - |f(0)|^2) - (\|f\|^2_{\mathscr{H}} - |f(0)|^2) : f \in \mathscr{D}\} \\
&= \sup\{\|zg + f\|^2 - \|f\|^2_{\mathscr{H}} : f \in \mathscr{D}\} \\
&\leq \|zg\|^2_{\mathscr{M}}.
\end{aligned}$$

As above, since the reverse inequality holds, multiplication by z is an isometry on $\mathscr{M}$, and hence the theorem follows in this case.

Next, suppose that 1 belongs to $\mathscr{D}$. There exists an integer $n \geq 1$ such that $\mathscr{D}$ contains $z^m (m = 0, \ldots, n-1)$ but not z^n. (Otherwise $i^*_{\mathscr{H}} z^m = z^m$ for every $m \geq 0$, and consequently $\mathscr{H} = \mathscr{C}(z)$ isometrically. But then T^* would be multiplication by z on $\mathscr{C}(z)$ so that $1 - TT^* = 0$, contradicting its rank hypothesis.) It follows that $T^{*n} 1 \neq z^n$ and $(1 - TT^*) z^{n-1} \neq 0$,

but if $n > 1$ then $T^{*m}1 = z^m$ and $(1 - TT^*)z^{m-1} = 0$, for every $m = 1, \dots, n-1$ ($(1-T^*T)z^n$ is nonzero and orthogonal to the kernel of T since $\langle (1 - T^*T)z^n, 1\rangle_{\mathscr{H}} = \langle z^n, 1\rangle_{\mathscr{H}} = \langle z^n, i^*_{\mathscr{H}}1\rangle_{\mathscr{H}} = \langle i_{\mathscr{H}}z^n, 1\rangle = \langle z^n, 1\rangle = 0$. Therefore $(1 - TT^*)z^{n-1} = T(1 - T^*T)z^n \neq 0$.).

The $\mathscr{M}$-norm may be simplified in this case. Specifically, we will show that if f is in $\mathscr{H}$ then there exists an element $\hat{f}$ in the kernel of $1 - T^nT^{*n}$ such that, for every g in $\mathscr{M}$,

$$\|g + f\|^2 - \|f\|^2_{\mathscr{H}} = \|g + \hat{f}\|^2 - \|\hat{f}\|^2_{\mathscr{H}}.$$

Let f be in $\mathscr{H}$, and define constants $c_0, \dots, c_{n-1}$ recursively as follows:

$$c_{n-1} = -\langle f, (1 - TT^*)z^{n-1}\rangle_{\mathscr{H}} / \langle z^{n-1}, (1 - TT^*)z^{n-1}\rangle_{\mathscr{H}};$$

if $n > 1$, use

$$c_{n-k} = -\left\langle f + \sum_{n-k+1}^{n-1} c_m z^m, T^{k-1}(1 - TT^*)T^{*^{k-1}} z^{n-k}\right\rangle_{\mathscr{H}} \div \langle z^{n-k}, T^{k-1}(1 - TT^*)T^{*^{k-1}} z^{n-k}\rangle_{\mathscr{H}},$$

for $k = 2, \dots, n$. By the previous paragraph, $T^{*^{k-1}} z^{n-k} = z^{n-1}$, and furthermore $\hat{f} = f + \sum_0^{n-1} c_m z^m$ is orthogonal to the range of $T^{k-1}(1 - TT^*)T^{*^{k-1}}$, which is spanned by $T^{k-1}(1 - TT^*)T^{*^{k-1}} z^{n-k}$, for every $k = 1, \dots, n$. Since $1 - T^nT^{*n} = \sum_1^n T^{k-1}(1 - TT^*)T^{*^{k-1}}$, we have that $\hat{f}$ is orthogonal to the range of $1 - T^nT^{*n}$.

Let g be in $\mathscr{M}$. Then $\langle g, z^m\rangle = \langle i_{\mathscr{M}}g, z^m\rangle = \langle g, i^*_{\mathscr{M}}z^m\rangle_{\mathscr{M}} = \langle g, 0\rangle_{\mathscr{M}} = 0$ for every $m = 0, \dots, n-1$, so the first n coefficients of g vanish. Let us write $f = (\sum_0^{n-1} a_m z^m) + z^nT^nf$ for some constants a_m and observe that $\|f\|^2_{\mathscr{H}} = \sum_0^{n-1} |a_m|^2 + \|z^nT^nf\|^2_{\mathscr{H}}$ because $i^*_{\mathscr{H}}z^m = z^m$, $m = 0, \dots, n-1$. Therefore,

$$\|g + f\|^2 - \|f\|^2_{\mathscr{H}} = \|g + z^nT^nf\|^2 - \|z^nT^nf\|^2_{\mathscr{H}},$$

and the same expression holds for $\|g + \hat{f}\|^2 - \|\hat{f}\|^2_{\mathscr{H}}$. Thus

$$\|g\|^2_{\mathscr{M}} = \sup\{\|g + \hat{f}\|^2 - \|\hat{f}\|^2_{\mathscr{H}}\},$$

where the supremum is taken over all elements $\hat{f}$ of the kernel of $1 - T^nT^{*n}$. Moreover, for these elements, $\|\hat{f}\|_{\mathscr{H}} = \|z^n\hat{f}\|_{\mathscr{H}}$ since $(1-T^{*n}T^n)z^n\hat{f}$ is simultaneously in the kernel of T^n and orthogonal to the kernel of T^n. Indeed, for $m = 0, \dots, n-1$, $\langle (1 - T^{*n}T^n)z^n\hat{f}, z^m\rangle_{\mathscr{H}} = \langle z^n\hat{f}, z^m\rangle_{\mathscr{H}} = \langle z^n\hat{f}, i^*_{\mathscr{H}}z^m\rangle_{\mathscr{H}} = \langle z^n\hat{f}, z^m\rangle = 0$. Consequently,

$$\|g\|^2_{\mathscr{M}} = \sup\{\|g + \hat{f}\|^2 - \|z^n\hat{f}\|^2_{\mathscr{H}} : (1 - T^nT^{*n})\hat{f} = 0\}.$$

Finally, since multiplication by z is bounded by one on $\mathscr{M}$, and $\mathscr{H} = T^n\mathscr{H}$, it follows that

$$\begin{aligned}
\|z^n g\|^2_{\mathscr{M}} &\le \|zg\|^2_{\mathscr{M}} \\
&\le \|g\|^2_{\mathscr{M}} \\
&= \sup\{\|g + T^n f\|^2 - \|z^n T^n f\|^2_{\mathscr{H}} : (1 - T^n T^{*n})T^n f = 0\} \\
&= \sup\{\|z^n g + f\|^2 - \|f\|^2_{\mathscr{H}} : (1 - T^n T^{*n})T^n f = 0\} \\
&\le \|z^n g\|^2_{\mathscr{M}}.
\end{aligned}$$

Therefore, multiplication by z is isometric on $\mathscr{M}$.

Remark. The referee has kindly pointed out that other equivalent conditions for $\mathscr{H} = \mathscr{H}(b)$ isometrically were recently obtained by R. B. Leech [5].

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