ASYMPTOTIC BEHAVIOR OF STABLE MANIFOLDS

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Abstract. The relation between local stable manifolds of an ordinary differential equation and its discretization is studied. We show that a local stable manifold of a hyperbolic fixed point of an ordinary differential equation is the limit of local stable manifolds of the same fixed point of its discretizations as the discretization parameter \( h > 0 \) approaches 0.

The main purpose of this paper is to investigate the following problem. Let us consider a mapping \( s_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n \),

\[
s_\varepsilon(z) = z + \varepsilon \cdot f(z)
\]

where \( f \in \mathcal{C}^2, \varepsilon \in \mathbb{R}\setminus\{0\}, f(0) = 0, Df(0) \) is hyperbolic, i.e., the eigenvalues of \( Df(0) \) have nonzero real parts. We can write the mapping \( s_\varepsilon \) in the form

\[
(\cdot) \quad s_\varepsilon(x, y) = (x + \varepsilon(Ax + g(x, y), y + \varepsilon(By + r(x, y)))
\]

where \( A \in \mathcal{L}(\mathbb{R}^k), B \in \mathcal{L}(\mathbb{R}^m) \) have positive, negative real parts of their eigenvalues, respectively. It is easy to see that \( I+\varepsilon B, I+\varepsilon A \) have no eigenvalues on the unit circle for all small \( \varepsilon \neq 0 \). Hence the mapping \( s_\varepsilon \) has the hyperbolic fixed point 0. For this point there exists a local stable manifold \( W^{s}_\varepsilon \) for each small \( \varepsilon \) [2]. The following theorem gives us information about their limit as \( \varepsilon > 0 \) approaches 0.

**Theorem 1.** There is \( \delta > 0 \) and a continuous mapping

\[
h : (0, \delta) \to C^1(B_\delta, \mathbb{R}^k)
\]

such that

(i) the graph of \( h(\varepsilon, \cdot) \in C^2(B_\delta, \mathbb{R}^k) \) is a local stable manifold \( W^{s}_\varepsilon \) of 0 for \( s_\varepsilon \) and \( \varepsilon > 0 \)

(ii) the graph of \( h(0, \cdot) \) is a local stable manifold of 0 for the differential equation \( z' = f(z) \)

\[
(B_\delta = \{x \in \mathbb{R}^m, |x| \leq \delta\}).
\]
We remark that this problem is directly related to the method of averaging (c.f. [6] for further details).

The plan of our paper is as follows. In the first part, we present the proof of Theorem 1. In the second part, we give a simple proof of a result from [3], where the author of that paper has considered a similar problem for periodic ordinary differential equations. Finally, we investigate a discrete version of this problem.

Proof of Theorem 1. By the well-known arguments used in [1] and [5], we can suppose the existence of $e_0 > 0$, $K > 0$ with the properties

$$1 + e \cdot K \leq |(I + e \cdot A)|, \quad |I + e \cdot B| \leq 1 - e \cdot K$$

for each $0 \leq e < e_0$.

Let

$$Z = \{(x_n, y_n) \}_{n=1}^\infty, \quad x_n \in \mathbb{R}^k, \quad y_n \in \mathbb{R}^m, \quad \sup(|x_n| + |y_n|) < \infty\}.$$

$Z$ is a Banach space with the norm

$$\|\{(x_n, y_n)\}\|_1 = \sup(|x_n| + |y_n|).$$

Let us consider the linear mappings

$$A : \mathbb{R}^k \to \mathbb{R}^k, \quad (A(x^\infty)_n = x_{n+1} - (I + eA)x_n)$$

$$B : \mathbb{R}^m \to \mathbb{R}^m, \quad (B(y^\infty)_n = y_{n+1} - (I + eB)y_n)$$

where

$$X = \{x^\infty_n, \quad x_n \in \mathbb{R}^k, \quad \sup|x_n| < \infty\},$$

$$\overline{X} = \{y^\infty_n, \quad y_n \in \mathbb{R}^m, \quad \sup|y_n| < \infty\}.$$

We solve the equation

$$x_{n+1} - (I + eA)x_n = h_n$$

in the space $X$. This equation has a unique bounded solution for all small $e > 0$ [7, p. 272] (i.e., $1 \gg e > 0$), namely,

$$x_n = -((I + eA)^{-1}h_n + (I + eA)^{-2}h_{n+1} + \cdots), \quad n \geq 1.$$

Hence

$$|x_n| \leq \frac{|h_n|}{1 + e \cdot K} + \frac{|h_{n+1}|}{(1 + e \cdot K)^2} + \cdots$$

$$\leq |h| \cdot \left(\frac{1}{1 + K e} + \cdots\right) = |h| \cdot \frac{1}{e \cdot K}.$$

