WEAKLY INFINITE-DIMENSIONAL PRODUCT SPACES

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ABSTRACT. It is shown that the product of a weakly infinite-dimensional compactum with a C-space is weakly infinite-dimensional. Some observations on the coincidence of weak infinite-dimensionality and property C are made. The question of when a weakly infinite-dimensional space has weakly infinite-dimensional product with all zero-dimensional spaces is investigated.

1. Introduction

By a space we mean a metric space, and by the dimension of a space we mean the Lebesgue covering dimension, for example as presented in [E]. In this sequel paper, we continue the line of investigation, initiated in [R2], into when the product of two weakly infinite-dimensional spaces is itself weakly infinite-dimensional. The reader is referred to that source for a more complete discussion of the history of this question and specifically for the definitions of countable-dimensional, weakly infinite-dimensional, and strongly infinite-dimensional spaces, as well as these of property C, C-space, and C-refinement.

While it is known that every countable-dimensional space, hence every finite-dimensional space, has property C and that every C-space is weakly infinite-dimensional, R. Pol has constructed a compact C-space which is not countable-dimensional [P1]. It remains unknown whether or not every weakly infinite-dimensional space must have property C.

Similarly, while it is also known that the product of two C-spaces can be strongly infinite-dimensional [EP], [P2], it is still unknown whether or not the product of two weakly infinite-dimensional compacta must always be weakly infinite-dimensional. On the other hand, the productivity of property C for compacta has been established [R2].

Theorem. The product of two C-spaces, one of which is compact, is itself a C-space.

In this paper we investigate the productivity of weak infinite-dimensionality for compacta, showing productivity for a large, perhaps the entire, class of...
weakly infinite-dimensional factors. Finally, some observations concerning the coincidence of weak infinite-dimensionality and property C are given, along with some remarks on when a weakly infinite-dimensional space has weakly infinite-dimensional product with all zero-dimensional spaces.

2. Results and proofs

In proving our main result, we will make use of the following characterization of weak infinite-dimensionality in terms of binary open covers:

**Lemma 1.** A space $X$ is weakly infinite-dimensional if and only if, for any sequence of binary open covers \( \{ U_n : n \in \mathbb{N} \} \) of $X$ of the form $U_n = \{ V_n^1, V_n^2 \}$, there exists a precise pairwise disjoint open refinement $\mathcal{V}_n$ of each $U_n$—i.e.:

1. For each $n \in \mathbb{N}$, $\mathcal{V}_n = \{ V_n^1, V_n^2 \}$, with $V_n^1$ and $V_n^2$ open in $X$.
2. For each $n \in \mathbb{N}$, $V_n^1 \cap V_n^2 = \emptyset$.
3. For each $n \in \mathbb{N}$, $V_n^1 \subset U_n$ and $V_n^2 \subset U_n$—so that the $\bigcup \{ \mathcal{V}_n : n \in \mathbb{N} \}$ forms an open cover of $X$.

As the proof is elementary, involving only complete normality, we omit it. For this characterization and further generalizations of the notion, we refer the reader to [R1, Chapter 3].

**Theorem 1.** The product of a weakly infinite-dimensional compactum with a $C$-space is again weakly infinite-dimensional.

**Proof.** Given a $C$-space $X$ and a weakly infinite-dimensional compactum $Y$, we show that the product $X \times Y$ is weakly infinite-dimensional. Let a countable collection of binary open covers of $X \times Y$ be given. We rewrite this collection as a sequence of such countable collections

\[ \{ \mathcal{U}_{m,n} : n \in \mathbb{N} \} : m \in \mathbb{N} \],

where each binary open cover has the form

\[ \mathcal{U}_{m,n} = \{ U_{m,n}^1, U_{m,n}^2 \} \).

Fix $m \in \mathbb{N}$, let $x \in X$ be fixed but arbitrary, and let $\pi : X \times Y \to X$ denote the projection mapping. For each $n \in \mathbb{N}$ and $\alpha \in \{1, 2\}$, we set

\[ U_{m,n}^\alpha(x) = U_{m,n}^\alpha \cap \pi^{-1}(x) \quad \text{and} \quad \mathcal{U}_{m,n}(x) = \{ U_{m,n}^1(x), U_{m,n}^2(x) \} \).

