THE GAP BETWEEN THE FIRST TWO EIGENVALUES
OF A ONE-DIMENSIONAL SCHRÖDINGER OPERATOR
WITH SYMMETRIC POTENTIAL

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Abstract. We prove the inequality \[ \lambda_2[V_1] - \lambda_1[V_1] \geq \lambda_2[V_0] - \lambda_1[V_0] \]
for the difference of the first two eigenvalues of one-dimensional Schrödinger operators
\[ -\frac{d^2}{dx^2} + V_i(x), \quad i = 0, 1, \]
where \( V_1 \) and \( V_0 \) are symmetric potentials on \((a, b)\) and on \((a, (a + b)/2)\), and \( V_0 - V_1 \) is decreasing on \((a, (3a + b)/4)\).

The gap between consecutive eigenvalues of Schrödinger operators has been the object of considerable attention recently (see [1-5] and many others).

In this note we use the same method established in [1].

We consider the two Schrödinger operators \( H_0 = -\frac{d^2}{dx^2} + V_0(x) \) and \( H_1 = -\frac{d^2}{dx^2} + V_1(x) \), both acting on \( L^2(0, \pi) \) with Dirichlet boundary conditions and with both \( V_0 \) and \( V_1 \) symmetric with respect to \( x = \pi/2 \) and in \( L^1(0, \pi) \).

Let \((\lambda_1, u_1)\) and \((\lambda_2, u_2)\) be the first two eigenvalues together with their associated eigenfunctions of \( H_1 \), and let \((\mu_1, \nu_1)\) and \((\mu_2, \nu_2)\) be the corresponding quantities for \( H_0 \). We will use the following lemma, which is part of Proposition 1 in [1].

Lemma. Let \( H_0 \) and \( H_1 \) be as described above; then

\[ \lambda_2 - \lambda_1 \geq \mu_2 - \mu_1 + \frac{4}{(u, u)} \int_0^{\pi/2} u_1 u_2 \left( \frac{\nu_1}{\nu_2} \right)' \left( \frac{u_1}{u_2} \right)' dx, \]

where \( u = (\nu_1/\nu_2) u_2 \).

Proof. See the proof of inequality (7) in Proposition 1 [1].

Definition. A potential \( V \) is a double-well potential on the closed interval \( I \) if there are \( c_1 \leq c_2 \leq c_3 \in I \) such that \( V \) is nonincreasing for \( x \leq c_1 \) and \( c_2 \leq x \leq c_3 \) and is nondecreasing otherwise.
Theorem 1. Let $H_0$ and $H_1$ be as described above. If $V_0 - V_1$ is a double-well potential and symmetric on $(0, \pi/2)$ then

$$\lambda_2 - \lambda_1 \geq \mu_2 - \mu_1$$

with equality if and only if $V_0 - V_1$ is constant on $[0, \pi]$.

Proof. (2) will follow if we can show that the integral in (1) satisfies

$$I = \int_0^{\pi/2} \nu_1 u_2 \left( \frac{\nu_1}{\nu_2} \right)' \left( \frac{u_2}{\nu_2} \right) dx \geq 0.$$

In order to prove (3), let us use the following:

$$\left( \frac{\nu_1}{\nu_2} \right)' = \frac{\mu_2 - \mu_1}{\nu_2^2} \int_0^x \nu_1 \nu_2 \, dt \geq 0, \quad 0 \leq x \leq \pi/2$$

Substituting (4) and (5) into (3), we get

$$I = \int_0^{\pi/2} \nu_1 u_2 \left( \frac{\nu_1}{\nu_2} \right)' \left( \frac{u_2}{\nu_2} \right) \, dx$$

$$= \int_0^{\pi/2} \left( \frac{\nu_1 u_2^2}{\nu_2^2} (\mu_2 - \mu_1) \int_0^x \nu_1 \nu_2 \, dt \int_0^x [(V_1 - V_0) - (\lambda_2 - \mu_2)] u_2 \nu_2 \, dt \right) dx.$$

We use the following properties:

(a) $G(x) = \int_0^x \nu_1 \nu_2 \, dt$ is a positive increasing function, and therefore

$$0 \leq G \left( \frac{\pi}{4} - x \right) \leq G \left( \frac{\pi}{4} + x \right), \quad 0 \leq x \leq \frac{\pi}{4}.$$

(b) $A(x) = (\nu_2 u_2' - u_2 \nu_2') = \int_0^x [(V_1 - V_0) - (\lambda_2 - \mu_2)] u_2 \nu_2 \, dt$ must vanish at $x = \pi/4$ because of the symmetry of $V_0 - V_1$ on $[0, \pi/2]$, as well as at $x = \pi/2$ because of the symmetry of $V_0 - V_1$ on $[0, \pi]$. Also, $V_1 - V_0$ is nondecreasing on $[0, \pi/4]$. Hence $A(x)$ is nonpositive on $[0, \pi/4]$ and $-A(\pi/4 - x) = A(\pi/4 + x) \geq 0, \quad 0 \leq x \leq \pi/4$.

(c) Using the symmetry of $V_0$ about $\pi/4$ and using $\nu_1(-\pi/2) = \nu_1'(0) = 0$, we get

$$0 \leq \nu_1 \left( \frac{\pi}{4} - x \right) \leq \nu_1 \left( \frac{\pi}{4} + x \right), \quad 0 \leq x \leq \frac{\pi}{4}.$$

(d) $u_2(x)$ and $v_2(x)$ are positive symmetric functions on $0 \leq x \leq \pi/2$.

Using properties (a), (b), (c), and (d) to evaluate $I$, we get immediately that $I \geq 0$; hence, $\lambda_2 - \lambda_1 \geq \mu_2 - \mu_1$. Equality occurs only when $V_1 - V_0 - (\lambda_2 - \mu_2) = 0$, which means $V_1 - V_0$ is a constant. Hence Theorem 1 is proven.
Theorem 2. Let $H = -\frac{d^2}{dx^2} + V(x)$ be an operator on $L^2(a, b)$ with Dirichlet boundary condition, and suppose that $V$ is a symmetric double-well potential and is symmetric also on $(a, (a+b)/2)$. Then the first two eigenvalues satisfy $\lambda_2 - \lambda_1 \leq \frac{3\pi^2}{(b-a)^2}$, with equality if and only if $V$ is a constant.

Proof. In Theorem 1 take $V_0 = V$ and $V_1 = 0$, and observe that for $V_1 = 0$, $\lambda_2 = \frac{4\pi^2}{(b-a)^2}$, $\lambda_1 = \frac{\pi^2}{(b-a)^2}$.

Remark. By reversing the roles of $V_0$ and $V_1$, we get that if $V$ is a symmetric double-barrier potential which is also symmetric on $(a, (a+b)/2)$ we get $\lambda_2 - \lambda_1 \geq \frac{3\pi^2}{(b-a)^2}$.

References


