A FURTHER GENERALIZATION OF THE KNASTER-KURATOWSKI-MAZURKIEWICZ THEOREM

NAOKI SHIOJI

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Abstract. Granas and Dugundji obtained the following generalization of the Knaster-Kuratowski-Mazurkiewicz theorem.

Let $X$ be a subset of a topological vector space $E$ and let $G$ be a set-valued map from $X$ into $E$ such that for each finite subset $\{x_1, \ldots, x_n\}$ of $X$, $\text{co}\{x_1, \ldots, x_n\} \subseteq \bigcup_{i=1}^{n} Gx_i$ and for each $x \in X$, $Gx$ is finitely closed, i.e., for any finite-dimensional subspace $L$ of $E$, $Gx \cap L$ is closed in the Euclidean topology of $L$. Then $\{Gx : x \in X\}$ has the finite intersection property.

By relaxing, among others, the condition that $X$ is a subset of $E$, we obtain a further generalization of the theorem and show some of its applications.

1. Introduction

In 1961 Fan [5] showed an infinite-dimensional version of the classical Knaster-Kuratowski-Mazurkiewicz theorem [13]. We can find some versions of Fan's theorem in [2, 7, 14, 18]. One of them is Theorem A which was proved by Dugundji and Granas [2]. First we state some definitions. Let $X$ be a subset of a vector space $E$. A set-valued map $G$ from $X$ into $E$ is called a Knaster-Kuratowski-Mazurkiewicz map or simply a KKM map if for each finite subset $\{x_1, \ldots, x_n\}$ of $X$,

$$\text{co}\{x_1, \ldots, x_n\} \subseteq \bigcup_{i=1}^{n} Gx_i.$$

We can find many examples of KKM maps in [10, 14]. A subset of a vector space $E$ is called finitely closed if its intersection with each finite-dimensional linear space $L \subseteq E$ is closed in the Euclidean topology of $L$.

Theorem A (Dugundji and Granas). Let $E$ be a vector space, $X$ an arbitrary subset of $E$, and $G : X \to 2^E$ a KKM map such that each $Gx$ is finitely closed. Then the family $\{Gx : x \in X\}$ of sets has the finite intersection property.

The object of this paper is to obtain a generalization of Theorem A by relaxing, among others, the condition that $X$ is a subset of $E$. The idea is inspired...
by Theorem 3 in [11] which is a simple result of our theorem. Our main result is Theorem 1 and its proof relies on Górniewicz’s fixed point theorem in [8]. As applications of our theorem, we show two system theorems concerning inequalities and a minimax theorem. In [16, 17], Simons classifies minimax theorems into two groups. One is of a fixed point type and the other is of a Hahn–Banach type. Theorem B, which is Theorem 1.4 in [17], is a minimax theorem of a typical fixed point type. Recall that a function \( f: X \to R \) is called quasi-convex if for any real number \( \alpha \), \( \{ x \in X : f(x) \leq \alpha \} \) is convex, where \( X \) is a convex subset of a vector space. \( f \) is called quasi-concave if \(-f\) is quasi-convex.

**Theorem B (Simons).** Let \( X \) be a nonempty convex subset of a topological vector space, let \( Y \) be a nonempty compact convex subset of a topological vector space, let \( f: X \times Y \to R \) be quasi-concave in its first variable and lower semicontinuous in its second variable, let \( g: X \times Y \to R \) be upper semicontinuous in its first variable and quasi-convex in its second variable, and let \( f \leq g \) on \( X \times Y \). Then

\[
\min_{y \in Y} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} \inf_{y \in Y} g(x, y).
\]

Simons deduced this theorem from the KKM theorem. Theorem 6 in this paper is a minimax theorem of a typical Hahn–Banach type. Simons deduced Theorem 6 from the Hahn–Banach theorem. We deduce Theorem 6 from our main result, Theorem 1. Since Theorem 1 is a generalization of the KKM theorem, minimax theorems of both types are easily obtained from Theorem 1.