Thus, $\mathcal{U}_{m,n}(x)$ is a binary open cover of $\pi^{-1}(x)$ for each $n \in \mathbb{N}$. Since $\pi^{-1}(x)$ is homeomorphic to $Y$, we will not, when the context is clear, distinguish between $\pi^{-1}(x)$ as a subspace of $X \times Y$, and $Y$.

In particular, $\pi^{-1}(x)$ is weakly infinite-dimensional, so that, using Lemma 1, we can choose subsets $V_{m,n}^1(x)$ and $V_{m,n}^2(x)$ of $\pi^{-1}(x)$ for each $n \in \mathbb{N}$ with

1. $V_{m,n}^1(x)$ and $V_{m,n}^2(x)$ open in $\pi^{-1}(x)$
2. $V_{m,n}^{1}(x) \cap V_{m,n}^{2}(x) = \emptyset$

3. $V_{m,n}^{1}(x) \subseteq U_{m,n}^{1}(x)$ and $V_{m,n}^{2}(x) \subseteq U_{m,n}^{2}(x)$,

so that $\{V_{m,n}^{\alpha}(x) : \alpha = 1, 2, n \in \mathbb{N}\}$ is a cover of $\pi^{-1}(x)$. We then use the compactness of $\pi^{-1}(x)$ to extract a finite subcover

$$\{V_{m,n}^{\alpha}(x) : \alpha = 1, 2, n = 1, \ldots, r_{m}(x)\}$$

for some positive integer $r_{m}(x)$, and, using normality, we “shrink” the elements of the finite subcover so that

$$V_{m,n}^{\alpha}(x) \subseteq \overline{V_{m,n}^{\alpha}(x)} \subseteq U_{m,n}^{\alpha}(x)$$

for each $\alpha \in \{1, 2\}$ and $n \in \{1, \ldots, r_{m}(x)\}$.

Next, we use an idea of Dieudonné [D] to construct an open cover of $X$.

**Claim.** For each $n \in \{1, \ldots, r_{m}(x)\}$, there is an open neighborhood $W_{m,n}(x)$ of $x$ in $X$ so that, for any $x' \in W_{m,n}(x)$ and $\alpha \in \{1, 2\}$, the inclusions

$$V_{m,n}^{\alpha}(x) \subseteq \overline{V_{m,n}^{\alpha}(x)} \subseteq U_{m,n}^{\alpha}(x')$$

hold.

Indeed if not, then we could choose a sequence $(x_k, y_k)$ in $X \times Y$ with $x_k \to x$ where, without loss of generality, for each $k \in \mathbb{N}$,

$$y_k \notin \overline{V_{m,n}^{1}(x)} \quad \text{but} \quad y_k \notin U_{m,n}^{1}(x_k).$$

By the compactness of $Y$, passing to a convergent subsequence if necessary, we have

$$y_k \to y \in \overline{V_{m,n}^{1}(x)} \subseteq U_{m,n}^{1}(x),$$

so that

$$(x_k, y_k) \to (x, y) \in U_{m,n}^{1}.$$ But then, since $U_{m,n}^{1}$ is open in $X \times Y$, we see that, for all sufficiently large $k,$

$$(x_k, y_k) \in U_{m,n}^{1}, \quad \text{so that} \ y_k \in U_{m,n}^{1}(x_k),$$

which is a contradiction.

We construct such an open set $W_{m,n}(x)$ for each $n \in \{1, \ldots, r_{m}(x)\}$ and set

$$W_{m}(x) = \bigcap\{W_{m,n}(x) : n = 1, \ldots, r_{m}(x)\}.$$ Then, $W_{m}(x)$ is an open neighborhood of $x \in X$, so that

$$\{W_{m}(x) \times V_{m,n}^{\alpha}(x) : \alpha = 1, 2, n = 1, \ldots, r_{m}(x)\}$$

is an open cover of $\pi^{-1}(x)$ in $X \times Y$. We form the open cover

$$\mathcal{W}_{m} = \{W_{m}(x) : x \in X\}$$

of $X$ by constructing such a neighborhood $W_{m}(x)$ for each $x \in X$. 