Thus

$$|x| \leq |h| \cdot \frac{1}{e \cdot K}, \quad x = \{x^\infty_n\}, \quad h = \{h^\infty_n\}.$$

In the same way we solve in $\overline{X}$

$$y_{n+1} - (I + eB)y_n = h_n.$$
This equation has a unique bounded solution for \( 1 \gg \varepsilon > 0 \) with the initial condition \( y_1 = c \):

\[
y_n = (I + \varepsilon B)^{n-1} c + (I + \varepsilon B)^{n-2} h_1 + \cdots + h_{n-1}, \quad n \geq 2.
\]

Hence

\[
|y_n| \leq |c| + |h| \cdot ((1 - \varepsilon \cdot K)^{n-2} + \cdots + 1)
\]

\[
< |c| + \frac{|h|}{\varepsilon \cdot K}.
\]

Finally, we solve the equation

\[
\begin{align*}
x_{n+1} &= (I + \varepsilon A)x_n + \varepsilon g(x_n, y_n) \\
y_{n+1} &= (I + \varepsilon B)y_n + \varepsilon r(x_n, y_n)
\end{align*}
\]

in \( Z \) near \((0,0)\). By the above results (1) can be written in the form

\[
\begin{pmatrix} x_n \end{pmatrix}_1 = e(A)^{-1} \begin{pmatrix} g(x_n, y_n) \end{pmatrix}_1
\]

\[
\begin{pmatrix} y_n \end{pmatrix}_2 = e(B)^{-1} \begin{pmatrix} r(x_n, y_n) \end{pmatrix}_2, \quad y_1 = c.
\]

But \( e(A)^{-1} \), \( e(B)^{-1} \) are uniformly bounded for \( 1 \gg \varepsilon > 0 \). Hence we can uniformly apply the implicit function theorem to obtain a solution of (+):

\[
\begin{pmatrix} x_n(e, c) \end{pmatrix}_1, \quad \{y_n(e, c) \}_2
\]

From this it follows that the graph \( W^s_\varepsilon \) of \( x_1(e, \cdot) \) in \( B_\delta \times \mathbb{R}^k \) contains all points from which bounded orbits of \( s_{\varepsilon} \) near \( 0 \in \mathbb{R}^k \times \mathbb{R}^m \) start. Obviously \( x_1(e, \cdot) \in C^2(B_\delta, \mathbb{R}^k) \) and the set

\[
\{x_1(e, \cdot)\}_{e \in (0, \delta_0)}
\]

is bounded in \( C^2(B_\delta, \mathbb{R}^k) \).

Now we show that \( W^s_\varepsilon \) is the local stable manifold of 0 for \( s_{\varepsilon} \). By (1) we have

\[
\begin{align*}
|x_n| &\leq |(I + \varepsilon A)^{-1}| \cdot |h_n| + |(I + \varepsilon A)^{-2}| \cdot |h_{n+1}| + \cdots , \quad h_n = \varepsilon \cdot g(x_n, y_n) \\
|y_n| &\leq |(I + \varepsilon B)^{n-1}| \cdot |c| + |(I + \varepsilon B)^{n-2}| \cdot |g_1| + \cdots + |g_{n-1}|, \quad g_n = \varepsilon \cdot r(x_n, y_n).
\end{align*}
\]

If \( \lim (|x_n| + |y_n|) = b > 0 \) for a small \( \{(x_n, y_n)\}_{1}^{\infty} \) then

\[
|h_n| \leq \frac{K}{3} \cdot b \cdot \varepsilon, \quad |g_n| \leq \frac{K}{3} \cdot b \cdot \varepsilon, \quad n \gg 1 \text{ (i.e., \( n \) is large)}.
\]

Hence

\[
\begin{align*}
|x_n| &\leq \frac{1}{1 + \varepsilon \cdot K} \cdot \frac{K}{3} \cdot b \cdot \varepsilon + \frac{1}{(1 + \varepsilon \cdot K)^2} \cdot \frac{K}{3} \cdot b \cdot \varepsilon + \cdots \\
&= \frac{1}{3} \cdot b, \quad n \gg 1
\end{align*}
\]
and for a fixed large \( p \)

\[
|y_n| \leq (1 - \varepsilon K)^{n-1} \cdot |c| + \cdots + (1 - \varepsilon K)^{n-p} \cdot |g_{p-1}| \\
+ (1 - \varepsilon K)^{n-p-1} \cdot \frac{K \varepsilon}{3} \cdot b + \cdots + \frac{K \varepsilon}{3} \cdot b.
\]

We see that

\[
\lim (|x_n| + |y_n|) \leq \frac{1}{3} \cdot b + \frac{1}{3} \cdot b < b.
\]

This proves that \( \lim (|x_n| + |y_n|) = 0 \).

