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A NOTE ON THE JACOBIAN CONDITION AND TWO POINTS AT INFINITY

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ABSTRACT. If two polynomials F and G satisfy the Jacobian condition and the Newton polygon of F has an edge of negative slope, then the sum of terms of F along this edge has at most two distinct irreducible factors and their exponents must be different. Moreover, the slope is either a (negative) integer and the edge touches the y-axis or a (negative) Egyptian fraction and the edge touches the x-axis. Furthermore, there is an elementary automorphism which reduces the size of the Newton polygon.

The main purpose of this note is to prove Theorem 1 and Corollary 4. Part of it was first independently proved by Abhyankar [1, (18.13) Theorem and (18.15) Corollary], Makar-Limanov [4] (which was cited by Vitushkin in [7, p. 416]), and Moh [6, Propositions 4.3 and 4.5], by using different methods. Then it was re-proved by Appelgate and Onishi [1, Lemmas 13, 14, 15, 31 and 33]. The theorem was partially generalized by Charzyński, Chadzyński and Skibiński [3, Theorem 5.1]. Their proofs usually involve many other lemmas. Our proof, as given here, shows that the same elementary ideas which were used in [5] can be used to prove that the Jacobian condition, $\frac{\partial(F,G)}{\partial(x,y)} = 1$, implies that F has at most two zeros at infinity. Recall that the Newton polygon for $F(x, y)$ is the convex hull of the origin together with the support of F.

Theorem 1. *If $F(x, y)$, $G(x, y) \in \mathbb{C}[x, y]$, $\frac{\partial(F,G)}{\partial(x,y)} = 1$ and the Newton polygon of F has an edge of negative slope $-\frac{\tilde{n}}{\tilde{m}}$, with $\tilde{n}$, $\tilde{m}$ as relatively prime positive integers, then this slope is either a (negative) integer or a (negative) Egyptian fraction, neither the right vertex nor the left vertex is on the $45°$-line through the origin, and $F^{+}_{(\tilde{n},\tilde{m})}(x, y)$, the sum of terms of F along this edge has at most two distinct irreducible factors. Furthermore if $d + 1$ is the number of lattice points on the edge (so that $d \geq 1$), then exactly one of the following*

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statements is true:

(a) $\tilde{n} = 1$, $\tilde{m} > 1$, *the right vertex of the edge is on the* x*-axis, and* $F^{+}_{(1,\tilde{m})}(x, y) = \mu x^{l_1}(x^{\tilde{m}} - \rho y)^d$ *with* $\rho \neq 0$ *and* $0 \leq l_1 \neq d$.

(b) $\tilde{m} = 1$, $\tilde{n} > 1$, *the left vertex of the edge is on the* y*-axis, and* $F^{+}_{(\tilde{n},1)}(x, y) = \mu y^{r_2}(x - \rho y^{\tilde{n}})^d$ *with* $\rho \neq 0$ *and* $0 \leq r_2 \neq d$.

(c) $\tilde{n} = \tilde{m} = 1$, *the edge has the right vertex on the* x*-axis and the left vertex on the* y*-axis, and* $F^{+}_{(1,1)}(x, y) = \mu(x - \rho_1 y)^{e_1}(x - \rho_2 y)^{e_2}$ *with* $\rho_1 \neq \rho_2$, $\rho_1\rho_2 \neq 0$, $e_1 \neq e_2$, $e_1 \geq 0$, $e_2 \geq 0$ *and* $e_1 + e_2 = d$.

Proof. Denote the right vertex of the edge by $R(r_1, r_2)$ and the left vertex of the edge by $L(l_1, l_2)$. Assign weight$(x) = \tilde{n}$, weight$(y) = \tilde{m}$. Then every point on this edge has weight $= l_1\tilde{n} + l_2\tilde{m} = r_1\tilde{n} + r_2\tilde{m} = a$ and

$$F(x, y) = \sum_{i\tilde{n}+j\tilde{m}\leq a} F_{i,j}\, x^i y^j$$

with $F_{i,j} \in \mathbb{C}$. The sum of terms in $F(x, y)$ along the specified edge is

$$F^{+}_{(\tilde{n},\tilde{m})}(x, y) = \sum_{i\tilde{n}+j\tilde{m}=a} F_{i,j}\, x^i y^j,$$

and there are at least two distinct non-zero terms in this sum. Let $d + 1$ be the number of lattice points on the edge (including the vertices of the edge), then

$$\begin{aligned} r_1 - l_1 &= d\tilde{m}, \\ r_2 - l_2 &= -d\tilde{n}. \end{aligned}$$