In this manner, we construct such an open cover \( \mathcal{U}_m \) of \( X \) for each \( m \in \mathbb{N} \). Since \( X \) has property \( C \), we can choose a \( C \)-refinement \( \mathcal{E}_m \) of \( \mathcal{U}_m \) for each \( m \in \mathbb{N} \) so that the \( \bigcup \{ \mathcal{E}_m : m \in \mathbb{N} \} \) covers \( X \). Since each \( \mathcal{E}_m \) refines \( \mathcal{U}_m \), we can choose a function \( \phi_m : \mathcal{E}_m \rightarrow X \) for each \( m \in \mathbb{N} \) so that if \( O \in \mathcal{E}_m \) we have

\[
O \subset W_m(\phi_m(O)).
\]

Thus, if \( n \in \{1, \ldots, r_m(\phi_m(O))\} \) for some \( O \in \mathcal{E}_m \), then

\[
O \subset W_m(\phi_m(O)) \subset W_{m,n}(\phi_m(O)),
\]

so that for \( \alpha \in \{1, 2\} \) we have

\[
O \times V_{m,n}^\alpha(\phi_m(O)) \subset W_{m,n}(\phi_m(O)) \times V_{m,n}^\alpha(\phi_m(O)) \subset U_{m,n}^\alpha.
\]

For each \( m, n \in \mathbb{N} \) and \( \alpha \in \{1, 2\} \), we define

\[
C_{m,n}^\alpha = \bigcup \{ O \times V_{m,n}^\alpha(\phi_m(O)) : n \in \{1, \ldots, r_m(\phi_m(O))\} \}
\]

and set

\[
\mathcal{E}_{m,n}^\alpha = \{ C_{m,n}^1, C_{m,n}^2 \}.
\]

If \( (x, y) \in C_{m,n}^\alpha \), then there exists \( O \in \mathcal{E}_m \) with \( n \in \{1, \ldots, r_m(\phi_m(O))\} \) so that

\[
(x, y) \in O \times V_{m,n}^\alpha(\phi_m(O)) \subset U_{m,n}^\alpha.
\]

Therefore, \( \mathcal{E}_{m,n}^\alpha \) is a precise open refinement of \( \mathcal{U}_{m,n} \). Furthermore, since the elements of \( \mathcal{E}_m \) are pairwise disjoint, and since

\[
V_{m,n}^1(\phi_m(O)) \cap V_{m,n}^2(\phi_m(O)) = \emptyset
\]

for any \( O \in \mathcal{E}_m \) with \( n \in \{1, \ldots, r_m(\phi_m(O))\} \), we see that

\[
C_{m,n}^1 \cap C_{m,n}^2 = \emptyset.
\]

Finally, since \( \bigcup \{ \mathcal{E}_m : m \in \mathbb{N} \} \) covers \( X \), given a point \((x, y) \in X \times Y\) we can find \( m \in \mathbb{N} \) and \( O \in \mathcal{E}_m \) so that \( x \in O \). Since \( \pi^{-1}(\phi_m(O)) \) is covered by

\[
\{ V_{m,n}^\alpha(\phi_m(O)) : \alpha = 1, 2, n = 1, \ldots, r_m(\phi_m(O)) \},
\]

we can also find \( \alpha \in \{1, 2\} \) and \( n \in \{1, \ldots, r_m(\phi_m(O))\} \) so that

\[
y \in V_{m,n}^\alpha(\phi_m(O)).
\]

Therefore, we see that

\[
(x, y) \in O \times V_{m,n}^\alpha(\phi_m(O)) \subset C_{m,n}^\alpha,
\]

so that \( \bigcup \{ \mathcal{E}_{m,n} : m, n \in \mathbb{N} \} \) forms an open cover of \( X \times Y \). By Lemma 1, we conclude that \( X \times Y \) is weakly infinite-dimensional. \( \square \)

We single out two special cases of separate interest. In the second corollary, which we were unable to find in the literature, \( I \) denotes the closed unit interval.
Corollary 1. The product of R. Pol's uncountable-dimensional compact C-space with any weakly infinite-dimensional compactum is again weakly infinite-dimensional.

Corollary 2. \( X \) is a weakly infinite-dimensional compactum if and only if \( X \times I \) is a weakly infinite-dimensional compactum.

Question 1. If \( X \times I \) is weakly infinite-dimensional, then must \( X \) have property C?

In the final part of this note, we consider the following question.

Question 2. What properties must an infinite-dimensional space possess to ensure that its product with every zero-dimensional, hence countable-dimensional, space is a C-space (is weakly infinite-dimensional)?

Necessarily, such a space must itself be a C-space (weakly infinite-dimensional), but it is also known that this is not a sufficient condition for productivity with zero-dimensional factors [P]. On the other hand, as the following example shows, while we have shown that compactness of a C-space (weakly infinite-dimensional) factor is a sufficient condition for such productivity, it is not a necessary condition.