Now we take a sequence \( \{x_1(\varepsilon_n, \cdot)\}_{n=1}^{\infty} \in C^2(B_{\delta}, R^k) \) such that \( \varepsilon_n > 0 \), \( \varepsilon_n \to 0 \). This sequence is bounded in \( C^2(B_{\delta}, R^k) \) and hence there exists a subsequence \( \{x_1(\varepsilon_{i}, \cdot)\}_{i=1}^{\infty} \) which has a limit point \( h \in C^1(B_{\delta}, R^k) \) in the space \( C^1(B_{\delta}, R^k) \).

On the other hand, since \( W^s_{\varepsilon} \) is the invariant manifold of \( s_{\varepsilon} \), \( x_1(\varepsilon, \cdot) \) must satisfy

\[
(1 + \varepsilon A)x_1(\varepsilon, y) + \varepsilon g(x_1(\varepsilon, y), y) = x_1(\varepsilon, (1 + \varepsilon B)y + \varepsilon r(x_1(\varepsilon, y), y)).
\]

Hence

\[
\varepsilon_n (A x_1(\varepsilon_n, y) + g(x_1(\varepsilon_n, y), y)) \\
= x_1(\varepsilon_n, (1 + \varepsilon_n B)y + \varepsilon_n r(x_1(\varepsilon_n, y), y)) - x_1(\varepsilon_n, y)
\]

and

\[
A x_1(\varepsilon, y) + g(x_1(\varepsilon_n, y), y) \\
= (D_y x_1(\varepsilon, y) + O(\varepsilon))(By + r(x_1(\varepsilon_n, y), y))
\]

where we used the mean value theorem. Hence

\[
Ah(y) + g(h(y), y) = D_y h(y)(By + r(h(y), y))
\]

\( h(0) = 0 \).

If \( \delta \) is sufficiently small, then (4) has a unique solution and the graph of \( h \) is a local stable manifold of 0 for the equation

\[
z' = f(z).
\]

Indeed, the point 0 is hyperbolic for \( z' = f(z) \). By the assumptions on \( f \) the point 0 has local stable and local unstable \( C^2 \)-manifolds. Using a \( C^2 \)-change of coordinates we can consider that the \( x \)-axis is the local unstable manifold and the \( y \)-axis is the local stable manifold. In these new coordinates the equation \( z' = f(z) \) has the form \( z' = f_1(z) \), i.e.,

\[
\begin{align*}
    x' &= A_1 x + g_1(x, y) \\
    y' &= B_1 y + r_1(x, y)
\end{align*}
\]
where $A_1$, $B_1$ have the above properties, $g_1(x, y) = O(|x|) \cdot O(|x| + |y|)$, $r_1(x, y) = O(|y|) \cdot O(|x| + |y|)$. We can assume that
\[(B_1 y, y) \leq -a \cdot |y|^2, \quad a > 0\]
where $(\cdot, \cdot)$ is a scalar product. The graph of $h$ in the new coordinates is the graph of some mapping $h_1 : B_\delta \to R^k$. We have for a small $|x| + |y|:
\[(B_1 y + r_1(x, y), y) \leq -a_1 \cdot |y|^2, \quad 0 < a_1 < a.\]

Using this property and the fact that the graph of $h_1$ is invariant by $z' = f_1(z)$, we see that the graph of $h$ is the local stable manifold of 0 for $z' = f(z)$.

From the above results we obtain the proof of Theorem 1.

We note that using the above method we can give a simple proof of a result from the paper [3]. The author of that paper considers an ordinary differential equation
\[(6) \quad z' = \varepsilon f(z, t, \varepsilon)\]
where by [3, Proposition 2.2] we can suppose that
\[f(z, t, \varepsilon) = (Ax + g(x, y, t, \varepsilon), By + r(x, y, t, \varepsilon))\]
where $A$, $B$ have the above properties, $r$, $g$ are $2\pi$-periodic in $t$, $r$, $g \in C^2$, $g(0, 0, \cdot, \cdot) = 0$, $D_x g(0, 0, \cdot, 0) = 0$, $D_y r(0, 0, \cdot, 0) = 0$.

