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AN INEQUALITY WITH APPLICATIONS TO THE SUBELLIPTICITY OF THE $\bar{\partial}$-NEUMANN PROBLEM

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ABSTRACT. We prove an interesting inequality in this note. This inequality will be used to remove an unnecessary assumption in [2]. That paper dealt with the sufficient condition for the subellipticity of the $\bar{\partial}$-Neumann problem on nonpseudoconvex domains. We will then state the revised theorem and show why the original assumption can be removed.

Proposition 1. *Let $\Omega \subset \mathbb{C}^n$, $x_0 \in b\Omega$, and $L \in T^{1,0}(b\Omega)$. Then given any $\varepsilon > 0$, there exists a neighbourhood U of x_0 and $C > 0$ such that for all u, $v \in C_0^\infty(U \cap \overline{\Omega})$ we have*

$$|(Lu, v)| \leq \varepsilon(\|\overline{L}u\|^2 + \|Lv\|^2) + C(\|u\|^2 + \|v\|^2).$$

Proof. We know that the adjoint L^* of L is given by

$$L^* = -\overline{L} + g, \tag{1}$$

where g is smooth. Hence

$$L + L^* = L - \overline{L} + g.$$

By a change of coordinates we may assume that

$$L - \overline{L} = -ia(x)\frac{\partial}{\partial x_1}$$

where $a(x) > 0$ in U. Hence the symbol of $L - \overline{L}$ is given by

$$\sigma(L - \overline{L}) = a(x)\xi_1.$$

We define operators P^+, P^-, and P^0 as follows:

$$\widehat{P^+u}(\xi, r) = \chi_1(\xi_1)\hat{u}(\xi, r)$$
$$\widehat{P^-u}(\xi, r) = \chi_2(\xi_1)\hat{u}(\xi, r)$$
$$P^0 = I - P^+ - P^-,$$

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where χ_1, $\chi_2 \in C^\infty(\mathbb{R})$, $0 \le \chi_i \le 1$, $i = 1, 2$, and

$$\chi_1(\xi) = \begin{cases} 1 & \xi \ge 1, \\ 0 & \xi < 0, \end{cases}$$

$$\chi_2(\xi) = \begin{cases} 0 & \xi > 0, \\ 1 & \xi \le -1. \end{cases}$$

We now define

$$\{u, v\}^+ = \langle (L + L^*)P^+u, P^+v\rangle + K\langle P^+u, P^+v\rangle,$$

where K is a large positive constant.

Lemma 2. *$\{u, v\}^+$ is an inner product for large K.*

Proof. Since $a(x)\xi_1$ is nonnegative on the support of χ_1, by Garding's inequality we have

$$\langle (L - \overline{L})P^+u, P^+u\rangle \ge -C\langle P^+u, P^+u\rangle.$$

Hence for some large K we have $\{u, u\}^+ \ge 0$.

It is easy to see that $\{u, u\}^+ = 0$ if and only if $u = 0$, and that $\{u, v\}^+ = \overline{\{v, u\}^+}$.

Since $\{u, v\}^+$ is an inner product, by Schwarz's inequality

$$|\{u, v\}^+| \le (\{u, u\}^+)^{1/2}(\{v, v\}^+)^{1/2}.$$

It is easily seen that

$$\begin{aligned} \{u, u\}^+ &= \langle (L + L^*)P^+u, P^+u\rangle + K\langle P^+u, P^+u\rangle \\ &\le \varepsilon\|\overline{L}u\|^2 + C\|u\|^2 \end{aligned} \tag{2}$$

by (1). Similarly,

$$\{v, v\}^+ \le \varepsilon\|Lv\|^2 + C\|v\|^2. \tag{3}$$

Hence

(4)
$$\begin{aligned} \langle LP^+u, P^+v\rangle &= \{u, v\}^+ - \langle L^*P^+u, P^+v\rangle - K\langle P^+u, P^+v\rangle \\ &= \{u, v\}^+ + \langle P^+u, LP^+v\rangle - \langle gP^+u, P^+v\rangle - K\langle P^+u, P^+v\rangle, \end{aligned}$$

and we get from (2), (3), and (4) that

$$|\langle LP^+u, P^+v\rangle| \le \varepsilon(\|\overline{L}u\|^2 + \|Lv\|^2) + C(\|u\|^2 + \|v\|^2). \tag{5}$$

Similarly, we define

$$\{u, v\}^- = -\langle (L + L^*)P^-u, P^-v\rangle + K\langle P^-u, P^-v\rangle.$$

Just as above, $\{u, v\}^-$ is an inner product, and again

$$|\langle LP^-u, P^-v\rangle| \le \varepsilon(\|\overline{L}u\|^2 + \|Lv\|^2) + C(\|u\|^2 + \|v\|^2).$$

Finally,

(7)
$$\begin{aligned}\langle Lu\,,\,v\rangle = \langle LP^{+}u\,,\,P^{+}v\rangle + \langle LP^{-}u\,,\,P^{-}v\rangle + \langle LP^{+}u\,,\,P^{0}v\rangle + \langle LP^{+}u\,,\,P^{-}v\rangle \\ + \langle LP^{-}u\,,\,P^{+}v\rangle + \langle LP^{-}u\,,\,P^{0}\rangle + \langle LP^{0}u\,,\,P^{+}v\rangle + \langle LP^{0}u\,,\,P^{0}v\rangle \\ + \langle LP^{0}u\,,\,P^{-}v\rangle\,.\end{aligned}$$

To deal with the third to ninth terms on the right-hand side of (7), we need the facts that

$$\|[L\,,\,P]u\|^{2} \leq \text{const}\,\|u\|^{2}\,,$$

where $P = P^{+}$, P^{-}, or P^{0} and that

$$\|(L - \overline{L})Pu\|^{2} \leq \text{const}\,\|u\|^{2}$$

if the symbol of P is a compactly supported function of ξ_1.

