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A PRECOMPOSITION ANALYSIS OF LINEAR OPERATORS ON ℓ^p

STEVE M. HUDSON

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ABSTRACT. Given a function g, the operator that sends the function $f(x)$ to the function $f(g(x))$ is called a precomposition operator. If g preserves measure on its domain, at least approximately, then this operator is bounded on all the L^p spaces. We ask which operators can be written as an average of precomposition operators. We give sufficient, almost necessary conditions for such a representation when the domain is a finite set. The class of operators studied approximate many commonly used positive operators defined on L^p of the real line, such as maximal operators.

A major tool is the combinatorial theorem of distinct representatives, commonly called the marriage theorem. A strong connection between this theorem and operators of weak-type 1 is demonstrated.

A very basic kind of linear operator on $L^p(R)$ is that of the form $Tf(x) = f(\phi(x))$, which we will call a precomposition operator. It is an isometry on L^p if ϕ preserves Lebesgue measure on R, and, likewise, by a theorem of Banach and Lamperti (see [4]),

Theorem. *If T is an isometry on $L^p(R)$ for some $p \neq 2$, with $1 \leq p < \infty$, then there are unique functions $\phi\colon \mathbf{R} \to \mathbf{R}$ and $h\colon \mathbf{R} \to \mathbf{C}$ such that*

$$Tf(x) = h(x)f(\phi(x)) \quad \textit{for all } x.$$

Of course, the Hilbert transform provides a counterexample to the case $p = 2$. Even if ϕ only preserves measure up to some constant, then $T\colon f \to f \circ \phi$ is still bounded on L^p. The aim of this paper is to show that the common positive linear operators of harmonic analysis, such as the maximal operator, can be written as averages of such basic operators. One immediate consequence is boundedness on L^p.

For technical reasons, this aim is not quite fulfilled; we must work on sequences in ℓ^p instead of on L^p (as in the original Hardy-Littlewood paper [2]). Of course, all the usual operators on L^p have analogues in ℓ^p that are

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equivalent for the purpose of studying boundedness. The advantage of ℓ^p is the possibility of using inductive and combinatorial methods. Before turning to general operators, we will analyze the maximal operator of [2].

We let $[1, N] = \{1, 2, 3, \ldots, N\}$ (which should be viewed as a substitute for R; no crucial inequality will depend on the size of N). For $A \subset [1, N]$, $\#A$ is the number of elements of A. If ϕ is a function from $[1, N]$ to itself, we say ϕ is *at most* C *to one* if $\#\phi^{-1}(k) \leq C$ for all $k \in [1, N]$. If $f(j)$, $j \in [1, N]$ is a sequence of positive real numbers, then the maximal operator applied to f is

$$Mf(j) = \max_{r>0} \frac{1}{r} \sum_{i=1}^{r} f(j-i)$$

for $j \in [1, N]$, where $f(k) = 0$ by convention if $k < 1$. Of course, M is sublinear. But if the r have been chosen (for each j) and fixed, then M may be regarded as a linear operator, called a *linearization of* M, which may be applied to any other sequence $g(j)$.

Theorem 1. *For any positive sequence* f *as above, there is a double sequence* $\phi(i, j)$ *so that*

$$Mf(j) = \frac{1}{N!} \sum_{i=1}^{N!} f(\phi(i, j))$$

for every $j \in [1, N]$. *For any fixed* i, *the function* $j \to \phi(i, j)$ *is at most* $3N!/i$ *to one.*

The choice of ϕ depends on the values of r, that is, on the linearization of M, but otherwise does not depend on f. The theorem does not extend in any clear way to functions on R, but the following well-known corollary does, by approximation in L^p.

Corollary. M *is bounded on* ℓ^p *for* $p > 1$, *that is*:

$$\left(\sum_{j=1}^{N} Mf(j)^p\right)^{1/p} \leq \frac{3p}{p-1}\left(\sum_{j=1}^{N} f(j)^p\right)^{1/p}.$$

