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SEPARATE CONVERGENCE OF GENERAL T-FRACTIONS

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ABSTRACT. This article is concerned with the separate convergence of the sequences of numerators $\{A_n(z)\}$ and denominators $\{B_n(z)\}$ of the approximants $A_n(z)/(B_n(z)$ of the general T-fraction

$$\underset{n=1}{\overset{\infty}{K}}\left(\frac{F_n z}{1+G_n z}\right).$$

Convergence results for sequences $\{A_n(z)/\Gamma_n(z)\}$ and $\{B_n(z)/\Gamma_n(z)\}$, where the sequence $\{\Gamma_n(z)\}$ is "sufficiently simple" are also derived.

INTRODUCTION

A continued fraction

$$\underset{n=1}{\overset{\infty}{K}}\left(\frac{a_n(z)}{b_n(z)}\right), \tag{1.1}$$

with nth approximant $A_n(z)/B_n(z)$, is said to *converge separately* for $z \in \Delta$ if there exists a sequence $\{\Gamma_n(z)\}$ such that $\{A_n(z)/\Gamma_n(z)\}$ and $\{B_n(z)/\Gamma_n(z)\}$ both converge for $z \in \Delta$. The concept of separate convergence has attracted a certain amount of attention recently (see [1, 2, 4, 5]).

Here we present an approach suggested by the article of Schwartz [3] in which results on separate convergence (the term was not used) for regular C-fractions and J-fractions were obtained. In this paper we shall derive results on general T-fractions. Since the orthogonal L-polynomials arising from strong moment problems can be realized as denominators of continued fractions equivalent to general T-fractions for all solutions of the strong Stieltjes moment problem and for many solutions of the strong Hamburger problem the relevance of our results for an analysis of the asymptotic behavior of such sequences is clear.

2. A SERIES OF BASIC LEMMAS

Let $R_n = R_n(d_1, \dots, d_n; g_1, \dots, g_n)$ be a polynomial in the indicated variables with nonnegative coefficients. Define

$$\widehat{R}_n := R_n(|d_1|, \dots, |d_n|; |g_1|, \dots, |g_n|). \tag{2.1}$$

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Then

$$|R_n| \le \widehat{R}_n. \tag{2.2}$$

Let $\{W_n\}$ be defined by the recursion relation

$$W_n = (1+d_n)W_{n-1} + g_n W_{n-2}, \qquad n \ge 1, \ W_0 = 1, \ W_{-1} = 0. \tag{2.3}$$

We can now prove the first of our series of lemmas.

Lemma 2.1. *If* $\{W_n\}$ *satisfies* (2.3), *then*

$$|W_n| \le \prod_{k=1}^{n}(1+|d_k|+|g_k|), \qquad n \ge 1. \tag{2.4}$$

Proof. We have

$$\widehat{W}_n = (1+|d_n|)\widehat{W}_{n-1} + |g_n|\widehat{W}_{n-2}, \qquad \widehat{W}_0 = 1, \ \widehat{W}_{-1} = 0.$$

From this one deduces by induction that

$$\widehat{W}_n \le \prod_{k=1}^{n}(1+|d_k|+|g_k|). \tag{2.5}$$

The inequality (2.5) together with (2.2) then yields (2.4).

Lemma 2.2. *If* $\{W_n\}$ *satisfies* (2.3) *and if*

$$\sum_{n=1}^{\infty}|d_n| < \infty, \qquad \sum_{n=1}^{\infty}|g_n| < \infty, \tag{2.6}$$

then $\lim_{n\to\infty} W_n$ *exists and equals* W.

Proof. It follows from (2.6) and (2.4) that there exists a constant M such that

$$|W_n| \le M, \qquad n \ge 1.$$

From (2.3) one deduces

$$\begin{aligned} |W_n - W_{n-1}| &\le |d_n|\,|W_{n-1}| + |g_n|\,|W_{n-2}| \\ &\le (|d_n|+|g_n|)M. \end{aligned} \tag{2.7}$$

Hence

$$|W_{n+m} - W_n| < \sum_{k=n+1}^{n+m} |W_k - W_{k-1}| \le M \sum_{k=n+1}^{n+m} (|d_k|+|g_k|) \tag{2.8}$$

and $\{W_n\}$ converges.

Lemma 2.3. *If* $\{W_n(z)\}$ *satisfies* (2.3) *where* $d_n = d_n(z)$, $g_n = g_n(z)$, $n \ge 1$, $z \in \Delta$, *and if*

$$\sum_{n=1}^{\infty}|d_n(z)|, \qquad \sum_{n=1}^{\infty}|g_n(z)|$$

converge uniformly on compact subsets of Δ, *then* $\{W_n(z)\}$ *converges uniformly on compact subsets of* Δ *to* $W(z)$.

