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THE RELATIONSHIPS OF SPANS OF CONVEX CONTINUA IN R^n

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ABSTRACT. It has been conjectured that $\sigma^*(X) \geq \frac{1}{2}\sigma_0^*(X)$ for each nonempty connected metric space X. In this paper we show that $\sigma^*(X) \geq \frac{\sqrt{13}}{4}\sigma_0^*(X)$ for each convex continuum X in R^n. We also show that under certain conditions a lower bound for the ratio $\sigma^*(X)/\sigma_0^*(X)$ is larger than $\frac{\sqrt{13}}{4}$. It has also been conjectured that $\sigma^*(X) \geq \sigma(X)/2$ and that $\sigma_0^*(X) \geq \sigma_0(X)/2$ for each nonempty connected metric space X. We show that these two inequalities hold when X is a convex continuum in R^n.

Generally speaking, the spans of an object are connectedness-type analogues of its diameter. We follow the definitions from [1]: Let X be a nonempty connected metric space. The standard projections of the product $X \times X$ onto X are denoted by p_1 and p_2; that is, $p_1(x, x') = x$ and $p_2(x, x') = x'$ for $(x, x') \in X \times X$. The *surjective span* $\sigma^*(X)$ of X is the least upper bound of real numbers α such that there exist nonempty connected sets $C_\alpha \subset X \times X$ with $\operatorname{dist}(x, x') \geq \alpha$ for $(x, x') \in C_\alpha$ and $p_1(C_\alpha) = p_2(C_\alpha) = X$. Relaxing the last condition to $p_1(C_\alpha) = p_2(C_\alpha)$, or $p_2(C_\alpha) = X$, or $p_1(C_\alpha) \subset p_2(C_\alpha)$, one obtains the definitions of the *span* $\sigma(X)$, the *surjective semispan* $\sigma_0^*(X)$, and the *semispan* $\sigma_0(X)$ of X, respectively. Hence,

$$0 \leq \sigma^*(X) \leq \sigma(X) \leq \sigma_0(X) \leq \operatorname{diam} X, \tag{1}$$

$$0 \leq \sigma^*(X) \leq \sigma_0^*(X) \leq \sigma_0(X) \leq \operatorname{diam} X. \tag{2}$$

From the definitions, it is clear that if T is an isometry, then $\sigma(X) = \sigma[T(X)]$, $\sigma^*(X) = \sigma^*[T(X)]$, $\sigma_0(X) = \sigma_0[T(X)]$, and $\sigma_0^*(X) = \sigma_0^*[T(X)]$. When X is a continuum, that is, a nonempty, compact, connected metric space, we need to consider only those sets C_α which are closed in $X \times X$. It has been shown that when X is a continuum, the requirement that $\operatorname{dist}(x, x') \geq \alpha$ for $(x, x') \in C_\alpha$ can be replaced in the above definitions by $\operatorname{dist}(x, x') = \alpha$ for $(x, x') \in C_\alpha$ [2, p. 169].

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There is a conjecture that $\sigma^*(X) \geq \sigma_0^*(X)/2$ for each nonempty connected metric space X [1, p. 38]. In this paper we show that $\sigma^*(X) \geq \frac{\sqrt{13}}{4}\sigma_0^*(X)$ for each convex continuum X in R^n. We also show that under certain conditions a lower bound for the ratio $\sigma^*(X)/\sigma_0^*(X)$ is larger than $\sqrt{13}/4$. It has also been conjectured that $\sigma^*(X) \geq \sigma(X)/2$ and that $\sigma_0^*(X) \geq \sigma_0(X)/2$ for each nonempty connected metric space X [1, p. 37]. We show that these two inequalities hold when X is a convex continuum in R^n.

We will use ρ to denote the usual metrics on R^2, R^n, and $R^n \times R^n$. It will be clear in all cases which metric ρ represents.

Lemma 1. *Let* $Y = \{(x_1, x_2, \dots, x_n) \mid x_2 = x_3 = \cdots = x_n = 0,\ 0 \leq x_1 \leq d\} \cup \{(x_1, x_2, \dots, x_n) \mid 0 \leq x_1 \leq d/2,\ (x_1 - d)^2 + x_2^2 + \cdots + x_n^2 = d^2\} \cup \{(x_1, x_2, \dots, x_n) \mid d/2 \leq x_1 \leq d,\ x_1^2 + x_2^2 + \cdots + x_n^2 = d^2\}$. *Then for each* $p \in Y$, $\rho(p, (d/2, 0, \dots, 0)) \leq \frac{\sqrt{3}}{2}d$.