**2. Main results**

In this paper all topological structures are implicitly assumed to satisfy the Hausdorff separation axiom and by homology, we understand the Čech homology [4, 9] with rational coefficients. Let \( G \) be a set-valued map from a set \( X \) into a set \( Y \); we denote by \( G(A) \) the subset \( \bigcup_{x \in A} Gx \) of \( Y \). Let \( D \) be a subset of a vector space; we denote by \( \text{co} D \) the convex hull of \( D \).

Let \( X \) and \( Y \) be topological spaces. A nonempty set-valued map \( G \) from \( X \) into \( Y \) is said to be upper semicontinuous if for any point \( x_0 \in X \) and any open set \( V \) in \( Y \) containing \( Gx_0 \), there is a neighborhood \( U \) of \( x_0 \) in \( X \) such that \( Gx \subset V \) for all \( x \in U \). If \( Y \) is compact and \( G \) is a compact set-valued map then \( G \) is upper semicontinuous if and only if the graph \( \{(x, y) \in X \times Y : y \in Gx\} \) of \( G \) is closed in \( X \times Y \). If \( X \) is compact and \( G \) is a compact set-valued map then \( G \) is upper semicontinuous if and only if the graph \( \{(x, y) \in X \times Y : y \in Gx\} \) of \( G \) is closed in \( X \times Y \) and \( G(X) \) is compact.

We start with the following lemma which is crucial to prove Theorem 1.

**Lemma 1** (Eilenberg and Montgomery, Górniewicz). Let \( Z \) be an \( n \)-simplex with the Euclidean simplex topology and let \( W \) be a compact space. Let \( p \) be a single-valued continuous map from \( W \) into \( Z \) and let \( T \) be a set-valued upper semicontinuous map from \( Z \) into \( W \) such that for each \( x \in Z \), \( Tx \) is
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a nonempty compact acyclic subset of $W$. Then there is a point $x_0$ in $Z$ such that $x_0 \in p(Tx_0)$.

For a proof, see the argument in [3, §3] with Theorem 6.3 in [8] instead of the coincidence theorem that is used in [3, §3].

**Theorem 1.** Let $X$ be a subset of a vector space $E$ and let $Y$ be a topological space. Let $G$ be a set-valued map from $X$ into $Y$ and let $T$ be a set-valued map from $coX$ into $Y$ such that

(i) for each $x \in coX$, $Tx$ is a nonempty compact acyclic subset of $Y$;

(ii) for each finite subset $\{x_1, \ldots, x_n\}$ of $X$, $T(co\{x_1, \ldots, x_n\}) \subset \bigcup_{i=1}^{n} Gx_i$;

(iii) for each finite subset $\{x_1, \ldots, x_n\}$ of $X$, $T|_Z : Z \to 2^Y$ is upper semicontinuous, where $Z = co\{x_1, \ldots, x_n\}$ and $Z$ is endowed with the Euclidean simplex topology; and

(iv) for each finite subset $\{x_1, \ldots, x_n\}$ of $X$, $Gx_i \cap T(Z)$ is relatively closed in $T(Z)$ for all $i = 1, \ldots, n$, where $Z = co\{x_1, \ldots, x_n\}$.

Then the family $\{Gx : x \in X\}$ of sets has the finite intersection property. More explicitly, for any finite subset $\{x_1, \ldots, x_n\}$ of $X$,

$$\left( \bigcap_{i=1}^{n} Gx_i \right) \cap T(co\{x_1, \ldots, x_n\}) \neq \emptyset.$$

**Remark.** If $E = Y$ and $T$ is the identity mapping, the condition (ii) implies that $G$ is a KKM map. If $G$ is a closed set-valued map then the condition (iv) holds trivially. The condition (iii) is useful when $E$ is not endowed with topology. In fact, prove Theorem 5 in [11], which is one of Fan's general fixed point theorems [6], as Ha did in [11].