The proof is analogous to that of Lemma 2.2.

Lemma 2.4. *If* $\{W_n\}$ *satisfies* (2.3) *and if*

$$d_n \to 0, \qquad \sum_{n=1}^{\infty} |g_n| < \infty,$$

then there exists a sequence of positive integers $\{p_n\}$ *such that*

$$\sum_{n=1}^{\infty} |d_n|^{p_n} < \infty. \tag{2.9}$$

Further, if we define

$$E_p(x) := \exp - \sum_{k=0}^{p-1} (-1)^k x^k / k$$

and

$$W_n^{\dagger} := W_n \prod_{k=1}^{n} E_{p_k}(d_k),$$

then $\{W_n^{\dagger}\}$ *converges.*

Proof. Let n_0 be such that $-\log|d_n| > 0$ for $n > n_0$. Then a possible choice for $\{p_n\}$ is

$$p_n := 1, \quad 1 \le n \le n_0;$$
$$p_n := \text{an integer greater than } -n/\log|d_n|, \quad n > n_0.$$

One easily verifies that for this choice $|d_n|^{p_n} < e^{-n}$, $n > n_0$, so that (2.9) holds.

For $|x| < 1$,

$$\begin{aligned}(1+x)E_{p_n}(x) &= \exp\left(\log(1+x) - \sum_{k=0}^{p_n-1} (-1)^k x^k / k\right) \\ &= \exp\left(\sum_{k=p_n}^{\infty} (-1)^k x^k / k\right) \\ &=: 1 + \gamma_n(x).\end{aligned}$$

Now choose n_1 so that

$$u_n := \sum_{k=p_n}^{\infty} d_n^k / k$$

satisfies $|u_n| < 1/2$ for $n > n_1$. For $n > \max(n_0, n_1)$ we then have

$$\begin{aligned}|\gamma_n(d_n)| &= |\exp u_n - 1| = \left|\sum_{k=1}^{\infty} u_n^k / k!\right| \\ &\le \sum_{k=1}^{\infty} \frac{|u_n|^k}{k!} < \frac{|u_n|}{1-|u_n|} < \frac{2|d_n|^{p_n}}{1-|d_n|}.\end{aligned}$$

It follows that $\sum |\gamma_n(d_n)| < \infty$ from which the convergence of $\{W_n^{\dagger}\}$ follows by Lemma 2.2.

Lemma 2.5. *If* $\{W_n\}$ *satisfies* (2.3) *and*

$$\sum_{n=1}^{\infty} |d_n^2| < \infty, \qquad \sum_{n=1}^{\infty} |g_n| < \infty$$

then $\{W_n \prod_{k=1}^{n}(1-d_k)\}$ *converges.*

Proof. Set $W_n \prod_{k=1}^{n}(1-d_k) =: W_n^x$, then

$$W_n^x = (1+(-d_n^2))W_{n-1}^x + g_n(1-d_n)(1-d_{n-1})W_{n-2}^x .$$

3. Applications to general T-fractions

By a general T-fraction we mean a continued fraction of the form

$$\underset{n=1}{\overset{\infty}{K}} \left(\frac{F_n z}{1+G_n z}\right), \qquad F_n \neq 0, \ n \geq 1 . \tag{3.1}$$

We then have from Lemma 2.3

Theorem 3.1. *Let* $A_n(z)$ *and* $B_z(z)$ *be the numerators and denominators, respectively, of the nth approximant of the general* T*-fraction* (3.1) *satisfying*

$$\sum_{n=1}^{\infty} |F_n| < \infty, \qquad \sum_{n=1}^{\infty} |G_n| < \infty .$$

Then the sequences $\{A_n(z)\}$ *and* $\{B_n(z)\}$ *converge, uniformly on compact subsets of* $\mathbf{C}$, *to entire functions* $A(z)$ *and* $B(z)$ *of order at most one. Further* $B(0) = 1$, $A(0) = 0$, $A'(0) = F_1 \neq 0$ *so that neither function is identically equal to zero.*

Proof. That $A(z)$ and $B(z)$ exist and are entire follows from Lemma 2.3. From (2.4) and (2.2) we have

$$|B_n(z)| < \prod_{k=1}^{n}(1+(|F_k|+|G_k|)|z|)$$

and hence

$$|B(z)| \leq \prod_{n=1}^{\infty}(1+(|F_n|+|G_n|)|z|) < e^{|z|(\sum_{n=1}^{\infty}(|F_n|+|G_n|))} . \tag{3.2}$$

It follows that $B(z)$ (and by an analogous argument $A(z)$) is at most of order one. The inequality (3.2), if it is applied to a tail of (3.1) also allows one to conclude that if $B(z)$ is of order one, then it is of minimal type. That $B(0) = 1$ follows from the fact that $B_n(0) = 1$, $n \geq 0$. Similarly one proves $A(0) = 0$ and $A'(0) = F_1$. Note that in order to apply the result of §2 to $\{A_n\}$ one needs to change the initial conditions in (2.3) but this makes no essential difference in the results.

