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LEVEL CROSSINGS OF A RANDOM TRIGONOMETRIC POLYNOMIAL

KAMBIZ FARAHMAND

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ABSTRACT. This paper provides an asymptotic estimate for the expected number of K-level crossings of the random trigonometric polynomial $g_1 \cos x + g_2 \cos 2x + \ldots + g_n \cos nx$, where g_j $(j = 1, 2, \ldots, n)$ are independent normally distributed random variables with mean μ and variance one. It is shown that the result for $K = \mu = 0$ remains valid for any finite constant μ and any K such that $(K^2/n) \to 0$ as $n \to \infty$.

1. INTRODUCTION

Let

$$T(x) \equiv T_n(x, \omega) = \sum_{j=1}^{n} g_j(\omega) \cos jx \tag{1.1}$$

where $g_1(\omega), g_2(\omega), \ldots, g_n(\omega)$ is a sequence of independent random variables defined on a probability space $(\Omega, \mathscr{A}, P)$, each normally distributed with finite mathematical expectation μ and variance one. The set of equations $y = T(x)$ represents a family of curves in the xy-plane. Some years ago Dunnage [2] showed that the number of times all save a certain exceptional set of these curves crosses the x-axis in the interval $0 \le x \le 2\pi$ is $(2n)/\sqrt{3} + O\{n^{11/13}(\log n)^{3/13}\}$. The measure of his exceptional set does not exceed $(\log n)^{-1}$. Later Sambandham and Renganathan [7] and Farahmand [3] considered the case when $\mu \neq 0$ and found that the expected number of crossings of this family with the x-axis is asymptotic to $2n\sqrt{3}$. This asymptotic number of crossings remains invariant in the work of Farahmand [4] when, for $\mu = 0$, he considered the crossings of this family with the line $y = K$ (i.e., with level K), where K is any real value constant such that $(K^2/n) \to 0$ as $n \to \infty$.

Here we consider the effect of μ being nonzero on the K-level crossings of this family of curves, and show even in this case the above asymptotic number

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of crossings persists. Denote by $N(\alpha, \beta) \equiv N_K(\alpha, \beta)$ the number of times that the family of curves $y = T(x)$ crosses the line $y = K$ on the interval $\alpha \le x \le \beta$, and let $EN(\alpha, \beta)$ be its expectation. We prove the following.

Theorem. *If the coefficients of $T(x)$ in (1.1) are independent normally distributed random variables with mean μ and variance one, then for all sufficiently large n and any constant K the expected number of real roots of the equation $T(x) = K$ satisfies*

$$EN(0, 2\pi) = (2n)/\sqrt{3} + O(n^{3/4}) \quad \text{if } K = O(n^{3/8})$$

and

$$EN(0, 2\pi) = (2n)/\sqrt{3} + o(n) \quad \text{if } K = o(\sqrt{n}).$$

2. A formula for the number of crossings

Let

$$\Phi(t) = (2\pi)^{-1/2} \int_{-\infty}^{t} \exp(-y^2/2)\, dy$$

and

$$\phi(t) = \Phi'(t) = (2\pi)^{-1/2} \exp(-t^2/2),$$

then by using the expected number of level crossings given by Cramer and Leadbetter [1, p. 285] for our equation $T(x) - K = 0$ we can obtain

$$EN(\alpha, \beta) = \int_{\alpha}^{\beta} (B/A)(1-\lambda^2)^{1/2} \phi(\xi/A)[2\phi(\eta) + \eta\{2\Phi(\eta) - 1\}]\, dx \tag{2.1}$$

where

$$A^2 = \operatorname{Var}\{T(x) - K\}, \qquad B^2 = \operatorname{Var}\{T'(x)\},$$
$$\lambda = (AB)^{-1} \operatorname{cov}[\{T(x) - K\}, T'(x)], \qquad \xi = E\{T(x) - K\},$$
$$\eta = B^{-1}(1-\lambda^2)^{-1/2}(\gamma - B\lambda\xi/A) \quad \text{and} \quad \gamma = E\{T'(x)\}.$$

