A NOTE ON THE JACOBIAN CONDITION
AND TWO POINTS AT INFINITY

JAMES H. MCKAY AND STUART SUI-SHENG WANG

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Abstract. If two polynomials $F$ and $G$ satisfy the Jacobian condition and
the Newton polygon of $F$ has an edge of negative slope, then the sum of terms
of $F$ along this edge has at most two distinct irreducible factors and their
exponents must be different. Moreover, the slope is either a (negative) integer
and the edge touches the $y$-axis or a (negative) Egyptian fraction and the edge
touches the $x$-axis. Furthermore, there is an elementary automorphism which
reduces the size of the Newton polygon.

The main purpose of this note is to prove Theorem 1 and Corollary 4. Part of
it was first independently proved by Abhyankar [1, (18.13) Theorem and (18.15)
Corollary], Makar-Limanov [4] (which was cited by Vitushkin in [7, p. 416]),
and Moh [6, Propositions 4.3 and 4.5], by using different methods. Then it was
re-proved by Appelgate and Onishi [1, Lemmas 13, 14, 15, 31 and 33]. The
theorem was partially generalized by Charzyński, Chadzyński and Skibiński [3,
Theorem 5.1]. Their proofs usually involve many other lemmas. Our proof, as
given here, shows that the same elementary ideas which were used in [5] can be
used to prove that the Jacobian condition, $\frac{\partial(F,G)}{\partial(x,y)} = 1$, implies that $F$ has at
most two zeros at infinity. Recall that the Newton polygon for $F(x,y)$ is the
convex hull of the origin together with the support of $F$.

Theorem 1. If $F(x,y), G(x,y) \in \mathbb{C}[x,y], \frac{\partial(F,G)}{\partial(x,y)} = 1$ and the Newton poly-
gon of $F$ has an edge of negative slope $-\frac{\tilde{n}}{\tilde{m}}$, with $\tilde{n}$, $\tilde{m}$ as relatively prime
positive integers, then this slope is either a (negative) integer or a (negative)
Egyptian fraction, neither the right vertex nor the left vertex is on the 45°–line
through the origin, and $F^+_{(\tilde{n},\tilde{m})}(x,y)$, the sum of terms of $F$ along this edge
has at most two distinct irreducible factors. Furthermore if $d + 1$ is the number
of lattice points on the edge (so that $d \geq 1$), then exactly one of the following

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statements is true:

(a) \( \tilde{n} = 1, \tilde{m} > 1 \), the right vertex of the edge is on the \( x \)-axis, and

\( F_{(1, \tilde{m})}^{+}(x, y) = \mu x^{l_1}(x^{\tilde{m}} - \rho y)^d \) with \( \rho \neq 0 \) and \( 0 \leq l_1 \neq d \).

(b) \( \tilde{m} = 1, \tilde{n} > 1 \), the left vertex of the edge is on the \( y \)-axis, and

\( F_{(\tilde{n}, 1)}^{+}(x, y) = \mu y^{r_2}(x - \rho y^{\tilde{n}})^d \) with \( \rho \neq 0 \) and \( 0 \leq r_2 \neq d \).

(c) \( \tilde{n} = \tilde{m} = 1 \), the edge has the right vertex on the \( x \)-axis and the left vertex on the \( y \)-axis, and

\( F_{(1, 1)}^{+}(x, y) = \mu (x - \rho_1 y)^{e_1}(x - \rho_2 y)^{e_2} \) with \( \rho_1 \neq \rho_2 \), \( \rho_1 \rho_2 \neq 0 \), \( e_1 \neq e_2 \), \( e_1 \geq 0 \), \( e_2 \geq 0 \) and \( e_1 + e_2 = d \).

\textbf{Proof.} Denote the right vertex of the edge by \( R(r_1, r_2) \) and the left vertex of the edge by \( L(l_1, l_2) \). Assign weight\((x) = \tilde{n} \), weight\((y) = \tilde{m} \). Then every point on this edge has weight \( = \tilde{n}r_1 + \tilde{m}l_1 = r_1 \tilde{n} + l_2 \tilde{m} = a \) and

\[
F(x, y) = \sum_{i \in \mathbb{N}, j \in \mathbb{N}, i + jm \leq a} F_{i,j} x^i y^j
\]

with \( F_{i,j} \in \mathbb{C} \). The sum of terms in \( F(x, y) \) along the specified edge is

\[
F_{(\tilde{n}, \tilde{m})}^{+}(x, y) = \sum_{i \in \mathbb{N}, j \in \mathbb{N}, i + jm = a} F_{i,j} x^i y^j,
\]

and there are at least two distinct non-zero terms in this sum. Let \( d + 1 \) be the number of lattice points on the edge (including the vertices of the edge), then

\[
\begin{align*}
\tilde{n} - n_1 &= d \tilde{m}, \\
\tilde{m} - l_2 &= -d \tilde{n}.
\end{align*}
\]