Proof. We need just to observe that

$$f : [d/2, d] \to R, \quad \text{where } f(t) = \sqrt{(t - \frac{d}{2})^2 + d^2 - t^2},$$

is a decreasing function. Hence the maximum value of f occurs when $t = d/2$. Also, $g : [0, d/2] \to R$ where $g(t) = \sqrt{(t - \frac{d}{2})^2 + d^2 - (t - d)^2}$ is an increasing function. Hence the maximum value of g occurs when $t = d/2$. Also, $g(d/2) = f(d/2) = \frac{\sqrt{3}}{2}d$. Consequently, if $p \in Y$, then $\rho(p, (d/2, 0, \dots, 0)) \leq \frac{\sqrt{3}}{2}d$.

Let $a, b \in R^n$; we denote the straight line segment in R^n with endpoints a and b by $\overline{ab}$.

Lemma 2. *If* $X \subset R^n$ *and* T *is an isometry from* X *into* R^n, *then for* $x, y \in X$, $T[\overline{xy}] = \overline{T(x)T(y)}$.

Lemma 3. *If* X *is a convex continuum in* R^n, *then* $\sigma_0^*(X) \leq \frac{\sqrt{3}}{2} \operatorname{diam} X$.

Proof. Let $a, b \in X$ such that $\rho(a, b) = \operatorname{diam} X = d$. Let q be the midpoint of $\overline{ab}$.

Let $C \subset X \times X$ such that $p_2(C) = X$ and, for each $(x, y) \in C$, $\rho(x, y) = \sigma_0^*(X)$. Hence, for some $p \in X$, $(p, q) \in C$, since $q \in \overline{ab} \subset X$ and $p_2(C) = X$. Clearly, there is an isometry T from X into R^n which is the composition of a translation and one or two rotations such that $T(a) = \theta$, $T(b) = (d, 0, \dots, 0)$, and $T(p)$ is in the plane $R \times R \times \{0\} \times \cdots \times \{0\}$. Note also that $T(q) = (d/2, 0, \dots, 0)$. The set

$$T[X] \subset \{(x_1, x_2, \dots, x_n) \mid x_1^2 + x_2^2 + \cdots + x_n^2 \leq d^2\}$$
$$\cap \{(x_1, x_2, \dots, x_n) \mid (x_1 - d)^2 + x_2^2 + \cdots + x_n^2 \leq d^2\} = Z,$$

since $\rho(a, b) = \rho(T(a), T(b)) = \operatorname{diam} X = d$. Let z be the point where the ray $\overrightarrow{T(q)T(p)}$, with initial point $T(q)$ through the point $T(p)$, inter-

sects the boundary of Z. Hence $\rho(q, p) = \rho(T(q), T(p)) \leq \rho(T(q), z) = \rho((d/2, 0, \ldots, 0), z)$ and $\rho((d/2, 0, \ldots, 0), z) \leq \frac{\sqrt{3}}{2}d$ by Lemma 1. Consequently, $\sigma_0^*(X) \leq \frac{\sqrt{3}}{2} \operatorname{diam} X$.

Lemma 4. *Let* $p_t \in \{(x, y) \in R^2 \mid (x-d)^2 + y^2 = t^2d^2$, *where* $0 \leq t \leq 1\} \cap \{(x, y) \mid x^2 + y^2 \leq d^2\}$. *Then* $\rho((d, 0), \overline{\theta p_t}) \geq \frac{dt\sqrt{4-t^2}}{2}$.

Proof. The lemma is trivial when $d \leq 0$. We just need to consider the case when $d > 0$. Clearly, the lemma is true when $t = 0$. We will just consider the case where $0 < t \leq 1$. Also, we will only consider the case when $\pi_2(p_t) \geq 0$, where π_2 is the standard projection from R^2 into the second factor space. The case when $\pi_2(p_t) < 0$ is similar.

Let $r_t = (\frac{d(2-t^2)}{2}, \frac{dt\sqrt{4-t^2}}{2})$. Let q be the point on $\overleftrightarrow{\theta r_t}$, the line through θ and r_t, such that $\overleftrightarrow{\theta r_t}$ and $\overleftrightarrow{q(d, 0)}$, the line through q and $(d, 0)$, are perpendicular to each other. The triangles $\triangle\theta(\frac{d(2-t^2)}{2}, 0)r_t$ and $\triangle\theta q(d, 0)$ are congruent, since angles $< \theta(\frac{d(2-t^2)}{2}, 0)r_t$ and $< \theta q(d, 0)$ are right angles, $< r_t\theta(\frac{d(2-t^2)}{2}, 0) =< q\theta(d, 0)$, and $\rho(\theta, r_t) = \rho(\theta, (d, 0)) = d$. Consequently, $\rho((d, 0), q) = \rho((\frac{d(2-t^2)}{2}, 0), r_t) = \frac{dt\sqrt{4-t^2}}{2}$.