Since the coefficients of $T(x)$ are independent normal random variables with mean μ and variance one we can easily find that

$$\xi = \mu \sum_{j=1}^{n} \cos jx - K = m_1 - K, \text{ say,}$$

$$\gamma = -\mu \sum_{j=1}^{n} j \sin jx = -m_2, \text{ say}$$

$$A^2 = \sum_{j=1}^{n} \cos^2 jx, \qquad B^2 = \sum_{j=1}^{n} j^2 \sin^2 jx,$$

$$C = \sum_{j=1}^{n} j \sin jx \cos jx, \qquad \lambda = -C/AB,$$

$$\eta = \{C(m_1 - K) - A^2 m_2\}/A\Delta \quad \text{and} \quad \Delta^2 = A^2B^2 - C^2.$$

Hence from (2.1) and since $\Phi(t) = 1/2 + (\pi)^{-1/2}\operatorname{erf}(t/\sqrt{2})$, we have the extension of the Kac-Rice formula [5].

$$
\begin{aligned}
&EN(\alpha, \beta) \\
&\quad = \int_\alpha^\beta (\Delta/\pi A^2)\exp[\{-B^2(m_1 - K)^2 - A^2 m_2^2 + 2m_2 C(m_1 - K)\}/(2\Delta^2)]\,dx \\
&\qquad + \int_\alpha^\beta (\sqrt{2}/\pi)A^{-3}|C(m_1 - K) - A^2 m_2|\exp\{-(m_1 - K)^2/2A^2\} \\
&\qquad\qquad \times \operatorname{erf}\{|C(m_1 - K) - A^2 m_2|/\sqrt{2}A\Delta\}\,dx \\
&\quad = \int_\alpha^\beta I_1(x)\,dx + \int_\alpha^\beta I_2(x)\,dx. \qquad (2.2)
\end{aligned}
$$

3. Overview of the proof of the theorem and some approximations

In order to evaluate the expected number of real roots we divide the real roots into two groups: (i) those lying in the intervals $(0, \varepsilon)$, $(\pi - \varepsilon, \pi + \varepsilon)$ and $(2\pi - \varepsilon, 2\pi)$ and (ii) those lying in the intervals $(\varepsilon, \pi - \varepsilon)$ and $(\pi + \varepsilon, 2\pi - \varepsilon)$. For the roots (i) which, it so happens, are negligible, we need some modification to apply Dunnage's [2] approach, which is based on an application of Jensen's theorem [8, p. 125] or [6, p. 332]. For roots (ii) we use the Kac-Rice formula (2.2). The ε should be chosen positive and small enough to facilitate handling type (i) roots, but large enough to allow for the approximation needed to discuss type (ii) roots. We choose $\varepsilon = n^{-1/4}$.

The function

$$S(x) = \sin(2n + 1)x/\sin x$$

and, defined by continuity at $x = j\pi$, will occur frequently and so it is convenient to collect together some related inequalities. Since for $\varepsilon \le x \le \pi - \varepsilon$ or $\pi + \varepsilon \le x \le 2\pi - \varepsilon$ we have $|S(x)| < 1/\sin\varepsilon$, we can obtain

$$S(x) = O(1/\varepsilon). \tag{3.1}$$

Further

$$S'(x) = (2n + 1)\cos(2n + 1)x/\sin x - \cot x S(x) = O(n/\varepsilon) \tag{3.2}$$

and

$$
\begin{aligned}
S''(x) = &-(2n + 1)^2 S(x) - (2n + 1)\cos x\cos(2n + 1)x\sin^{-2} x \\
&- \cot x S'(x) + \operatorname{cosec}^2 x S(x) = O(n^2/\varepsilon).
\end{aligned} \tag{3.3}
$$

Hence from (3.1) and since by expanding $\sin x(1 + 2\sum_{j=1}^n \cos 2jx)$, we can show that

$$\sum_{j=1}^n \cos 2jx = (1/2)\{S(x) - 1\} \tag{3.4}$$

for this range of x we have

$$(3.5)\quad A^2 = (1/2)\sum_{j=1}^{n}(1+\cos 2jx) = n/2 + (1/4)\{S(x)-1\} = n/2 + O(1/\varepsilon).$$

Also from (3.3) and since from (3.4)

$$S''(x) = -8\sum_{j=1}^{n} j^2 \cos 2jx = 4\sum_{j=1}^{n} j^2(2\sin^2 jx - 1)$$

we have

$$(3.6)\qquad B^2 = n(n+1)(2n+1)/12 + (1/8)S''(x) = n^3/6 + O(n^2/\varepsilon).$$