**Proof.** Let $\{x_1, \ldots, x_n\}$ be any finite subset of $X$ and let $Z = co\{x_1, \ldots, x_n\}$ which is endowed with the Euclidean simplex topology. Suppose that $\left( \bigcap_{i=1}^{n} Gx_i \right) \cap T(Z) = \emptyset$ then $\{T(Z) \setminus Gx_1, \ldots, T(Z) \setminus Gx_n\}$ is an open covering of $T(Z)$. Since $T$ is upper semicontinuous on $Z$ and compact valued, $T(Z)$ is compact. Hence there is a partition of unity $\{\alpha_1, \ldots, \alpha_n\}$ corresponding to the covering. Let $p$ be the function from $T(Z)$ into $Z$ defined by

$$p(y) = \sum_{i=1}^{n} \alpha_i(y)x_i, \quad y \in T(Z).$$

For each $y \in T(Z)$, let $I_y = \{i : \alpha_i(y) > 0\}$. If $i \in I_y$ then $\alpha_i(y) > 0$ and so $y \notin Gx_i$. Hence the condition (ii) implies that

$$y \notin T(co\{x_i : i \in I_y\}) \text{ for any } y \in T(Z).$$

On the other hand by Lemma 1, there is a point $x_0 \in Z$ such that $x_0 \in p(Tx_0)$. Let $y_0$ be a point in $T(Z)$ such that $y_0 \in Tx_0$ and $p(y_0) = x_0$. Since $p(y_0) = \sum_{i \in I_{y_0}} \alpha_i(y_0)x_i$, $x_0$ is contained in $co\{x_i : i \in I_{y_0}\}$. Hence $y_0 \in T(co\{x_i : i \in I_{y_0}\})$. This contradicts (1). □
We show some conditions which guarantee the whole intersection property, and hence we show some generalizations of KKM theorems in [5, 7, 14, 18].

**Theorem 2.** Let $X$ be a subset of a vector space $E$ and let $Y$ be a topological space. Let $G$ be a set-valued map from $X$ into $Y$ and $T$ be a set-valued map from $\text{co} X$ into $Y$ such that

(i) for each $x \in \text{co} X$, $T_x$ is a nonempty compact acyclic subset of $Y$ and for each $x \in X$, $G_x$ is a closed subset of $Y$;

(ii) for each finite subset $\{x_1, \ldots, x_n\}$ of $X$, $T(\text{co}\{x_1, \ldots, x_n\}) \subset \bigcup_{i=1}^n G_x$; and

(iii) for each finite subset $\{x_1, \ldots, x_n\}$ of $X$, $T|_Z : Z \to 2^Y$ is upper semicontinuous, where $Z = \text{co}\{x_1, \ldots, x_n\}$ and $Z$ is endowed with the Euclidean simplex topology.

Furthermore suppose that there exists a compact subset $K$ of $Y$ such that $T(X) \subset K$. Then $\bigcap_{x \in X} G_x \cap K \neq \emptyset$.

**Proof.** By Theorem 1, $\{G_x \cap K : x \in X\}$ is a family of subsets of $K$ that has the finite intersection property. Since $K$ is compact, the conclusion holds. \hfill $\square$

**Theorem 3.** Let $X$ be a subset of a topological vector space $E$ and let $Y$ be a topological space. Let $G$ be a set-valued map from $X$ into $Y$ and $T$ be an upper semicontinuous set-valued map from $\text{co} X$ into $Y$ such that

(i) for each $x \in \text{co} X$, $T_x$ is a nonempty compact acyclic subset of $Y$ and for each $x \in X$, $G_x$ is a closed subset of $Y$;

(ii) for each finite subset $\{x_1, \ldots, x_n\}$ of $X$, $T(\text{co}\{x_1, \ldots, x_n\}) \subset \bigcup_{i=1}^n G_x$.

Furthermore suppose that one of the following conditions is satisfied:

(a) $X$ is compact or

(b) $X$ is convex or closed and there exists a compact convex subset $X_0$ of $X$ such that $\bigcap_{x \in X_0} G_x$ is compact.

Then $\bigcap_{x \in X} G_x \neq \emptyset$.