Recall that R. Pol's compactum, when constructed as a subspace of the Hilbert cube, has the form \( P = X \cup B_1 \cup B_2 \), where \( X \) is a topologically complete, totally disconnected, strongly infinite-dimensional subspace of the Hilbert cube with countable-dimensional remainder \( B_1 \cup B_2 = P/X \). So constructed, \( B_1 \) and \( B_2 \) are disjoint Bernstein sets; i.e., all compact subsets of \( B_1 \) and \( B_2 \) are countable [P2]. It is known that \( B_1 \cup X \) and \( B_2 \cup X \) are noncompact C-spaces [EP].

We will also need the following classical result.

Lemma 2 [E, 4.3.6]. If \( f : X \to Y \) is a closed mapping between spaces \( X \) and \( Y \) where \( \dim f^{-1}(y) \leq 0 \) for each \( y \in Y \), then \( \dim X \leq \dim Y \).

Theorem 2. Given \( B_i \), where \( i \in \{1, 2\} \), and \( X \) as above, the product of \( B_i \cup X \) with any zero-dimensional space \( Z \) is a C-space and thus is weakly infinite-dimensional.

Proof. The proof follows ideas of [EP] and [P2]. We assume, without loss of generality, that \( i = 1 \), set \( X_1 = (B_1 \cup X) \), and let \( Z \) be any zero-dimensional space. We will show that \( X_1 \times Z \) has property C as a subspace of \( P \times Z \).

Given a sequence \( \{ \mathcal{U}_n : n \in \mathbb{N} \} \) of covers of \( X_1 \times Z \) by open subsets of \( P \times Z \), for each fixed \( n \in \mathbb{N} \) we set

\[
K_n = (P \times Z) \setminus \bigcup \{ U : U \in \mathcal{U}_n \}.
\]

Each \( K_n \) is a closed subset of \( P \times Z \) contained entirely in \( B_2 \times Z \). Since the restricted projection \( \pi : K_n \to Z \) is a closed mapping with countable, hence at most zero-dimensional, fibers, we apply Lemma 2 to see that \( \dim K_n \leq 0 \).
Being a countable union of closed zero-dimensional sets,
\[
\dim \bigcup\{K_n : n \in \mathbb{N}\} \leq 0.
\]
Thus, we can choose an at most zero-dimensional \(G_\delta\)-subset \(A \subset P \times Z\) [E, 4.1.19] so that
\[
\bigcup\{K_n : n \in \mathbb{N}\} \subset A.
\]
Then, we also have
\[
(P \times Z) \setminus A \subset (P \times Z) \setminus \bigcup\{K_n : n \in \mathbb{N}\}
\]
\[
\subset \bigcap\{(P \times Z) \setminus K_n : n \in \mathbb{N}\}
\]
\[
\subset \bigcap\{\cup\{U : U \in \mathcal{U}_n\} : n \in \mathbb{N}\},
\]
and, in particular, for each \(n \in \mathbb{N}\) we see that \(\mathcal{U}_n\) is an open cover of \((P \times Z) \setminus A\) in \(P \times Z\). Since \(P\) is a compact \(C\)-space, \(P \times Z\) is a \(C\)-space, and, being an \(F_\sigma\) subset of \(P \times Z\), \((P \times Z) \setminus A\) is also a \(C\)-space [AG]. Thus, we can choose a \(C\)-refinement \(\mathcal{V}_n\) of \(\mathcal{U}_n\) for each \(n > 1\) so that the \(\bigcup\{\mathcal{V}_n : n > 1\}\) is a cover of \((P \times Z) \setminus A\), hence also a cover of \((X_1 \times Z) \setminus A\).

Finally, since \((X_1 \times Z) \cap A \subset A\), we see that \((X_1 \times Z) \cap A\) can be at most zero-dimensional. Therefore, we can choose a \(C\)-refinement \(\mathcal{Y}_1\) of the remaining cover \(\mathcal{W}_1\), so that \(\mathcal{Y}_1\) still covers \((X_1 \times Z) \cap A\). Then, the \(\bigcup\{\mathcal{Y}_n : n \in \mathbb{N}\}\) is now a cover of all of \(X_1 \times Z\), which completes the proof that \(X_1 \times Z\) has property \(C\).  \(\square\)

**References**


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