From (3.2) we obtain

$$(3.7)\qquad C = -(1/2)\tfrac{d}{dx}(A^2) = -(1/8)S'(x) = O(n/\varepsilon).$$

Also from (3.1), (3.2), and (3.4) we can easily obtain the two following estimates:

$$(3.8)\qquad m_1 = (\mu/2)\{S(x/2)-1\} = O(1/\varepsilon)$$

and

$$(3.9)\qquad m_2 = (-\mu/4)S'(x/2) = O(n/\varepsilon).$$

Then (3.5), (3.6), and (3.7) give

$$(3.10)\qquad \Delta^2 = n^4/12 + O(n^3/\varepsilon).$$

4. Proof of the theorem

From (2.2) and (3.5)–(3.10) we can obtain

$$(4.1)\qquad \int_{\varepsilon}^{\pi-\varepsilon} I_1(x)\,dx = (n/\sqrt{3})\{1+O(\varepsilon)\}\exp\{-K^2/n + O(1/n\varepsilon^2)\}$$

and

$$(4.2)\qquad \int_{\varepsilon}^{\pi-\varepsilon} I_2(x)\,dx = O\{(n^{1/2}/\varepsilon)\exp(-1/n\varepsilon^2 - K^2/n)\}.$$

Hence from (2.2), (4.1), and (4.2) for $K = O(n^{3/8})$, we have

$$(4.3)\qquad EN(\varepsilon, \pi-\varepsilon) = EN(\pi+\varepsilon, 2\pi-\varepsilon) = n/\sqrt{3} + O(n^{3/4})$$

and for $K = o(\sqrt{n})$

$$(4.4)\qquad EN(\varepsilon, \pi-\varepsilon) = EN(\pi+\varepsilon, 2\pi-\varepsilon) = n/\sqrt{3} + o(n).$$

Now we show that the expected number of real roots in the intervals $(0, \varepsilon)$, $(\pi-\varepsilon, \pi+\varepsilon)$, and $(2\pi-\varepsilon, 2\pi)$ is negligible. The period of $T(x)$ is 2π and so the number of zeros in $(0, \varepsilon)$ and $(2\pi-\varepsilon, 2\pi)$ is the same as the number in $(-\varepsilon, \varepsilon)$. We shall therefore confine ourselves to this last interval; the interval

$(\pi - \varepsilon, \pi + \varepsilon)$ can be treated in exactly the same way to give the same result. We consider the random integral function of the complex variable z,

$$T(z, \omega) - K = \sum_{j=1}^{n} g_i(\omega) \cos jz - K.$$

We are seeking an upper bound to the number of real roots in the segment of the real axis joining the points $\pm\varepsilon$, and this certainly does not exceed the number of real roots in the circle $|z| < \varepsilon$. Let $N(r) \equiv N(r, \omega, K)$ denote the number of real roots of $T(z, \omega) - K = 0$ in $|z| < r$. By Jensen's theorem [8, p. 125] or [6, p. 332]

$$\begin{aligned}\int_{\varepsilon}^{2\varepsilon} r^{-1} N(r)\, dr &\le \int_{0}^{2\varepsilon} r^{-1} N(r)\, dr \\ &= (2\pi)^{-1} \int_{0}^{2\pi} \log |[\{T(2\varepsilon e^{ix}, \omega) - K\}/\{T(0) - K\}]|\, dx,\end{aligned}$$

assuming that $T(0) \neq K$, from which we have

$$N(\varepsilon) \log 2 \le (2\pi)^{-1} \int_{0}^{2\pi} \log |[\{T(2\varepsilon e^{ix}, \omega) - K\}/\{T(0) - K\}]|\, dx. \tag{4.5}$$

Now since the distribution function of $T(0, \omega) = \sum_{j=1}^{n} g_j(\omega)$ is

$$G(x) = (2\pi n)^{-1/2} \int_{-\infty}^{x} \exp\{-(t-\mu)^2/2n\}\, dt$$

we can see that for any positive ν

$$\begin{aligned}\operatorname{Prob}(-e^{-\nu} \le T(0) - K < e^{-\nu}) &= (2\pi n)^{-1/2} \int_{K-e^{-\nu}}^{K+e^{-\nu}} \exp\{-(t-\mu)^2/2n\}\, dt \\ &< (2/n\pi)^{1/2} e^{-\nu}.\end{aligned} \tag{4.6}$$