Proof. The basic idea is that M is an average of operators bounded on ℓ^p (even for $p = 1$, though the bounds are not summable in this case). To be more precise, we apply the theorem, then Minkowski's inequality:

$$\begin{aligned}\left(\sum_{j=1}^{N} Mf(j)^p\right)^{1/p} &= \left(\sum_{j=1}^{N}\left[\frac{1}{N!}\sum_{i=1}^{N!} f(\phi(i,j))\right]^p\right)^{1/p} \\ &\le \frac{1}{N!}\sum_{i=1}^{N!}\left[\sum_{j=1}^{N} f(\phi(i,j))^p\right]^{1/p} \\ &\le \frac{1}{N!}\sum_{i=1}^{N!}\left((3N!/i)\sum_{k=1}^{N} f(k)^p\right)^{1/p} \\ &\le \frac{1}{N!}\sum_{i=1}^{N!}(3N!/i)^{1/p}\left[\sum_{k=1}^{N} f(k)^p\right]^{1/p} \\ &\le \frac{3p}{p-1}\left[\sum_{k=1}^{N} f(k)^p\right]^{1/p}\end{aligned}$$

which proves the corollary. □

Proof of Theorem 1. Let $\{r_j\}$ linearize Mf. So $Mf(j) = (1/r_j)\sum_{i=1}^{r_j} f(j-i)$, and we may assume that $r_j \le N$ for each j. Let $[x]$ denote the greatest integer less than or equal to x. Since $N!$ is divisible by r_j,

$$\begin{aligned} Mf(j) &= \frac{1}{N!}\sum_{i=1}^{N!} f\left(\left[j - \frac{r_j i}{N!}\right]\right) \\ &= \frac{1}{N!}\sum_{i=1}^{N!} f\left(\left[j - r_j\frac{\psi(i,j)}{N!}\right]\right),\end{aligned}$$

where $\psi(\cdot, j)$ is any bijection of $[1, N!]$. To prove the theorem, it is enough to construct ψ for $j = 1, 2, \ldots, N$ so that

$$i \to \psi(i, j) \quad \text{is one to one for each } j$$

and

$$j \to \phi(i,j) = [j - r_j\psi(i,j)/(N!)] \quad \text{is at most } \frac{3N!}{i} \text{ to one for each } i.$$

Fix j and suppose this has been done for all $j' < j$. For each $i \in [1, N!]$, $m \in [1, N]$, let $\gamma(i, m) = \#\{j' < j : \phi(i, j') = m\}$. By hypothesis, this is at most $3N!/i$. Let G_{ij} be the set of "good" candidates for $\phi(i, j)$ and B_{ij} the "bad":

$$\begin{aligned} G_{ij} &= \left\{m \in [j - r_j, j-1] : \gamma(i, m) \le \frac{2N!}{i}\right\} \\ B_{ij} &= [j - r_j, j-1] - G_{ij}.\end{aligned}$$

Applying Chebyshev's inequality to $\gamma(i, m)$,

$$\begin{aligned}
\#B_{ij} &\le \frac{i}{2N!} \sum_{m=j-r_j}^{j-1} \gamma(i, m) \\
&= \frac{i}{2N!} \#\{j' < j : \phi(i, j') \in [j - r_j, j-1]\} \\
&\le \frac{i}{2N!} \#\{j' < j : j' \ge j - r_j\} \\
&= \frac{i r_j}{2N!}.
\end{aligned}$$

Then $\#G_{ij} \ge r_j - \#B_{ij} \ge r_j(1 - i/(2N!))$. We choose values for $\psi(i, j)$ one at a time, in the order $i = N!, N! - 1, \dots, 1$. Since $\psi(\cdot, j)$ is to be a bijection, we cannot choose any value in $[1, N!]$ twice. So, given i, let $\psi(i, j)$ be an unchosen number in $[1, N!]$ such that $\phi(i, j) \in G_{ij}$. This is possible because each element of G_{ij} corresponds to one possible $\phi(i, j)$ and $N!/r_j$ possibilities for $\psi(i, j)$. $N! - i$ choices are already taken. So, the number of allowable choices for ψ is always at least

$$\begin{aligned}
\frac{N!}{r_j} \#G_{ij} - (N! - i) &\ge N!(1 - i/2N!) - N! + i \\
&= i(1 - 1/2) = i/2 > 0.
\end{aligned}$$

Thus, it is possible to make such a choice for each i. This constructs a bijection $\psi(\cdot, j)$ on $[1, N!]$. To prove that the second required condition on ψ still holds, it is enough to check that $\gamma(i, m) \le 3N!/i$ still holds for all i, m. But if $\gamma(i, m)$ is increased (by at most 1) during the construction, then it was because $\phi(i, j) = m$, which implies $m \in G_{ij}$. So $\gamma(i, m)$ was at most $2N!/i$ before. And now, $\gamma(i, m) \le 2N!/i + 1 \le 3N!/i$, as desired, and the construction may continue to $j + 1, \dots, N$. This completes the proof. □

The classical marriage theorem (or, theorem of distinct representatives; see [1]) concerns matching up elements of two finite sets A and B of the same size. The following version applies to the case $\#A \le \#B$ and is easily proved from the original by adding dummy elements to A. It is useful in the proof of Theorem 2.

