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THE GAP BETWEEN THE FIRST TWO EIGENVALUES OF A ONE-DIMENSIONAL SCHRÖDINGER OPERATOR WITH SYMMETRIC POTENTIAL

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ABSTRACT. We prove the inequality $\lambda_2[V_1] - \lambda_1[V_1] \geq \lambda_2[V_0] - \lambda_1[V_0]$ for the difference of the first two eigenvalues of one-dimensional Schrödinger operators $-\frac{d^2}{dx^2} + V_i(x)$, $i = 0, 1$, where V_1 and V_0 are symmetric potentials on (a, b) and on $(a, (a+b)/2)$, and $V_0 - V_1$ is decreasing on $(a, (3a+b)/4)$.

The gap between consecutive eigenvalues of Schrödinger operators has been the object of considerable attention recently (see [1–5] and many others).

In this note we use the same method established in [1].

We consider the two Schrödinger operators $H_0 = -\frac{d^2}{dx^2} + V_0(x)$, and $H_1 = -\frac{d^2}{dx^2} + V_1(x)$, both acting on $L^2(0, \pi)$ with Dirichlet boundary conditions and with both V_0 and V_1 symmetric with respect to $x = \pi/2$ and in $L^1(0, \pi)$.

Let (λ_1, u_1) and (λ_2, u_2) be the first two eigenvalues together with their associated eigenfunctions of H_1, and let (μ_1, ν_1) and (μ_2, ν_2) be the corresponding quantities for H_0. We will use the following lemma, which is part of Proposition 1 in [1].

Lemma. *Let H_0 and H_1 be as described above; then*

$$\lambda_2 - \lambda_1 \geq \mu_2 - \mu_1 + \frac{4}{(u, u)} \int_0^{\pi/2} \nu_1 u_2 \left(\frac{\nu_1}{\nu_2}\right)' \left(\frac{u_2}{\nu_2}\right)' dx, \tag{1}$$

where $u = (\nu_1/\nu_2)u_2$.

Proof. See the proof of inequality (7) in Proposition 1 [1].

Definition. A potential V is a *double-well potential* on the closed interval I if there are $c_1 \leq c_2 \leq c_3 \in I$ such that V is nonincreasing for $x \leq c_1$ and $c_2 \leq x \leq c_3$ and is nondecreasing otherwise.

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Theorem 1. *Let H_0 and H_1 be as described above. If $V_0 - V_1$ is a double-well potential and symmetric on $(0, \pi/2)$ then*

$$\lambda_2 - \lambda_1 \geq \mu_2 - \mu_1 \tag{2}$$

with equality if and only if $V_0 - V_1$ is constant on $[0, \pi]$.

Proof. (2) will follow if we can show that the integral in (1) satisfies

$$I = \int_0^{\pi/2} \nu_1 u_2 \left(\frac{\nu_1}{\nu_2}\right)' \left(\frac{u_2}{\nu_2}\right)' dx \geq 0. \tag{3}$$

In order to prove (3), let us use the following:

$$\left(\frac{\nu_1}{\nu_2}\right)' = \frac{\mu_2 - \mu_1}{\nu_2^2} \int_0^x \nu_1 \nu_2 \, dt \geq 0, \quad 0 \leq x \leq \pi/2 \tag{4}$$

$$\left(\frac{u_2}{\nu_2}\right)' = \frac{1}{\nu_2^2}(\nu_2 u_2' - u_2 \nu_2') = \frac{1}{\nu_2^2} \int_0^x [(V_1 - V_0) - (\lambda_2 - \mu_2)] u_2 \nu_2 \, dt. \tag{5}$$

Substituting (4) and (5) into (3), we get

$$\begin{aligned} I &= \int_0^{\pi/2} \nu_1 u_2 \left(\frac{\nu_1}{\nu_2}\right)' \left(\frac{u_2}{\nu_2}\right)' dx \\ &= \int_0^{\pi/2} \left(\frac{\nu_1 u_2}{\nu_2^4}(\mu_2 - \mu_1) \int_0^x \nu_1 \nu_2 \, dt \int_0^x [(V_1 - V_0) - (\lambda_2 - \mu_2)] u_2 \nu_2 \, dt\right) dx. \end{aligned}$$

We use the following properties:

(a) $G(x) = \int_0^x \nu_1 \nu_2 \, dt$ is a positive increasing function, and therefore

$$0 \leq G\left(\frac{\pi}{4} - x\right) \leq G\left(\frac{\pi}{4} + x\right), \quad 0 \leq x \leq \frac{\pi}{4}.$$

(b) $A(x) = (\nu_2 u_2' - u_2 \nu_2') = \int_0^x [(V_1 - V_0) - (\lambda_2 - \mu_2)] u_2 \nu_2 \, dt$ must vanish at $x = \pi/4$ because of the symmetry of $V_0 - V_1$ on $[0, \pi/2]$, as well as at $x = \pi/2$ because of the symmetry of $V_0 - V_1$ on $[0, \pi]$. Also, $V_1 - V_0$ is nondecreasing on $[0, \pi/4]$. Hence $A(x)$ is nonpositive on $[0, \pi/4]$ and $-A(\pi/4 - x) = A(\pi/4 + x) \geq 0$, $0 \leq x \leq \pi/4$.

(c) Using the symmetry of V_0 about $\pi/4$ and using $\nu_1(-\pi/2) = \nu_1'(0) = 0$, we get

$$0 \leq \nu_1\left(\frac{\pi}{4} - x\right) \leq \nu_1\left(\frac{\pi}{4} + x\right), \quad 0 \leq x \leq \frac{\pi}{4}.$$

(d) $u_2(x)$ and $\nu_2(x)$ are positive symmetric functions on $0 \leq x \leq \pi/2$.

Using properties (a), (b), (c), and (d) to evaluate I, we get immediately that $I \geq 0$; hence $\lambda_2 - \lambda_1 \geq \mu_2 - \mu_1$. Equality occurs only when $V_1 - V_0 - (\lambda_2 - \mu_2) = 0$, which means $V_1 - V_0$ is a constant. Hence Theorem 1 is proven.

Theorem 2. *Let* $H = -\frac{d^2}{dx^2} + V(x)$ *be an operator on* $L^2(a, b)$ *with Dirichlet boundary condition, and suppose that* V *is a symmetric double-well potential and is symmetric also on* $(a, (a+b)/2)$. *Then the first two eigenvalues satisfy* $\lambda_2 - \lambda_1 \le 3\pi^2/(b-a)^2$, *with equality if and only if* V *is a constant.*

Proof. In Theorem 1 take $V_0 = V$ and $V_1 = 0$, and observe that for $V_1 = 0$, $\lambda_2 = 4\pi^2/(b-a)^2$, $\lambda_1 = \pi^2/(b-a)^2$.

Remark. By reversing the roles of V_0 and V_1, we get that if V is a symmetric double-barrier potential which is also symmetric on $(a, (a+b)/2)$ we get $\lambda_2 - \lambda_1 \ge 3\pi^2/(b-a)^2$.

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