Also we have

$$|T(2\varepsilon e^{ix})| = \left| \sum_{j=1}^{n} g_j \cos(2j\varepsilon e^{ix}) \right| \le n e^{2n\varepsilon} \max |g_j| \tag{4.7}$$

where the maximum is taken over $1 \le j \le n$. The distribution function of $|g_j|$ is

$$F(x) = \begin{cases} \sqrt{2}/\pi \displaystyle\int_{0}^{x} \exp\{-(t-\mu)^2/2\}\, dt, & x \ge 0, \\ 0, & x < 0, \end{cases}$$

and so for any positive ν and all sufficiently large n

$$\begin{aligned}\operatorname{Prob}\{\max |g_j| > ne^{\nu}\} &\le n \operatorname{Prob}\{|g_1| > ne^{\nu}\} \\ &= n\sqrt{2\pi} \int_{ne^{\nu}}^{\infty} \exp\{-(t-\mu)^2/2\}\, dt \\ &\sim \sqrt{2/\pi} \exp\{-\nu - (ne^{\nu} - \mu)^2/2\}.\end{aligned} \tag{4.8}$$

Therefore from (4.7) and (4.8) except for sample functions in an ω-set of measure not exceeding $(2/\pi)^{1/2}\exp\{-\nu-(ne^{\nu}-\mu)^2/2\}$,

$$|T(2\varepsilon e^{ix})| < n^2\exp(2n\varepsilon+\nu). \tag{4.9}$$

Hence from (4.6), (4.9) and since

$$|n^2\exp(2n\varepsilon+\nu)-K| < 2n^2\exp(2n\varepsilon+\nu)$$

if $K=o(\sqrt{n})$ or if $K=O(n^{3/8})$, we obtain

$$\begin{aligned}|\{T(2\varepsilon e^{ix},\omega)-K\}/\{T(0,\omega)-K\}| &\le e^{\nu}|2n^2\exp(2n\varepsilon+\nu)-K| \\ &\le 2n^2\exp(2n\varepsilon+2\nu)\end{aligned} \tag{4.10}$$

except for sample functions in an ω-set of measure not exceeding

$$(2/n\pi)^{1/2}e^{-\nu}+(2/\pi)^{1/2}\exp\{-\nu-(ne^{\nu}-\mu)^2/2\}.$$

Therefore from (4.5) and (4.10) we can show that outside the exceptional set

$$N(\varepsilon)\le(\log 2+2\log n+2n\varepsilon+2\nu)/\log 2. \tag{4.11}$$

Because $\varepsilon=n^{-1/4}$, it follows from (4.11) and for all sufficiently large n

$$\operatorname{Prob}\{N(\varepsilon)>3n\varepsilon+2\nu\}\le(2/n\pi)^{1/2}e^{-\nu}+(2/\pi)^{1/2}\exp\{-\nu-(ne^{\nu}-\mu)^2/2\}. \tag{4.12}$$

Let $n'=[3n^{3/4}]$ be the greatest integer less than or equal to $3n^{3/4}$; then from (4.12) and for n large enough we obtain

$$\begin{aligned}EN(\varepsilon) &= \sum_{j>0}\operatorname{Prob}\{N(\varepsilon)\ge j\} \\ &= \sum_{1\le j\le n'}\operatorname{Prob}\{N(\varepsilon)>j\}+\sum_{j\ge 1}\operatorname{Prob}\{N(\varepsilon)>n'+j\} \\ &\le n'+(2/n\pi)^{1/2}\sum_{j\ge 1}e^{-j/2}+(2/\pi)^{1/2}\sum_{j\ge 1}\exp\{-j/2-(ne^{j}-\mu)^2/2\} \\ &= O(n^{3/4}).\end{aligned} \tag{4.13}$$

Finally (4.3), (4.4), and (4.13) complete proof of the theorem.

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DEPARTMENT OF MATHEMATICAL STATISTICS, UNIVERSITY OF CAPE TOWN, RONDEBOSCH 7700, SOUTH AFRICA