Let $p_t \in \{(x, y) \mid (x-d)^2+y^2 = t^2d^2$, where $0 < t \leq 1$ and $y \geq 0\} \cap \{(x, y) \mid x^2 + y^2 \leq d^2\}$. There are two cases to consider.

Case 1. $m(\overline{\theta p_t}) \geq m(\overline{\theta r_t})$.

Here $m(\overline{\theta p_t})$ and $m(\overline{\theta r_t})$ represent the slopes of the corresponding line segments. Clearly, $\rho((d, 0), \overline{\theta p_t}) \geq \rho((d, 0), \overline{\theta r_t}) = \frac{dt\sqrt{4-t^2}}{2}$.

Case 2. $m(\overline{\theta p_t}) < m(\overline{\theta r_t})$.

Let $(\frac{d(2-t^2)}{2}, s)$ be a point on $\overleftrightarrow{\theta p_t}$. Then $0 \leq s < \frac{dt\sqrt{4-t^2}}{2}$, since $m(\overleftrightarrow{\theta p_t}) = m(\overline{\theta p_t}) < m(\overline{\theta r_t})$ and $\pi_2(p_t) \geq 0$. Consequently, $\overleftrightarrow{\theta p_t}$ intersects the interior of the circle $(x-d)^2 + y^2 = t^2d^2$. Hence $\overleftrightarrow{\theta p_t}$ intersects this circle in two points, u and v, and either $u = p_t$ or $v = p_t$. Either $\pi_1(u) < \frac{d(2-t^2)}{2} < \pi_1(v)$, where π_1 is the standard projection from R^2 into the first factor space, or $\pi_1(v) < \frac{d(2-t^2)}{2} < \pi_1(u)$. We will assume that $\frac{d(2-t^2)}{2} < \pi_1(v)$. Either $d < \pi_1(v)$ or $\frac{d(2-t^2)}{2} < \pi_1(v) \leq d$. If $d < \pi_1(v)$, then $v \notin \{(x, y) \mid x^2+y^2 \leq d^2\}$. Hence $v \neq p_t$. If $\frac{d(2-t^2)}{2} < \pi_1(v) \leq d$, then $\frac{dt\sqrt{4-t^2}}{2} < \pi_2(v) \leq td$, where π_2 is the standard projection map from R^2 into the second factor space. However, if $(x, y) \in \{(x, y) \mid x^2 + y^2 = d^2\}$ and $\frac{d(2-t^2)}{2} < x \leq d$, then $0 \leq y < \frac{dt\sqrt{4-t^2}}{2}$. Hence, $v \neq p_t$. So $u = p_t$. Observe that $\overline{\theta p_t}$ does not intersect the

interior of the circle $(x-d)^2+y^2=t^2d^2$. Hence, $\rho(\overline{\theta p_t},(d,0))=td>\frac{dt\sqrt{4-t^2}}{2}$.

In both cases we see that $\rho(\overline{\theta p_t},(d,0))\geq\frac{dt\sqrt{4-t^2}}{2}$, as desired.

Theorem 1. *Let X be a nondegenerate convex continuum in R^n. Then $\sigma^*(X)\geq[\sqrt{4-[\sigma_0^*(X)/\operatorname{diam}X]^2}/2]\sigma_0^*(X)$.*

Proof. Let $C\subset X\times X$ such that C is closed and connected, $p_2(C)=X$, and $\rho(x,y)=\sigma_0^*(X)$ for each $(x,y)\in C$. Let $a,b\in X$ such that $\rho(a,b)=\operatorname{diam}X=d$. There must be points $a_1,b_1\in X$ such that (a_1,b) and (b_1,a) are elements of C, since $p_2(C)=X$. The line segment $\overline{aa_1}\subset X$, since X is convex. Also, $\overline{b_1b}\subset X$. Hence, $\overline{aa_1}\times\{b\}\subseteq X\times X$ and $\{a\}\times\overline{b_1b}\subseteq X\times X$.