In terms of this notation,

$$F^{+}_{(\tilde{n},\tilde{m})}(x, y) = \mu x^{l_1} y^{r_2} \prod_{i=1}^{d} (x^{\tilde{m}} - \rho_i y^{\tilde{n}})$$

with μ, $\rho_i \in \mathbb{C} \setminus \{0\}$.

Pick a pair of positive integers u, v such that $u\tilde{m} - v\tilde{n} = 1$ as in [5, §4]. After the change of variables

$$\begin{aligned} x &= U^u V^{\tilde{n}}, \\ y &= U^v V^{\tilde{m}}, \end{aligned}$$

$F(x, y)$ and $G(x, y)$ become $A(U, V)$ and $B(U, V)$ respectively. We can write $A(U, V)$ and $B(U, V)$ as Laurent series in V^{-1} with coefficients in $\mathbb{C}(U)$ (so the terms are written in decreasing powers of V). Note that the *weight* of any monomial in x, y is exactly the exponent of V, after the change

of variables. Since

$$\begin{aligned} A(U, V) &= F(U^u V^{\tilde{n}}, U^v V^{\tilde{m}}) \\ &= \sum_{i\tilde{n}+j\tilde{m}\le a} F_{i,j}\, U^{iu+jv}\, V^{i\tilde{n}+j\tilde{m}} \\ &= \alpha(U)V^a + \text{ (terms of } \deg_V < a), \end{aligned}$$

the leading term $\alpha(U)V^a$ of $A(U, V)$ is

$$\begin{aligned} \alpha(U)V^a &= \sum_{i\tilde{n}+j\tilde{m}=a} F_{i,j}\, U^{iu+jv}\, V^{i\tilde{n}+j\tilde{m}} \\ &= \mu\, (U^u V^{\tilde{n}})^{l_1} (U^v V^{\tilde{m}})^{r_2} \prod_{i=1}^{d} [(U^u V^{\tilde{n}})^{\tilde{m}} - \rho_i (U^v V^{\tilde{m}})^{\tilde{n}}] \\ &= \mu\, U^{ul_1+vr_2+dv\tilde{n}} \left[\prod_{i=1}^{d}(U-\rho_i)\right] V^{\tilde{n}l_1+\tilde{m}r_2+d\tilde{m}\tilde{n}} \\ &= \mu\, U^{ul_1+vl_2} \left[\prod_{i=1}^{d}(U-\rho_i)\right] V^{\tilde{n}l_1+\tilde{m}l_2}. \end{aligned}$$

Thus

$$\alpha(U) = \mu\, U^{ul_1+vl_2} \prod_{i=1}^{d}(U-\rho_i)$$

and

$$a = \tilde{n}l_1 + \tilde{m}l_2 = \tilde{n}r_1 + \tilde{m}r_2.$$

By the chain rule,

$$\frac{\partial(A, B)}{\partial(U, V)} = \frac{\partial(F, G)}{\partial(x, y)} \frac{\partial(x, y)}{\partial(U, V)} = U^{u+v-1}\, V^{\tilde{n}+\tilde{m}-1}.$$

Thus the leading term $\gamma(U)V^{c-1}$ of $\frac{\partial(A, B)}{\partial(U, V)}$, as a Laurent series in V^{-1}, is given by

$$\begin{aligned} \gamma(U) &= U^{u+v-1}, \\ c &= \tilde{n} + \tilde{m}. \end{aligned}$$

According to [5, Theorem 2], there is a $\delta(U) \in \mathbb{C}(U)$ such that

$$(*) \qquad \frac{\gamma(U)}{\delta(U)} = \frac{U^{u+v-1}}{\delta(U)} = \frac{\psi'(U)}{\psi(U)}$$

with

$$\begin{aligned} \psi(U) &= \alpha(U)^c\, \delta(U)^{-a} \\ &= \mu^{\tilde{n}+\tilde{m}}\, U^{(ul_1+vl_2)(\tilde{n}+\tilde{m})} \left[\prod_{i=1}^{d}(U-\rho_i)\right]^{\tilde{n}+\tilde{m}} \delta(U)^{-a}. \end{aligned}$$