Theorem 3.2. *Let a general* T*-fraction* (3.1) *be given, let* $\{\alpha_n\}$ *be a sequence of complex numbers such that*

$$\sum_{n=1}^{\infty}\left|\frac{G_n-\alpha_n}{1+\alpha_n z}\right|<\infty, \qquad \sum_{n=1}^{\infty}\left|\frac{F_n}{(1+\alpha_n z)(1+\alpha_{n-1}z)}\right|<\infty,$$

$$z\in\mathbf{C}\sim\left[-\frac{1}{\alpha_1},-\frac{1}{\alpha_2},\ldots\right].$$

Then the sequences

$$\left\{A_n(z)/\prod_{k=1}^{n}(1+\alpha_k z)\right\} \quad \text{and} \quad \left\{B_n(z)/\prod_{k=1}^{n}(1+\alpha_k z)\right\}$$

both converge for $z\in\mathbf{C}\sim[-1/\alpha_1,-1/\alpha_2,\ldots]$. *Both limit functions are not identically equal to zero.*

Proof. One has

$$\frac{A_n(z)}{\prod_{k=1}^{n}(1+\alpha_k z)}=\left(\frac{1+\alpha_n z}{1+\alpha_n z}+\frac{G_n-\alpha_n z}{1+\alpha_n z}\right)\cdot\frac{A_{n-1}(z)}{\prod_{k=1}^{n-1}(1+\alpha_k z)}$$
$$+\frac{F_n z}{(1+\alpha_n z)(1+\alpha_{n-1}z)}\frac{A_{n-2}(z)}{\prod_{k=1}^{n-2}(1+\alpha_k z)}.$$

This and the analogous recursion relation for $B_n(z)/\prod_{k=1}^{n}(1+\alpha_k z)$ establish the convergence of the two sequences using Lemma 2.2. As before one shows that for the two limits, say $A^*(z)$ and $B^*(z)$ one has $A^*(0)=0$, $dA^*(z)|_{z=0}/dz=F_1$, $B^*(0)=1$.

Remark. Possibly the most important application of this theorem is that it provides an asymptotic expression for $A_n(z)$ and $B_n(z)$, not just an upper bound as in (2.4).

If the $G_n\to\infty$ the following result may be useful.

Theorem 3.3. *In the general* T*-fraction* (3.1) *let* $G_n\neq 0$, $n\geq 1$, *then the expressions*

$$C_n(z)=A_n(z)/z^n\prod_{k=1}^{n}G_k, \qquad D_n(z)=B_n(z)/z^n\prod_{k=1}^{n}G_k$$

are polynomials in $u=1/z$. *Set*

$$S_n(u)=C_n\left(\tfrac{1}{u}\right), \qquad T_n(u)=D_n\left(\tfrac{1}{u}\right),$$

then

$$S_n(u)=\left(1+\frac{u}{G_n}\right)S_{n-1}(u)+\frac{F_n}{G_nG_{n-1}}uS_{n-2}(u), \qquad n\geq 2$$

$$S_1(u)=\frac{F_1}{G_1}u, \qquad S_0(u)=0.$$

Analogous formulas hold for $\{T_n(u)\}$. *Further, if*

$$\sum_{n=1}^{\infty} \frac{1}{|G_n|} < \infty, \qquad \sum_{n=1}^{\infty} \left| \frac{F_n}{G_n G_{n-1}} \right| < \infty,$$

then $\{S_n(u)\}$, $\{T_n(u)\}$ *converge to entire functions* $S(u)$ *and* $T(u)$ *of order at most one in* u. *Finally,* $S(0) = 0$, $S'(0) = F_1/G_1$, *and* $T(0) = 1$.

The proof proceeds by now well established arguments.

We conclude the article with an application of Lemma 2.5.

Theorem 3.4. *If in the general* T*-fraction* (3.1)

$$\sum |G_n^2| < \infty, \qquad \sum |F_n| < \infty$$

then

$$\left\{ A_n(z) \prod_{k=1}^{n} (1 - G_k z) \right\}, \qquad \left\{ B_n(z) \prod_{k=1}^{n} (1 - G_k z) \right\}$$

converge to entire functions of order at most one, which are not identically equal to zero.

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