In terms of this notation,

\[
F_{(\tilde{n}, \tilde{m})}^{+}(x, y) = \mu x^{l_1} y^{r_2} \prod_{i=1}^{d} (x^{\tilde{m}} - \rho_i y^{\tilde{n}})
\]

with \( \mu, \rho_i \in \mathbb{C} \setminus \{0\} \).

Pick a pair of positive integers \( u, v \) such that \( u \tilde{m} - v \tilde{n} = 1 \) as in \([5, \S 4]\). After the change of variables

\[
\begin{align*}
x &= U^u V^{\tilde{n}}, \\
y &= U^v V^{\tilde{m}},
\end{align*}
\]

\( F(x, y) \) and \( G(x, y) \) become \( A(U, V) \) and \( B(U, V) \) respectively. We can write \( A(U, V) \) and \( B(U, V) \) as Laurent series in \( V^{-1} \) with coefficients in \( \mathbb{C}(U) \) (so the terms are written in decreasing powers of \( V \)). Note that the \textit{weight} of any monomial in \( x, y \) is exactly the exponent of \( V \), after the change

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of variables. Since

$$A(U, V) = F(U^u V^n, U^v V^m)$$

$$= \sum_{i \in \mathbb{N}, j \in \mathbb{N}} F_{i,j} U^{iu+jv} V^{i+n+jm}$$

$$= \alpha(U) V^a + \text{(terms of deg}_V < a),$$

the leading term $\alpha(U) V^a$ of $A(U, V)$ is

$$\alpha(U) V^a = \sum_{i \in \mathbb{N}, j \in \mathbb{N}} F_{i,j} U^{iu+jv} V^{i+n+jm}$$

$$= \mu (U^u V^n)^{l_1} (U^v V^m)^{l_2} \prod_{i=1}^{d} [((U^u V^n)^{m} - \rho_i(U^v V^m)^{\tilde{m}}]$$

$$= \mu U^{ul_1 + vl_2 + d v n} \left[ \prod_{i=1}^{d} (U - \rho_i) \right] V^{\tilde{n}_l + \tilde{m}_l + d\tilde{m}_n}$$

Thus

$$\alpha(U) = \mu U^{ul_1 + vl_2} \prod_{i=1}^{d} (U - \rho_i)$$

and

$$a = \tilde{n}_l + \tilde{m}_l = \tilde{n}_1 + \tilde{m}_2.$$

By the chain rule,

$$\frac{\partial(A, B)}{\partial(U, V)} = \frac{\partial(F, G)}{\partial(x, y)} \frac{\partial(x, y)}{\partial(U, V)} = U^{u+v-1} V^{\tilde{n}+\tilde{m}-1}.$$

Thus the leading term $\gamma(U) V^{c-1}$ of $\frac{\partial(A, B)}{\partial(U, V)}$, as a Laurent series in $V^{-1}$, is given by

$$\gamma(U) = U^{u+v-1},$$

$$c = \tilde{n} + \tilde{m}.$$

According to [5, Theorem 2], there is a $\delta(U) \in \mathbb{C}(U)$ such that

\begin{align*}
\gamma(U) &= \delta(U) \quad \left( \ast \right) \\
\psi(U) &= \alpha(U)^c \delta(U)^{-a} \\
\psi(U) &= \mu^{l_1 + l_2} U^{ul_1 + vl_2} \left[ \prod_{i=1}^{d} (U - \rho_i) \right]^{\tilde{n} + \tilde{m}} \delta(U)^{-a}.
\end{align*}

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For $\delta(U) \in \mathbb{C}(U)$ and $\theta \in \mathbb{C}$, we let $v_\theta(\delta)$ denote the discrete valuation of $\delta$ at the place $\theta$; in other words,

$$v_\theta(\delta) = \text{order of } \delta \text{ as a Laurent series in } (U - \theta).$$

Writing $\delta$ as a quotient of two polynomials, $\deg(\delta)$ denotes the degree of the numerator minus the degree of the denominator; in other words,

$$-\deg(\delta) = \text{order of } \delta \text{ as a Laurent series in } U^{-1}.$$