For $\delta(U) \in \mathbb{C}(U)$ and $\theta \in \mathbb{C}$, we let $v_\theta(\delta)$ denote the discrete valuation of δ at the place θ; in other words,

$$v_\theta(\delta) = \text{ order of } \delta \text{ as a Laurent series in } (U-\theta).$$

Writing δ as a quotient of two polynomials, $\deg(\delta)$ denotes the degree of the numerator minus the degree of the denominator; in other words,

$$-\deg(\delta) = \text{ order of } \delta \text{ as a Laurent series in } U^{-1}.$$

Applying [5, Lemma 4] to Equation $(*)$ with $\theta = 0$, we have

$$(\sharp) \quad (u+v-1) - v_0(\delta) \text{ is } \begin{cases} -1 & \iff (ul_1 + vl_2)(\tilde{n}+\tilde{m}) - av_0(\delta) \neq 0, \\ > -1 & \iff (ul_1 + vl_2)(\tilde{n}+\tilde{m}) - av_0(\delta) = 0; \end{cases}$$

and

$$(\flat) \quad (u+v-1) - \deg(\delta) \text{ is} \begin{cases} -1 & \iff (ul_1 + vl_2 + d)(\tilde{n}+\tilde{m}) - a\deg(\delta) \neq 0, \\ < -1 & \iff (ul_1 + vl_2 + d)(\tilde{n}+\tilde{m}) - a\deg(\delta) = 0. \end{cases}$$

There are two options for $v_0(\delta)$, and two options for $\deg(\delta)$:

$$v_0(\delta) = \begin{cases} u+v, \\ \dfrac{(ul_1 + vl_2)(\tilde{n}+\tilde{m})}{a} = u + v + \dfrac{l_1 - l_2}{a}; \end{cases}$$

and

$$\deg(\delta) = \begin{cases} u+v, \\ \dfrac{(ul_1 + vl_2 + d)(\tilde{n}+\tilde{m})}{a} = u + v + \dfrac{r_1 - r_2}{a}. \end{cases}$$

There are now four options:

$$(**) \qquad \deg(\delta) - v_0(\delta) = \begin{cases} 0, & (1) \\ \dfrac{r_1 - r_2}{a}, & (2) \\ \dfrac{l_2 - l_1}{a}, & (3) \\ \dfrac{(r_1 - l_1) - (r_2 - l_2)}{a} = \dfrac{d(\tilde{n}+\tilde{m})}{a}. & (4) \end{cases}$$

Let e_i be the multiplicity of ρ_i in $\alpha(U)$. Then

$$(\dagger) \qquad -v_{\rho_i}(\delta) \text{ is } \begin{cases} -1 & \iff (\tilde{n}+\tilde{m})e_i - av_{\rho_i}(\delta) \neq 0, \\ > -1 & \iff (\tilde{n}+\tilde{m})e_i - av_{\rho_i}(\delta) = 0. \end{cases}$$

Either $v_{\rho_i}(\delta) = 1$ or $1 > v_{\rho_i}(\delta) = \frac{(\tilde{n}+\tilde{m})e_i}{a} =$ a positive number; and so $v_{\rho_i}(\delta) = 1$, i.e., $U - \rho_i$ is a factor of $\delta(U)$ of multiplicity 1. Hence every nonzero root of $\alpha(U)$ is a simple root of $\delta(U)$. Now consider any $\theta \in \mathbb{C}$ with $\theta \neq 0, \rho_1, \rho_2, \ldots, \rho_d$, in which case

$$-v_\theta(\delta) \text{ is } \begin{cases} -1 & \iff -av_\theta(\delta) \neq 0, \\ > -1 & \iff -av_\theta(\delta) = 0. \end{cases}$$

Thus $v_\theta(\delta) = 1$ or 0.

These last two applications of [5, Lemma 4] imply:

(i) $\delta(U)$ is a power of U times a polynomial in U, i.e., $\delta(U)$ is a Laurent polynomial;

(ii) any non-zero root of $\delta(U)$ is a simple root;

(iii) any non-zero root of $\alpha(U)$ is a (simple) root of $\delta(U)$; and
(iv) $\deg(\delta) - v_0(\delta) \geq$ (number of distinct roots ρ_i's of $\alpha(U)$) ≥ 1.