**Proof.** First suppose that the condition (a) is satisfied. Since $X$ is compact and $T$ is upper semicontinuous, $T(X)$ is compact. By Theorem 2, $\bigcap_{x \in X} G_x \cap T(X) \neq \emptyset$ and hence the conclusion holds.

Next suppose that the condition (b) is satisfied. Since $X_0$ is compact, $\bigcap_{x \in X_0} G_x \neq \emptyset$. We show that the family $\{G_x \cap (\bigcap_{z \in X_0} G_z) : x \in X\}$ has the finite intersection property. For any finite subset $\{x_1, \ldots, x_n\}$ of $X \backslash X_0$, if $X$ is convex then let $X_1 = \text{co}\{X_0 \cup \{x_1, \ldots, x_n\}\}$; otherwise let $X_1 = X \cap \text{co}\{X_0 \cup \{x_1, \ldots, x_n\}\}$. Since $X_1$ is compact, $\bigcap_{x \in X_1} G_x \neq \emptyset$ and hence $\bigcap_{i=1}^n (G_{x_i} \cap (\bigcap_{z \in X_0} G_z)) \neq \emptyset$. Therefore the conclusion holds. \hfill $\square$

The following corollary is a slight generalization of Ha's theorem [11] and the theorem of Ben-El-Mechaiekh, Deguire, and Granas [1]. We can see some of its applications in [11] and [12].
Corollary 1 (Ha). Let $E, F$ be topological vector spaces, $X \subseteq E$, $Y \subseteq F$ be nonempty convex subsets and let $A \subseteq B \subseteq C$ be subsets of $X \times Y$ such that

(i) for each $x \in X$, the set $\left\{ y \in Y : (x, y) \in C \right\}$ is closed in $Y$;
(ii) for each $x \in X$, the set $\left\{ y \in Y : (x, y) \notin B \right\}$ is empty or convex;
(iii) $A$ is closed in $X \times Y$; and
(iv) there exists a compact subset $K$ of $Y$ such that for each $x \in X$, the set $\left\{ y \in K : (x, y) \in A \right\}$ is nonempty and convex.

Then there exists a point $y_0 \in K$ such that $X \times \{y_0\} \subseteq C$.

Proof. For each $x \in X$, let $T_x = \left\{ y \in K : (x, y) \in A \right\}$, let $G_x = \left\{ y \in Y : (x, y) \in B \right\}$ and let $H_x = \left\{ y \in Y : (x, y) \in C \right\}$. Then for each $x \in X$, $H_x$ is closed and $T_x$ is nonempty, compact, and convex. And also for each finite subset $\{x_1, \ldots, x_n\}$ of $X$, $T(\{x_1, \ldots, x_n\}) \subseteq \bigcup_{i=1}^n G_{x_i} \subseteq \bigcup_{i=1}^n H_{x_i}$ and $T|_Z : Z \rightarrow 2^Y$ is upper semicontinuous, where $Z = \text{co}\{x_1, \ldots, x_n\}$. Furthermore, there is the compact set $K \subseteq Y$ such that $T(Z) \subseteq K$. Hence by Theorem 2, $(\bigcap_{x \in X} H_x) \cap K \neq \emptyset$. This implies that there exists a point $y_0 \in K$ such that $X \times \{y_0\} \subseteq C$. □

3. Some applications

We show two theorems concerning inequalities and a minimax theorem. First, the following theorem is a special case of Theorem 1 in [15], but the proof is simpler than that in [15]. For each $n \in \mathbb{N}$, we define sets $W_n$ and $S_n$ in $\mathbb{R}^n$ as

$$W_n = \left\{ e_j = (0, \ldots, 1, \ldots, 0) \in \mathbb{R}^n : j = 1, \ldots, n \right\};$$

$$S_n = \text{co} W_n = \left\{ \alpha = \sum_{j=1}^n \alpha_j e_j \in \mathbb{R}^n : \sum_{j=1}^n \alpha_j = 1, \alpha_j \geq 0 \right\}.$$

Theorem 4. Let $X$ be a nonempty compact convex subset of a topological vector space. Let $f_1, \ldots, f_n$ be lower semicontinuous convex functions on $X$ with values in $(-\infty, +\infty]$. Then the following are equivalent:

(i) The system of convex inequalities

$$f_i(x) \leq 0 \quad \text{for all } i \in I$$

is consistent on $X$; that is, there exists an $x \in X$ satisfying (2).