Applying [5, Lemma 4] to Equation (\ast) with $\theta = 0$, we have

$$(u + v - 1) - v_0(\delta) \begin{cases} -1 & \iff (ul_1 + vl_2)(\tilde{n} + \tilde{m}) - \alpha v_0(\delta) \neq 0, \\
> -1 & \iff (ul_1 + vl_2)(\tilde{n} + \tilde{m}) - \alpha v_0(\delta) = 0; \end{cases}$$

and

$$(u + v - 1) - \deg(\delta) \begin{cases} -1 & \iff (ul_1 + vl_2 + d)(\tilde{n} + \tilde{m}) - a \deg(\delta) \neq 0, \\
< -1 & \iff (ul_1 + vl_2 + d)(\tilde{n} + \tilde{m}) - a \deg(\delta) = 0. \end{cases}$$

There are two options for $v_0(\delta)$, and two options for $\deg(\delta)$:

$$v_0(\delta) = \begin{cases} u + v, \\
\frac{(ul_1 + vl_2)(\tilde{n} + \tilde{m})}{a} = u + v + \frac{l_1 - l_2}{a}; \end{cases}$$

and

$$\deg(\delta) = \begin{cases} u + v, \\
\frac{(ul_1 + vl_2 + d)(\tilde{n} + \tilde{m})}{a} = u + v + \frac{r_1 - r_2}{a}. \end{cases}$$

There are now four options:

$$\deg(\delta) - v_0(\delta) = \begin{cases} 0, \\
r_1 - r_2 \frac{a}{a}, \\
\frac{l_2 - l_1}{a}, \\
\frac{(r_1 - l_1) - (r_2 - l_2)}{a} = \frac{d(\tilde{n} + \tilde{m})}{a}. \end{cases}$$

Let $e_i$ be the multiplicity of $\rho_i$ in $\alpha(U)$. Then

$$-v_{\rho_i}(\delta) = \begin{cases} -1 & \iff (\tilde{n} + \tilde{m})e_i - \alpha v_{\rho_i}(\delta) \neq 0, \\
> -1 & \iff (\tilde{n} + \tilde{m})e_i - \alpha v_{\rho_i}(\delta) = 0. \end{cases}$$

Either $v_{\rho_i}(\delta) = 1$ or $1 > v_{\rho_i}(\delta) = \frac{(\tilde{n} + \tilde{m})e_i}{a}$ is a positive number; and so $\nu_{\rho_i}(\delta) = 1$, i.e., $U - \rho_i$ is a factor of $\delta(U)$ of multiplicity 1. Hence every nonzero root of $\alpha(U)$ is a simple root of $\delta(U)$. Now consider any $\theta \in \mathbb{C}$ with $\theta \neq 0$, $\rho_1, \rho_2, \ldots, \rho_d$, in which case

$$-v_\theta(\delta) = \begin{cases} -1 & \iff -\alpha v_\theta(\delta) \neq 0, \\
> -1 & \iff -\alpha v_\theta(\delta) = 0. \end{cases}$$

Thus $v_\theta(\delta) = 1$ or 0.

These last two applications of [5, Lemma 4] imply:

(i) $\delta(U)$ is a power of $U$ times a polynomial in $U$, i.e., $\delta(U)$ is a Laurent polynomial;

(ii) any non-zero root of $\delta(U)$ is a simple root;
(iii) any non-zero root of $\alpha(U)$ is a (simple) root of $\delta(U)$; and
(iv) $\deg(\delta) - v_0(\delta) \geq \text{(number of distinct roots $\rho_i$'s of $\alpha(U)$)} \geq 1$.

We return to a further consideration of the four options in (**). The fact that $\deg(\delta) - v_0(\delta) \geq 1$ rules out the first option immediately.

Case (a). In the case of option (2),
\[
\deg(\delta) - v_0(\delta) = \frac{r_1 - r_2}{a}
\]
\[
= \frac{r_1 - r_2}{r_1 n + r_2 m}
\]
\[
= \frac{1}{n} - \frac{r_2(n + m)}{(r_1 n + r_2 m)n}
\]
\[
\leq \frac{1}{n}.
\]
However, $1 \leq \text{(number of distinct roots $\rho_i$'s of $\alpha(U)$)} \leq \deg(\delta) - v_0(\delta) \leq \frac{1}{n}$.

Consequently, $n = 1$, $r_2 = 0$ and there is only one $\rho_i$. Thus a vertex is on the $x$-axis, and
\[
F_{(1, m)}^+(x, y) = \mu x^{l_1} (x^m - \rho y)^d.
\]
The option (2) comes from the first option for $v_0(\delta)$ in (a) which says that $v_0(\delta) = u + v$ and $v_0(\delta) \neq u + v + \frac{l_1 - l_2}{a}$. Hence $l_1 \neq l_2 = d$.