The marriage theorem. *Let A and B be finite sets and $G \subset A \times B$. There is a one-to-one function $\theta: A \to B$ such that $\{(a, \theta(a)) : a \in A\} \subset G$ if and only if $\# \bigcup_{j \in J} \{b \in B : (j, b) \in G\} \ge \#J$ for every $J \subset A$.*

As for the more usual operators on $L^1(R)$ (see [5]), we say that an operator T is *weak-type* 1 if there is a constant C so that, for every $\alpha > 0$ and every sequence f,

$$\#\{k : Tf(k) \ge \alpha\} \le \frac{C\|f\|_{\ell^1}}{\alpha}.$$

We will adopt the conventions that N is some large integer, that $f(j) = 0$ for $j \notin [1, N]$, and that $f(\infty) = 0$ for every f.

In the following theorem, linear operators are analyzed as sums of pre-composition operators. The class of operators T and sequences f is general enough to approximate most of the classical positive operators and functions of harmonic analysis in $L^p(\mathbf{R})$, such as the linearized maximal function in Theorem 1. Decomposing convolution operators is rather trivial however; Theorem 2 is mainly of interest when T is not translation-invariant. Certain fractional integration operators fit into this category, for example (see [3]). This class does not approximate singular integral operators, which have non-positive kernels, and also are not bounded on L^∞.

Theorem 2. *Let T be the linear operator with kernel K;*

$$Tf(j) = \sum_{i=1}^{N} K(i, j) f(i)$$

for $j = 1, 2, 3, \dots, N$. Suppose, for every i, j, that $K(i, j)$ is a positive rational number. Suppose that $\|Tf\|_{\ell^\infty} \le \|f\|_{\ell^\infty}$ for all f and that T is weak-type 1 with constant C. Then there is an integer M and a function ϕ from $[1, M] \times [1, N]$ to $[1, N] \cup \{\infty\}$ such that

(1) *$Tf(j) = (1/M) \sum_{i=1}^{M} f(\phi(i, j))$ for all f and*
(2) *$j \to \phi(i, j)$ is at most CM/i to one.*

Proof. The ℓ^∞ inequality above, with $f \equiv 1$, shows that $\sum_{i=1}^{N} K(i, j) \le 1$ for all j. Let M be a common denominator of all the rational values of K, and also a multiple of $N!$. For each fixed j, there is a function $\psi(\cdot, j)$ from $[1, M]$ to $[1, N] \cup \{\infty\}$ such that $\#\{k : \psi(k, j) = i\} = MK(i, j)$ for all $i \in [1, N]$. We can construct ψ as follows: Let $\psi(k, j) = 1$ if $0 < k \le MK(1, j)$ and, in general, let $\psi(k, j) = i$ if $M \sum_{\ell=1}^{i-1} K(\ell, j) < k \le M \sum_{\ell=1}^{i} K(\ell, j)$. If $k > M \sum_{i=1}^{N} K(i, j)$, then set $\psi(k, j) = \infty$. This construction guarantees that

$$\begin{aligned} M \cdot Tf(j) &= \sum_{i=1}^{N} MK(i, j) f(i) \\ &= \sum_{i=1}^{N} \#\{k : \psi(k, j) = i\} \cdot f(i) \\ &= \sum_{k=1}^{M} f(\psi(k, j)), \end{aligned}$$

so ψ satisfies condition (1) of the theorem. However, ψ is unlikely to satisfy the second condition. So, as in the proof of Theorem 1, it must be rearranged in the k variable to form a ϕ that does.