(ii) For any $\alpha \in S_n$, there exists an $x \in X$ such that

$$\sum_{i=1}^n \alpha_i f_i(x) \leq 0.$$

Proof. It is clear that (i) implies (ii). We prove that (ii) implies (i). We define set-valued maps $T : S_n \rightarrow 2^X$ and $G : W_n \rightarrow 2^X$ by

$$T\alpha = \left\{ x \in X : \sum_{i=1}^n \alpha_i f_i(x) \leq 0 \right\} \quad (\alpha \in S_n).$$
and
\[ Ge_i = \{ x \in X : f_i(x) \leq 0 \} \quad (e_i \in W_n). \]

It is easy to see that for each \( \alpha \in S_n \), \( T\alpha \) is a nonempty compact convex subset of \( X \) and for each \( i = 1, \ldots, n \), \( Ge_i \) is closed. For any finite subset \( \{ e_{i_1}, \ldots, e_{i_m} \} \) of \( W_n \),
\[
\left\{ x \in X : \text{there exists a } \beta \in S_m \text{ such that } \sum_{j=1}^{m} \beta_j f_{i_j}(x) \leq 0 \right\} \subseteq \bigcup_{j=1}^{m} Ge_{i_j};
\]
that is,
\[
T(\text{co}\{e_{i_1}, \ldots, e_{i_m}\}) \subseteq \bigcup_{j=1}^{m} Ge_{i_j}.
\]

By Lemma 1 in [15], it is also easy to see that \( T \) is upper semicontinuous. Hence, by Theorem 1, there is a point \( x_0 \in X \) such that
\[
x_0 \in \bigcap_{i=1}^{n} Ge_i.
\]
Therefore we have
\[
f_i(x_0) \leq 0 \quad \text{for all } i \in I. \quad \Box
\]

Next we show Takahashi's system theorem [18] concerning inequalities. Let \( X \) and \( Y \) be arbitrary sets. A function \( f : X \times Y \to \mathbb{R} \) is convexlike in its second variable if for any \( y_1, y_2 \in Y \) and \( 0 < a < 1 \), there exists a \( y_0 \in Y \) such that
\[
f(x, y_0) \leq af(x, y_1) + (1 - a)f(x, y_2)
\]
for all \( x \in X \). Also, a function \( g : X \times Y \to \mathbb{R} \) is concavelike in its first variable if for any \( x_1, x_2 \in X \) and \( 0 < a < 1 \), there exists an \( x_0 \in X \) such that
\[
g(x_0, y) \geq ag(x_1, y) + (1 - a)g(x_2, y)
\]
for all \( y \in Y \).

**Theorem 5 (Takahashi).** Let \( X \) be a set, let \( f_1, \ldots, f_n \) be real valued functions on \( X \) and suppose that the function \( F \) on \( I \times X \), defined by \( F(i, x) = f_i(x) \) for \( x \in X \) and \( i \in I \), is convexlike in its second variable, where \( I = \{1, \ldots, n\} \). Let \( c \in \mathbb{R} \). If for any \( \alpha \in S_n \), there exists an \( x_0 \in X \) such that \( \sum_{i=1}^{n} \alpha_i f_i(x_0) \leq c \), then
\[
\inf_{x \in X} \max_{i \in I} f_i(x) \leq c.
\]

**Proof.** Let \( d = \inf_{x \in X} \max_{i \in I} f_i(x) \). If \( d = -\infty \) the conclusion holds trivially. So we may assume that \( d \) is a real number. Choose any finite subset \( \{ x_1, \ldots, x_m \} \) of \( X \) and let \( f(\alpha, x) = \sum_{i=1}^{n} \alpha_i f_i(x) \) for any \( x \in X \) and \( \alpha \in S_n \). We define set-valued maps \( T : S_m \to 2^{S_n} \) and \( G : W_m \to 2^{S_n} \) by
\[
T \beta = \left\{ \alpha \in S_n : \sum_{j=1}^{m} \beta_j f(\alpha, x_j) \geq d \right\} \quad (\beta \in S_m)
\]