We return to a further consideration of the four options in ($**$). The fact that $\deg(\delta) - v_0(\delta) \geq 1$ rules out the first option immediately.

Case (a). In the case of option (2),

$$\begin{aligned}\deg(\delta) - v_0(\delta) &= \frac{r_1 - r_2}{a} \\ &= \frac{r_1 - r_2}{r_1\tilde{n} + r_2\tilde{m}} \\ &= \frac{1}{\tilde{n}} - \frac{r_2(\tilde{n} + \tilde{m})}{(r_1\tilde{n} + r_2\tilde{m})\tilde{n}} \\ &\leq \frac{1}{\tilde{n}}.\end{aligned}$$

However, $1 \leq$ (number of distinct roots ρ_i's of $\alpha(U)$) $\leq \deg(\delta) - v_0(\delta) \leq \frac{1}{\tilde{n}}$. Consequently, $\tilde{n} = 1$, $r_2 = 0$ and there is only one ρ_i. Thus a vertex is on the x-axis, and

$$F^{+}_{(1,\tilde{m})}(x, y) = \mu x^{l_1}(x^{\tilde{m}} - \rho y)^d.$$

The option (2) comes from the first option for $v_0(\delta)$ in ($\sharp$) which says that $v_0(\delta) = u + v$ and $v_0(\delta) \neq u + v + \frac{l_1 - l_2}{a}$. Hence $l_1 \neq l_2 = d$.

Case (b). The case of option (3) is similar, with

$$\begin{aligned}\frac{l_2 - l_1}{a} &= \frac{l_2 - l_1}{l_1\tilde{n} + l_2\tilde{m}} \\ &= \frac{1}{\tilde{m}} - \frac{l_1(\tilde{n} + \tilde{m})}{(l_1\tilde{n} + l_2\tilde{m})\tilde{m}}.\end{aligned}$$

The conclusion is: $\tilde{m} = 1$, $l_1 = 0$ and there is only one ρ_i. Thus a vertex is on the y-axis, and

$$F^{+}_{(\tilde{n},1)}(x, y) = \mu y^{r_2}(x - \rho y^{\tilde{n}})^d.$$

The option (3) comes from the first option for $\deg(\delta)$ in ($\flat$) which says that $\deg(\delta) = u + v$ and $\deg(\delta) \neq u + v + \frac{r_1 - r_2}{a}$. Hence $r_2 \neq r_1 = d$.

Case (c). In the case of option (4), we need to note that this option comes from the combination of

$$u + v - 1 - v_0(\delta) > -1 \quad \text{and} \quad u + v - 1 - \deg(\delta) < -1,$$

i.e.,

$$u + v - 1 - v_0(\delta) \geq 0 \quad \text{and} \quad u + v - 1 - \deg(\delta) \leq -2,$$

which implies that

$$\deg(\delta) - v_0(\delta) \geq 2.$$

However, option (4) is the sum of options (2) and (3) which implies that option (4) requires

$$\deg(\delta) - v_0(\delta) \leq \frac{1}{\tilde{n}} + \frac{1}{\tilde{m}} \leq 2,$$

with equality only if $r_2 = l_1 = 0$, and $\tilde{n} = \tilde{m} = 1$.

Consequently, $\deg(\delta) - v_0(\delta) = 2$, and $r_2 = l_1 = 0$. There is a vertex of the edge on each axis; the number of distinct ρ_i in $\alpha(U)$ is at most two; and