Case (b). The case of option (3) is similar, with
\[
\frac{l_2 - l_1}{a} = \frac{l_2 - l_1}{l_1 n + l_2 m}
\]
\[
= \frac{1}{m} - \frac{l_1(n + m)}{(l_1 n + l_2 m)m}.
\]
The conclusion is: $m = 1$, $l_1 = 0$ and there is only one $\rho_i$. Thus a vertex is on the $y$-axis, and
\[
F_{(n, 1)}^+(x, y) = \mu y^{l_2} (x - \rho y)^d.
\]
The option (3) comes from the first option for $\deg(\delta)$ in (b) which says that $\deg(\delta) = u + v$ and $\deg(\delta) \neq u + v + \frac{l_1 - l_2}{a}$. Hence $r_2 \neq r_1 = d$.

Case (c). In the case of option (4), we need to note that this option comes from the combination of
\[
u + v - 1 - v_0(\delta) > -1 \quad \text{and} \quad u + v - 1 - \deg(\delta) < -1,
\]
i.e.,
\[
u + v - 1 - v_0(\delta) \geq 0 \quad \text{and} \quad u + v - 1 - \deg(\delta) \leq -2,
\]
which implies that
\[
\deg(\delta) - v_0(\delta) \geq 2.
\]
However, option (4) is the sum of options (2) and (3) which implies that option (4) requires

\[ \text{deg}(\delta) - v_0(\delta) \leq \frac{1}{n} + \frac{1}{m} \leq 2, \]

with equality only if \( r_2 = l_1 = 0 \), and \( \tilde{n} = \tilde{m} = 1 \).

Consequently, \( \text{deg}(\delta) - v_0(\delta) = 2 \), and \( r_2 = l_1 = 0 \). There is a vertex of the edge on each axis; the number of distinct \( \rho_i \) in \( \alpha(U) \) is at most two; and

\[ F_{(1,1)}^+(x, y) = \mu (x - \rho_1 y)^{e_1} (x - \rho_2 y)^{e_2}, \]

for nonnegative integers \( e_1, e_2 \) with \( e_1 + e_2 = d \). Note that \( e_i \) is the multiplicity of \( \rho_i \) in \( \alpha(U) \). From \( \tilde{n} = \tilde{m} = 1 \) and \( l_1 = r_2 = 0 \), we derive that \( a = l_1 \tilde{n} + l_2 \tilde{m} = l_1 + l_2 = l_2 = d \). We have already shown that \( v_{\rho_i}(\delta) = 1 \) for each non-zero root \( \rho_i \) of \( \alpha(U) \). Hence by Equation (†), \( (\tilde{n} + \tilde{m})e_i - av_{\rho_i}(\delta) \neq 0 \), i.e., \( 2e_i - d \neq 0 \). Thus \( e_1 \neq e_2 \).

It may help to visualize the three cases of Theorem 1 in terms of edges of negative slope \( -\frac{\tilde{n}}{m} \), which are "shallow", i.e., \( -1 \leq -\frac{\tilde{n}}{m} < 0 \), or "steep", i.e., \( -\frac{\tilde{n}}{m} \leq -1 \).

**Corollary 2.** Under the hypotheses of Theorem 1, any shallow edge of the Newton polygon must meet the \( x \)-axis and any steep edge must meet the \( y \)-axis.

**Corollary 3.** Under the hypotheses of Theorem 1, exactly one of the following is true:

(a) The Newton polygon of \( F \) is either contained within the trapezoid with vertices \((0, 0), (l_1 + \tilde{m}d, 0), (l_1, d), (0, d)\) or is a triangle with vertices \((0, 0), (\tilde{m}d, 0), (0, d)\).

(b) The Newton polygon of \( F \) is either contained within the trapezoid with vertices \((0, 0), (d, 0), (d, r_2), (0, r_2 + \tilde{n}d)\) or is a triangle with vertices \((0, 0), (d, 0), (0, \tilde{n}d)\).
(c) The Newton polygon of $F$ is the isosceles triangle with vertices $(0, 0)$, $(e_1 + e_2, 0)$, $(0, e_1 + e_2)$.

Proof. Case (a). If $l_1 = 0$, the Newton polygon is a triangle. If $l_1 > 0$ and the Newton polygon is not contained within the stated trapezoid, then the next vertex of the Newton polygon after the vertex $(l_1, d)$ in a counterclockwise direction determines an edge which is contained within the triangle with vertices $(0, d)$, $(l_1, d)$, $(0, \frac{l_1 + mad}{m})$. This edge is shallow and thus meets the $x$–axis. This is a contradiction because $d \neq 0$.