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and
\[ Ge_j = \{ \alpha \in S_n : f(\alpha, x_j) \geq d \} \quad (e_j \in W_m). \]

Since it is easy to check that the conditions of Theorem 1 hold, we have
\[ \bigcap_{j=1}^m Ge_j \neq \varnothing. \]
Hence, by compactness of \( S_n \),
\[ \bigcap_{x \in X} \{ \alpha \in S_n : f(\alpha, x) \geq d \} \neq \varnothing; \]
that is, there exists an \( \hat{\alpha} \in S_n \) such that
\[ f(\hat{\alpha}, x) \geq d \quad \text{for all } x \in X. \]

From the hypothesis, this implies
\[ \inf_{x \in X} \max_{i \in I} f_i(x) \leq c. \quad \square \]

Finally, we show Simons's minimax theorem [16]. Takahashi obtained a similar result in [18]. Compare the proof of the next theorem with those of Simons and Takahashi.

**Theorem 6 (Simons).** Let \( X \) and \( Y \) be arbitrary sets and let \( f, g \) be real valued functions on \( X \times Y \) satisfying
\begin{itemize}
  \item[(i)] \( f(x, y) \leq g(x, y) \) for each \( (x, y) \in X \times Y \);
  \item[(ii)] \( f \) is convex-like in its second variable, and
  \item[(iii)] \( g \) is concave-like in its first variable.
\end{itemize}

Then for any nonempty finite subset \( X_0 \) of \( X \),
\[ \inf_{y \in Y} \max_{x \in X_0} f(x, y) \leq \sup_{x \in X} \inf_{y \in Y} g(x, y). \]

**Proof.** Let \( X_0 \) be a finite subset \( \{x_1, \ldots, x_n\} \) of \( X \) and
\[ d = \inf_{y \in Y} \max_{x \in X_0} f(x, y). \]
If \( d = -\infty \) the conclusion holds trivially. So we may assume that \( d \) is a real number. Choose any finite subset \( \{y_1, \ldots, y_m\} \) of \( Y \). We define set-valued maps \( T : S_m \to 2^{S_\ast} \) and \( G : W_m \to 2^{S_\ast} \) by
\[ T\beta = \left\{ \alpha \in S_n : \sum_{j=1}^m \beta_j \sum_{i=1}^n \alpha_i f(x_i, y_j) \geq d \right\} \quad (\beta \in S_m) \]
and
\[ Ge_j = \left\{ \alpha \in S_n : \sum_{i=1}^n \alpha_i f(x_i, y_j) \geq d \right\} \quad (e_j \in W_m). \]

Since it is easy to check that the conditions of Theorem 1 hold, we have
\[ \bigcap_{j=1}^m Ge_j \neq \varnothing. \]
Hence, by compactness of \( S_n \),
\[ \bigcap_{y \in Y} \left\{ \alpha \in S_n : \sum_{i=1}^n \alpha_i f(x_i, y) \geq d \right\} \neq \varnothing; \]
that is, there exists an \( \alpha \in S_n \) such that
\[
d \leq \sum_{i=1}^{n} \alpha_i f(x_i, y) \leq \sum_{i=1}^{n} \alpha_i g(x_i, y)
\]
for all \( y \in Y \).

From hypothesis (iii), there exists an \( x_0 \in X \) such that
\[
d \leq g(x_0, y) \quad \text{for all } y \in Y.
\]

Therefore we have
\[
\inf_{y \in Y} \max_{x \in X} f(x, y) \leq \sup_{x \in X} \inf_{y \in Y} g(x, y). \quad \Box
\]

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**References**


**DEPARTMENT OF INFORMATION SCIENCE, TOKYO INSTITUTE OF TECHNOLOGY, OOKAYAMA, MEGUROKU, TOKYO 152, JAPAN**