$$F^{+}_{(1,1)}(x, y) = \mu(x - \rho_1 y)^{e_1}(x - \rho_2 y)^{e_2},$$

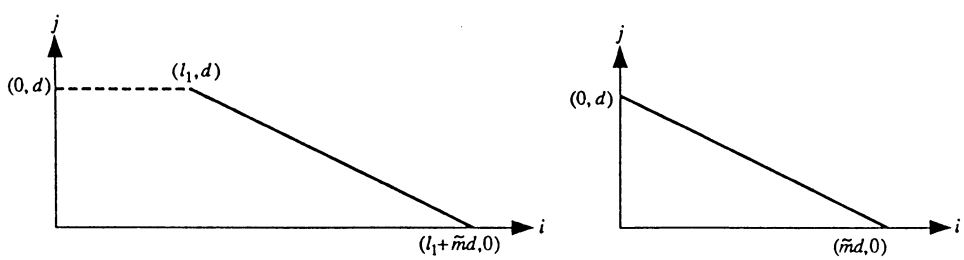
for nonnegative integers e_1, e_2 with $e_1 + e_2 = d$. Note that e_i is the multiplicity of ρ_i in $\alpha(U)$. From $\tilde{n} = \tilde{m} = 1$ and $l_1 = r_2 = 0$, we derive that $a = l_1\tilde{n} + l_2\tilde{m} = l_1 + l_2 = l_2 = d$. We have already shown that $v_{\rho_i}(\delta) = 1$ for each non-zero root ρ_i of $\alpha(U)$. Hence by Equation (†), $(\tilde{n} + \tilde{m})e_i - av_{\rho_i}(\delta) \neq 0$, i.e., $2e_i - d \neq 0$. Thus $e_1 \neq e_2$. □

It may help to visualize the three cases of Theorem 1 in terms of edges of negative slope $-\frac{\tilde{n}}{\tilde{m}}$ which are "shallow", i.e., $-1 \leq -\frac{\tilde{n}}{\tilde{m}} < 0$, or "steep", i.e., $-\frac{\tilde{n}}{\tilde{m}} \leq -1$.

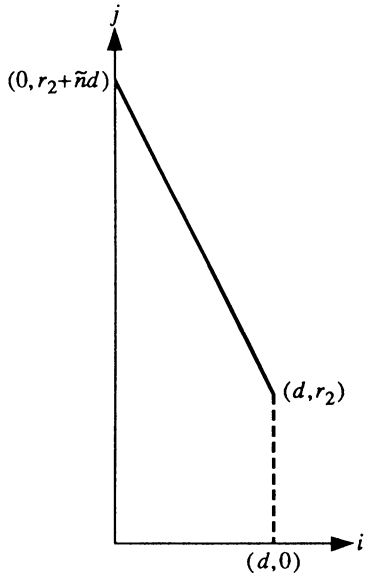
Corollary 2. *Under the hypotheses of Theorem* 1, *any shallow edge of the Newton polygon must meet the* x*-axis and any steep edge must meet the* y*-axis.* □

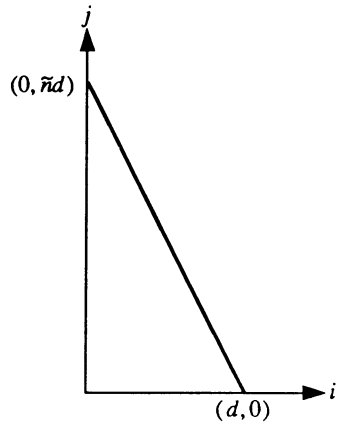
Corollary 3. *Under the hypotheses of Theorem* 1, *exactly one of the following is true*:

(a) *The Newton polygon of* F *is either contained within the trapezoid with vertices* $(0, 0)$, $(l_1 + \tilde{m}d, 0)$, (l_1, d), $(0, d)$ *or is a triangle with vertices* $(0, 0)$, $(\tilde{m}d, 0)$, $(0, d)$.



(b) *The Newton polygon of* F *is either contained within the trapezoid with vertices* $(0, 0)$, $(d, 0)$, (d, r_2), $(0, r_2 + \tilde{n}d)$ *or is a triangle with vertices* $(0, 0)$, $(d, 0)$, $(0, \tilde{n}d)$.





(c) *The Newton polygon of* F *is the isosceles triangle with vertices* $(0, 0)$, $(e_1 + e_2, 0)$, $(0, e_1 + e_2)$.

Proof. Case (a). If $l_1 = 0$, the Newton polygon is a triangle. If $l_1 > 0$ and the Newton polygon is not contained within the stated trapezoid, then the next vertex of the Newton polygon after the vertex (l_1, d) in a counterclockwise direction determines an edge which is contained within the triangle with vertices $(0, d)$, (l_1, d), $(0, \frac{l_1+\tilde{m}d}{\tilde{m}})$. This edge is shallow and thus meets the x-axis. This is a contradiction because $d \neq 0$.