Let $C'=C\cup(\overline{aa_1}\times\{b\})\cup(\{a\}\times\overline{b_1b})\cup C^{-1}$. So, $p_1(C')=p_2(C')=X$, since $X=p_2(C)\subseteq p_2(C')$ and $X=p_2(C)=p_1(C^{-1})\subseteq p_1(C')$. Each of the sets C, $\overline{aa_1}\times\{b\}$, $\{a\}\times\overline{bb_1}$, and C^{-1} is connected. Also, $(a_1,b)\in C\cap(\overline{aa_1}\times\{b\})$, $(a,b)\in(\overline{aa_1}\times\{b\})\cap(\{a\}\times\overline{b_1b})$, and $(a,b_1)\in(\{a\}\times\overline{b_1b})\cap C^{-1}$. Hence, C' is connected.

Let T be an isometry from X into R^n such that $T(a)=\theta$, $T(b)=(d,0,\dots,0)$, and $T(a_1)=p$ where p is a point in the plane $R\times R\times\{0\}\times\cdots\times\{0\}$. Hence $T[\overline{aa_1}]=\overline{\theta p}$ and $\rho(a_1,b)=\rho((d,0,\dots,0),p)=\sigma_0^*(X)$. Also, $\sigma_0^*(X)=dt$ where $0\leq t\leq\sqrt{3}/2$, by Lemma 3 and the fact that $\sigma_0^*(X)\geq 0$. Note also that $T[X]\subset\{(x_1,\dots,x_n)\mid x_1^2+\cdots+x_n^2\leq d^2\}\cap\{(x_1,\dots,x_n)\mid (x_1-d)^2+x_2^2+\cdots+x_n^2\leq d^2\}$. Consequently, $p\in\{(x_1,x_2,0,\dots,0)\mid (x_1-d)^2+x_2^2=t^2d^2\}\cap\{(x_1,x_2,0,\dots,0)\mid x_1^2+x_2^2\leq d^2\}$. Hence, $\rho(\overline{aa_1},b)=\rho(\overline{\theta p},(d,0,\dots,0))\geq\frac{dt\sqrt{4-t^2}}{2}$, by Lemma 4. Similarly, $\rho(\overline{b_1b},a)\geq\frac{dt\sqrt{4-t^2}}{2}$. So $\sigma^*(X)\geq\frac{dt\sqrt{4-t^2}}{2}$ and $\frac{dt\sqrt{4-t^2}}{2}=\left[\sqrt{4-[\sigma_0^*(X)/\operatorname{diam}X)]^2}/2\right]\sigma_0^*(X)$.

Corollary 1.1. *If X is a convex continuum in R^n, then $\sigma^*(X)\geq\frac{\sqrt{13}}{4}\sigma_0^*(X)$.*

Proof. If $\operatorname{diam}X\neq 0$, then $0\leq\sigma_0^*(X)/\operatorname{diam}X\leq\sqrt{3}/2$, by Lemma 3 and the fact that $\sigma_0^*(X)$ and $\operatorname{diam}X$ are both nonnegative. Consequently, the minimal value of $\sqrt{4-[\sigma_0^*(X)/\operatorname{diam}X]^2}/2$ is $\sqrt{13}/4$. If $\operatorname{diam}X=0$, then $\sigma^*(X)=\sigma_0^*(X)=0$, and the inequality holds.

Theorem 2. *Let X be a nondegenerate convex continuum in R^n. Then $\sigma^*(X)\geq\sigma_0(X)/2$.*

Proof. Let $C\subset X\times X$ such that C is closed and connected, $p_1(C)\subset p_2(C)$, and $\rho(x,y)=\sigma_0(X)$ for each $(x,y)\in C$. Let $a,b\in p_2(C)$ such that $\rho(a,b)=\operatorname{diam}p_2(C)=d$. Clearly, $d\geq\sigma_0(X)$. There must be points $a_1,b_1\in p_1(C)$ such that $(a_1,b)\in C$ and $(b_1,a)\in C$, since $a,b\in p_2(C)$. Let T be an isometry from X into R^n such that $T(a)=\theta$, $T(b)=(d,0,\dots,0)$, and $T(a_1)=p$ where p is a point in the plane $R\times R\times\{0\}\times\cdots\times\{0\}$. Let q be the

midpoint of the straight line segment $\overline{ab}$. Clearly, $\pi_1[T(q)] = (d/2, 0, \dots, 0)$.