Thus, combining (5), (6), (7), and the above two facts, we get

$$|(Lu\,,\,v)| \leq \varepsilon(\|\overline{L}u\|^{2} + \|Lv\|^{2}) + C(\|u\|^{2} + \|v\|^{2})\,.$$

We can now restate Theorem 2.2 in [2]. We refer the reader to [2] for the details and for definitions of $A^{(k)}$ and I_k^q. We assume that the reader is familiar with the $\overline{\partial}$-Neumann problem and subelliptic estimates. A detailed formulation of the problem can be found in [1] or [3].

Theorem 3. *Let Ω be a domain in $\mathbb{C}^n$ with C^∞ boundary, $x_0 \in b\Omega$, and $L_1, \dots, L_n$ a C^∞ basis for $T^{1,0}$ so that $L_1, \dots, L_{n-1}$ are tangential on $b\Omega$. Assume that there exists a neighborhood U of x_0 such that for some $k \geq n-q$ the matrix $A^{(k)}$ associated with the matrix of the Levi form is positive semidefinite in U, then if $1 \in I_m^q(x_0)$ for some m, there is a subelliptic estimate for $(p\,,\,q)$ forms at x_0.*

The following extra assumption is made in Theorem 2.2 in [2]:

For all $\varepsilon > 0$, there exists $C > 0$ (C depends on S^0 but not on ϕ) such that

$$|\langle D\phi_I\,,\,S^0\phi_J\rangle| \leq \varepsilon(\|\overline{\partial}\phi\|^{2} + \|\overline{\partial}^{*}\phi\|^{2}) + C\|\phi\|^{2}$$

for all $\phi \in D_U^{p,q}$ where $D \in \{\overline{L_{k+1}}, \dots, \overline{L_{n-1}}\}$, S^0 is a tangential pseudodifferential operator of order zero.

In [2] this assumption is used to verify the inequality

$$\tag{9} \begin{aligned}\sum_{\substack{j\notin J \text{ or} \\ j\in\{1,\dots,k,n\}}} \|\overline{L}_j\phi_J\|^{2} + \sum_{\substack{j\in J \text{ and} \\ k<j<n}} \|L_j\phi_J\|^{2} + \sum_{I,J}\int_{b\Omega} A_{IJ}^{(k)}\phi_I\overline{\phi_J}\,dS \\ \leq C(\|\overline{\partial}\phi\|^{2} + \|\overline{\partial}^{*}\phi\|^{2} + \|\phi\|^{2})\end{aligned}$$

for all $\phi \in D_U^{p,q}$ from the inequality

$$(10)\quad \sum_{\substack{j\notin J \text{ or} \\ j\in\{1,\dots,k,n\}}} \|\overline{L}_j\phi_J\|^2 + \sum_{\substack{j\in J \text{ and} \\ k<j<n}} \|L_j\phi_J\|^2 + \sum_{I,J}\int_{b\Omega} A_{IJ}^{(k)}\phi_I\overline{\phi_J}\,dS + R(\phi) \leq C(\|\overline{\partial}\phi\|^2 + \|\overline{\partial}^*\phi\|^2 + \|\phi\|^2),$$

where $R(\phi) = \sum_{j=1}^n \langle \overline{L}_j\phi_J, h_j\phi_K\rangle + O(\|\phi\|^2)$ for some smooth functions h_j.

We prove the following to remove assumption (8):

Lemma 4. *The inequality* (9) *is true without assumption* (8).

Proof. We want to prove (9) using (10) and Proposition 1. Clearly what we need to prove is that for all $j = 1, 2, \dots, n$,

$$(11)\quad |\langle \overline{L}_j\phi_J, h_j\phi_K\rangle| \leq \varepsilon\left(\sum_{\substack{j\notin J \text{ or} \\ j\in\{1,\dots,k,n\}}} \|\overline{L}_j\phi_J\|^2 + \sum_{\substack{j\in J \text{ and} \\ k<j<n}} \|L_j\phi_J\|^2\right) + C\|\phi\|^2.$$

When $j \in \{1, 2, \dots, k, n\}$, we have

$$(12)\qquad |\langle \overline{L}_j\phi_J, h_j\phi_K\rangle| \leq \varepsilon\|\overline{L}_j\phi_J\|^2 + C\|\phi\|^2.$$

When $j \in \{k+1, \dots, n-1\}$, if $j \notin J$ or $j \in K$, then $\|\overline{L}_j\phi_J\|^2$ or $\|L_j\phi_K\|^2$ is in the right-hand side of (11). Hence, using the type of inequality in (12) or by integrating $\overline{L}_j$ by parts on the left-hand side, we can absorb the term $\langle \overline{L}_j\phi_J, h_j\phi_K\rangle$ in the right-hand side of (11). Finally, when $j \in J$ and $j \notin K$, we use Proposition 1, and we have

$$|\langle \overline{L}_j\phi_J, h_j\phi_K\rangle| \leq \varepsilon(\|\overline{L}_j\phi_J\|^2 + \|L_j\phi_J\|^2) + C\|\phi\|^2.$$

We see that the terms in the right-hand side of the above inequality are in the right-hand side of (11). This finishes the proof.

References

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