The case (b) is similar and the case (c) is clear. □

Corollary 4. *The hypotheses of Theorem* 1 *imply that there is an elementary automorphism which reduces the area of the Newton polygon for* $F(x, y)$.

Proof. Since case (a) and case (b) are similar, we will treat case (b) only.

Case (b). From the statement of Theorem 1,

$$F^{+}_{(\tilde{n},1)}(x, y) = \mu y^{r_2}(x - \rho y^{\tilde{n}})^d.$$

Consider the elementary automorphism

$$\begin{aligned} \phi : \mathbb{C}[x, y] &\to \mathbb{C}[x, y] \\ x &\mapsto x + \rho y^{\tilde{n}}, \\ y &\mapsto y. \end{aligned}$$

We will examine its effect on the Newton polygon. Note that

$$\begin{aligned}\phi(x^i y^j) &= (x + \rho y^{\tilde{n}})^i y^j \\ &= \sum_{0 \le k \le i} \binom{i}{k} \rho^k x^{i-k} y^{\tilde{n}k+j}.\end{aligned}$$

Under the weight assignment $\text{weight}(x) = \tilde{n}$, $\text{weight}(y) = 1$, every term in the summation has weight

$$(i-k)\tilde{n} + (\tilde{n}k + j)1 = i\tilde{n} + j1 = \text{weight}(x^i y^j).$$

This implies that ϕ is weight preserving, i.e., points (i, j) in the Newton polygon contribute only to points $(i-k, \tilde{n}k + j)$ on the line of slope $-\frac{\tilde{n}}{1}$ through (i, j). Furthermore, since $0 \le k \le i$, the point $(i-k, \tilde{n}k + j)$ is to the left of (i, j) and hence inside the Newton polygon for $F(x, y)$. Thus the Newton polygon for $\phi(F(x, y))$ is contained in the Newton polygon for $F(x, y)$. The automorphism ϕ also has the property that $\phi(F^{+}_{(\tilde{n}, 1)}(x, y)) = \mu x^d y^{r_2}$. Therefore the Newton polygon for $\phi(F(x, y))$ does not include the point $(0, r_2 + \tilde{n}d)$ which is a vertex of the Newton polygon for $F(x, y)$. Thus the area for the Newton polygon is decreased by this automorphism.

Case (c). From the statement of Theorem 1,

$$F^{+}_{(1,1)}(x, y) = \mu (x - \rho_1 y)^{e_1} (x - \rho_2 y)^{e_2}.$$

Consider the elementary automorphism

$$\begin{aligned}\psi : \mathbb{C}[x, y] &\to \mathbb{C}[x, y] \\ x &\mapsto \frac{\rho_2 x - \rho_1 y}{\rho_2 - \rho_1}, \\ y &\mapsto \frac{x - y}{\rho_2 - \rho_1}.\end{aligned}$$

The Newton polygon for $\psi(F(x, y))$ is contained in the Newton polygon for $F(x, y)$. The automorphism ψ also has the property that $\psi(F^{+}_{(1,1)}(x, y)) = \mu x^{e_1} y^{e_2}$. Therefore at least one of $(e_1 + e_2, 0)$, $(0, e_1 + e_2)$ is not included in the Newton polygon for $\psi(F(x, y))$. Thus the area for the Newton polygon is decreased by this automorphism. □

Remark. This still leaves the question of whether or not the Newton polygon for a polynomial $F(x, y)$, which satisfies the Jacobian condition, must have an edge of negative slope.

A similar argument to the proof of Theorem 1 with the change of variables $x = UV$ and $y = V$ establishes

Theorem 5. *If $F(x, y)$, $G(x, y) \in \mathbb{C}[x, y]$, $\frac{\partial(F, G)}{\partial(x, y)} = x^p y^q +$ lower degree terms, then the leading form of F has at most $p + q + 2$ distinct irreducible factors.*

The next example shows that the bound $p + q + 2$ in Theorem 5 is almost the best possible.

Example 6. Let $F(x, y) = x^n - y^n$ and $G(x, y) = y$. Then $\frac{\partial(F, G)}{\partial(x, y)} = nx^{n-1}$ so that $p = n - 1$ and $q = 0$. Now F has n distinct irreducible factors and $p + q + 2 = n + 1$.

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