Let $X_L = \{x \in X \mid \pi_1[T(x)] \le d/2\}$ and $X_R = \{x \in X \mid \pi_1[T(x)] \ge d/2\}$. Let $SX_L = \cup\{\overline{xq} \mid x \in X_L\}$ and $SX_R = \cup\{\overline{qx} \mid x \in X_R\}$. The set SX_L is connected, since it is the union of arcs and each of these arcs contains the point q. Similarly, SX_R is connected. Also, for each line segment $\overline{xq}$, where $x \in X_L$, if $y \in \overline{xq}$, then $\pi_1[T(y)] \le d/2$ since $\pi_1[T(x)] \le d/2$, $\pi_1[T(q)] = d/2$, and $T(\overline{xq}) = \overline{T(x)T(q)}$. Similarly, for each line segment $\overline{qx}$, where $x \in X_R$, if $y \in \overline{qx}$, then $\pi_1[T(y)] \ge d/2$.

Let $C' = C \cup (\overline{aa_1} \times \{b\}) \cup (SX_L \times \{b\}) \cup (\{a\} \times SX_R) \cup (\overline{b_1 b} \times \{a\}) \cup (SX_R \times \{a\}) \cup (\{b\} \times SX_L)$. The set $p_2(C') = X$, since $p_2(\{a\} \times SX_R) = SX_R$, $p_2(\{b\} \times SX_L) = SX_L$, and $X = SX_L \cup SX_R$. Similarly, $p_1(C') = X$.

Each of the sets C, $\overline{aa_1} \times \{b\}$, $SX_L \times \{b\}$, $\{a\} \times SX_R$, $\overline{b_1 b} \times \{a\}$, $SX_R \times \{a\}$, and $\{b\} \times SX_L$ is connected. The set C' is connected since $(a_1, b) \in C \cap (\overline{aa_1} \times \{b\})$, $(a, b) \in (\overline{aa_1} \times \{b\}) \cap (SX_L \times \{b\}) \cap (\{a\} \times SX_R)$, $(b_1, a) \in C \cap (\overline{b_1 b} \times \{a\})$, and $(b, a) \in (\overline{b_1 b} \times \{a\}) \cap (SX_R \times \{a\}) \cap (\{b\} \times SX_L)$.

Next, we will see that if $(x, y) \in C'$, then $\rho(x, y) \ge \sigma_0(X)/2$. Let $(x, y) \in C'$. If $(x, y) \in C$, then $\rho(x, y) = \sigma_0(X)$. If $(x, y) = (x, b) \in \overline{aa_1} \times \{b\}$, then $\rho(x, b) = \rho(T(x), T(b)) = \rho(T(x), (d, 0, \dots, 0))$. In addition, $\rho(T(x), (d, 0, \dots, 0)) \ge \rho(\overline{\theta p}, (d, 0, \dots, 0))$, since $x \in \overline{aa_1}$ and $T(x) \in T[\overline{aa_1}] = \overline{\theta p}$. Clearly, $\sigma_0(X) = dt$ where $0 \le t \le 1$. Hence, $\sigma_0(X) = \rho(a_1, b) = \rho(p, (d, 0, \dots, 0)) = dt$. So $p \in \{(x_1, x_2, 0, \dots, 0) \mid (x_1 - d)^2 + x_2^2 = t^2 d^2\} \cap \{(x_1, x_2, 0, \dots, 0) \mid x_1^2 + x_2^2 \le d^2\}$. So, $\rho(\overline{\theta p}, (d, 0, \dots, 0)) \ge \frac{dt\sqrt{4-t^2}}{2} = \sqrt{4 - t^2}(\sigma_0(X)/2) \ge \sqrt{3}(\sigma_0(X)/2)$, by Lemma 4. So, $\rho(x, b) \ge \sigma_0(X)/2$. Similarly, if $(x, y) = (x, a) \in \overline{b_1 b} \times \{a\}$, then $\rho(x, a) \ge \sigma_0(X)/2$. If $(x, y) \in SX_L \times \{b\}$, then $\rho(x, y) = \rho(T(x), T(b)) \ge d/2 \ge \sigma_0(X)/2$, since $\pi_1[T(x)] \le d/2$ and $\pi_1[T(b)] = d$. Similarly, if $(x, y) \in (\{a\} \times SX_R) \cup (SX_R \times \{a\}) \cup (\{b\} \times SX_L)$, then $\rho(x, y) \ge \sigma_0(X)/2$. Consequently, in each case $\rho(x, y) \ge \sigma_0(X)/2$ and $\sigma^*(X) \ge \sigma_0(X)/2$.

Corollary 2.1. *Let X be a convex continuum in R^n. Then $\sigma^*(X) \ge \sigma(X)/2$.*

Proof. This follows from Theorem 2 and (1).

Corollary 2.2. *Let X be a convex continuum in R^n. Then $\sigma_0^*(X) \ge \sigma_0(X)/2$.*

Proof. This follows from Theorem 2 and (2).

References

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