We construct, for each $j \in [1, N]$, a bijection $\eta_j(i)$ on $[1, M]$ such that $j \to \psi(\eta_j(i), j) = \phi(i, j)$ is CM/i to one. We proceed in the order $i = M$,

$i = M - 1, \dots, i = 1$ (handling all the j at once in each stage). Fix $i \leq M$ and suppose $\eta_j(i)$ defined for all j and all $i' > i$. In order to define $\eta_j(i)$ for all j, let $F_{ij} = \{\eta_j(i'): i' > i\}$. Notice $\#F_{ij} = M - i$. Let

$$G_{ij} = \{\psi(h, j): h \notin F_{ij}\}.$$

For $J \subset [1, N]$, let

$$E_{iJ} = \bigcup_{j \in J} G_{ij}.$$

If $j \in J$ then

$$\begin{aligned} T\chi_{E_{iJ}}(j) &= \frac{1}{M}\sum_{k=1}^{M} \chi_{E_{iJ}}(\psi(k, j)) \\ &= \frac{1}{M}\#\{k \in [1, M]: \psi(k, j) \in E_{iJ}\} \\ &\geq \frac{1}{M}\#\{k \in [1, M]: \psi(k, j) \in G_{ij}\} \\ &\geq \frac{1}{M}\#\{k \in [1, M]: k \notin F_{ij}\} \\ &= \frac{i}{M}. \end{aligned}$$

Therefore, using the weak-type 1 hypothesis,

$$\#J \leq \#\left\{j: T\chi_{E_{iJ}}(j) \geq \frac{i}{M}\right\} \leq \frac{CM}{i}\#E_{iJ}.$$

Referring to the marriage theorem, let $A = [1, N]$, $B = [1, N] \times [1, CM/i]$, and $G = \{(j, k, l) \in A \times B: k \in G_{ij}\}$. Notice that

$$\begin{aligned} \# \cup_{j \in J} \{b \in B: (j, b) \in G\} &= \# \cup_{j \in J} G_{ij} \times [1, CM/i] \\ &= \frac{CM}{i}\#E_{iJ} \\ &\geq \#J \end{aligned}$$

by the weak-type 1 result above. So the marriage theorem guarantees an injection

$$\theta_i: [1, N] \to [1, N] \times \left[1, \frac{CM}{i}\right]$$

such that $\theta_i(j) \in G_{ij} \times [1, CM/i]$ for all j. Since, for each j, the first coordinate of $\theta_i(j)$ is in G_{ij}, it is equal to some $\psi(h_j, j)$, where $h_j \notin F_{ij}$. Set $\eta_j(i) = h_j$. Letting π_1 denote projection on the first coordinate, we have $\pi_1\theta_i(j) = \psi(\eta_j(i), j)$. From the definition of F_{ij} we see that $\eta_j(i) \neq \eta_j(i')$ for all $i' > i$. Continuing the construction through $i = 1$ defines $\eta_j(i)$, for all i, j, so that η_j will be injective, and consequently bijective.

Finally, recall that $\phi(i, j) = \psi(\eta_j(i), j) = \pi_1\theta_i(j)$. Since ψ satisfies condition (1) and each η_j is a bijection of $[1, M]$, it is clear that ϕ satisfies (1), too. For fixed i, θ_i is injective, so $\pi_1\theta_i$ is at most CM/i to one. Thus, ϕ also satisfies condition (2), and the theorem is proved. □

Of course, this result shows that T is bounded on l^p for $p > 1$ as in the corollary to Theorem 1. Theorem 2 does not have a converse. It is possible for (1) and (2) to hold while T is not weak-type 1. Such a T has the same l^p bounds as given by interpolation between weak-type 1 and l^∞. This shows that "extrapolation" is impossible from l^p to weak-type 1.

Also, it is possible for T to satisfy the hypotheses of the theorem without being bounded on l^1 (independently of N). For example, T could be a linearization of the maximal operator M. To be more specific, let $K(i, j) = j^{-1}$ for $0 < i \leq j \leq N$ and $K(i, j) = 0$ otherwise. Let $f(i) = 1$ if $i = 1$ and $f(i) = 0$ otherwise. Then the l^1 bound for T is at least $\log N$, while the weak-type constant is $C = 1$. By approximation, a similar K can be defined that has only positive values.

The weak-type 1 hypothesis could be replaced by *weak-type* p if CM/i were also replaced by $C(M/i)^p$, with the same proof, and a corollary similar to ours and to the usual interpolation results (see [5]). The L^q bound obtained would be $C^{1/q}q/(q-p)$ for $q > p$.

It is natural to conjecture that Theorem 2 has an analogue on R, but this remains an open question. Also, it would probably be difficult, but useful, to obtain an explicit formula for some choice of the $\phi(i, j)$ that arise.

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Department of Mathematics, Florida International University, Miami, Florida 33199