The case (b) is similar and the case (c) is clear. □

Corollary 4. The hypotheses of Theorem 1 imply that there is an elementary automorphism which reduces the area of the Newton polygon for $F(x, y)$.

Proof. Since case (a) and case (b) are similar, we will treat case (b) only.

Case (b). From the statement of Theorem 1,

$$F_{(n, 1)}^{+}(x, y) = \mu y^{r_2} (x - \rho y^{\bar{n}})^d.$$ 

Consider the elementary automorphism

$$\phi : \mathbb{C}[x, y] \to \mathbb{C}[x, y]$$

$$x \mapsto x + \rho y^{\bar{n}},$$

$$y \mapsto y.$$
We will examine its effect on the Newton polygon. Note that

\[
\phi(x^i y^j) = (x + \rho y^{\tilde{n}})^j y^j = \sum_{0 \leq k \leq i} \binom{i}{k} (\rho)^k x^{i-k} y^{\tilde{n}k+j}.
\]

Under the weight assignment weight(x) = \tilde{n}, weight(y) = 1, every term in the summation has weight

\[(i - k)\tilde{n} + (\tilde{n}k + j)1 = i\tilde{n} + j1 = \text{weight}(x^i y^j).\]

This implies that \(\phi\) is weight preserving, i.e., points \((i, j)\) in the Newton polygon contribute only to points \((i - k, \tilde{n}k + j)\) on the line of slope \(-\frac{\tilde{n}}{1}\) through \((i, j)\). Furthermore, since \(0 \leq k \leq i\), the point \((i - k, \tilde{n}k + j)\) is to the left of \((i, j)\) and hence inside the Newton polygon for \(F(x, y)\). Thus the Newton polygon for \(\phi(F(x, y))\) is contained in the Newton polygon for \(F(x, y)\). The automorphism \(\phi\) also has the property that \(\phi(F(x^\pm_1, y)) = \mu x^{\epsilon_1}y^2\). Therefore the Newton polygon for \(\phi(F(x, y))\) does not include the point \((0, r_2 + \tilde{n}d)\) which is a vertex of the Newton polygon for \(F(x, y)\). Thus the area for the Newton polygon is decreased by this automorphism.

Case (c). From the statement of Theorem 1,

\[F_{(1,1)}^+(x, y) = \mu (x - \rho_1 y)^{e_1}(x - \rho_2 y)^{e_2}.\]

Consider the elementary automorphism

\[\psi: \mathbb{C}[x, y] \to \mathbb{C}[x, y] \quad x \mapsto \frac{\rho_2 x - \rho_1 y}{\rho_2 - \rho_1}, \quad y \mapsto \frac{x - y}{\rho_2 - \rho_1}.\]

The Newton polygon for \(\psi(F(x, y))\) is contained in the Newton polygon for \(F(x, y)\). The automorphism \(\psi\) also has the property that \(\psi(F_{(1,1)}^+(x, y)) = \mu x^{e_1}y^{e_2}\). Therefore at least one of \((e_1 + e_2, 0)\), \((0, e_1 + e_2)\) is not included in the Newton polygon for \(\psi(F(x, y))\). Thus the area for the Newton polygon is decreased by this automorphism. □

**Remark.** This still leaves the question of whether or not the Newton polygon for a polynomial \(F(x, y)\), which satisfies the Jacobian condition, must have an edge of negative slope.

A similar argument to the proof of Theorem 1 with the change of variables \(x = UV\) and \(y = V\) establishes

**Theorem 5.** If \(F(x, y), G(x, y) \in \mathbb{C}[x, y], \frac{\partial(F, G)}{\partial(x, y)} = x^p y^q + \) lower degree terms, then the leading form of \(F\) has at most \(p + q + 2\) distinct irreducible factors.
The next example shows that the bound $p + q + 2$ in Theorem 5 is almost the best possible.

Example 6. Let $F(x, y) = x^n - y^n$ and $G(x, y) = y$. Then

$$\frac{\partial (F, G)}{\partial (x, y)} = nx^{n-1}$$

so that $p = n - 1$ and $q = 0$. Now $F$ has $n$ distinct irreducible factors and $p + q + 2 = n + 1$.

References


Department of Mathematical Sciences, Oakland University, Rochester Hills, Michigan 48309

Department of Mathematical Sciences, Oakland University, Rochester Hills, Michigan 48309

Department of Mathematics, Cornell University, Ithaca, New York 14853