The appropriate spaces are the following:
\[Z = \{(x, y), x \in C^0(\mathbb{R}^+, \mathbb{R}^k), y \in C^0(\mathbb{R}^+, \mathbb{R}^m), \sup(|x| + |y|) < \infty\}\]
\[X = \{x \in C^0(\mathbb{R}^+, \mathbb{R}^k), \sup |x| < \infty\}\]
\[\overline{X} = \{y \in C^0(\mathbb{R}^+, \mathbb{R}^m), \sup |y| < \infty\}.\]

By the variation of the constants formula we can obtain an equation similar to (6). Solving this equation we have a mapping
\[h(\varepsilon, \cdot, \cdot) \in C^2(B_\delta \times R, R^k)\]
where $(h(\varepsilon, y, t), y)$ is the initial value from which a bounded orbit of (6) near $0 \in R^k \times R^m$ starts. Moreover, the set
\[\{h(\varepsilon, \cdot, \cdot)\}_{\varepsilon \in (0, \varepsilon_0)}\]
is bounded in $C^2(B_\delta \times R, R^k)$ and $h(\varepsilon, \cdot, \cdot)$ is $2\pi$-periodic in $t$. A result of Hale [4, pp. 166–167] gives us that the graph $W^s_\varepsilon$ of $h(\varepsilon, \cdot, \cdot)$ is a local stable manifold of 0 for (6). Hence $W^s_\varepsilon$ is invariant by (6) and this implies:
\[D_x h(\varepsilon, \cdot, \cdot) + D_y h(\varepsilon, \cdot, \cdot)(\varepsilon B \cdot + \varepsilon r(h, \cdot, \cdot, \varepsilon)) = \varepsilon A h + \varepsilon g(h, \cdot, \cdot, \varepsilon).\]

Since
\[\frac{1}{2\pi} \int_0^{2\pi} D_x h(\cdot, \cdot, s) ds = 0\]
we obtain
\[
\frac{1}{2\pi} \int_0^{2\pi} D_y h(e, \cdot, s)(B \cdot + r(h, \cdot, s, e)) \, ds
= \frac{1}{2\pi} \int_0^{2\pi} (Ah(e, \cdot, s) + g(h, \cdot, s, e)) \, ds.
\]

We take a sequence \( \{h(e_n, \cdot, \cdot)\}_{n=1}^\infty \), \( e_n \to 0 \). Then there is a subsequence \( \{h(e_{n_i}, \cdot, \cdot)\}_{i=1}^\infty \) which tends to \( h \in C^1(B_\delta \times R, R^k) \) in this space. Hence
\[
\frac{1}{2\pi} \int_0^{2\pi} D_y h(\cdot, s)(B \cdot + r(h, \cdot, s, 0)) \, ds = \frac{1}{2\pi} \int_0^{2\pi} (Ah(\cdot, s) + g(h, \cdot, s, 0)) \, ds
= \frac{1}{2\pi} \int_0^{2\pi} r(h, \cdot, s, 0) \, ds.
\]

We see that \( h \) is independent of \( t \), \( h(y, t) = h(y) \) and
\[
D_y h(B, \frac{1}{2\pi} \int_0^{2\pi} r(h, \cdot, s, 0) \, ds) = Ah + \frac{1}{2\pi} \int_0^{2\pi} g(h, \cdot, s, 0) \, ds
= \bar{h}(0) = 0.
\]

This gives us that the graph of \( \bar{h} \) is a local stable manifold of 0 of the equation
\[
(7) \quad z' = \frac{1}{2\pi} \int_0^{2\pi} f(z, s, 0) \, ds.
\]

Summing up we have [3]:

**Theorem 2.** There exists \( \delta > 0 \) and a \( C^0 \)-mapping
\[
h : (0, \delta) \to C^1(B_\delta \times R, R^k)
\]
such that

(i) \( h \) is \( 2\pi \)-periodic in \( t \),

(ii) the graph of \( h(e, \cdot, \cdot) \in C^2(B_\delta \times R, R^k) \) in \( R^n \times R \) is a local stable manifold of 0 of (6) for \( e > 0 \),

(iii) the graph of \( h(0, \cdot, \cdot) \) in \( R^n \times R \) is \( W \times R \), where \( W \) is a local stable manifold of 0 for (7).

Finally, we shall investigate a discrete version of (6). For each initial point \((x, m) \in R^n \times Z\), where the set \( Z \) is the set of integers, we define an orbit \( \{x_i\}_{i=1}^\infty \) in the following way
\[
(8) \quad x_{i+1} = x_i + \varepsilon g(x_i, i, e), \quad i \geq m, \ x_m = x
\]
where \( g \in C^2, \ g : R^n \times Z \times R \to R^n \) and \( g \) is \( p \)-periodic in the second variable, i.e., \( g(\cdot, i + p, \cdot) = g(\cdot, i, \cdot) \).

Consider also the averaged mapping
\[
\bar{g}(x) = \frac{1}{p} \sum_{i=1}^p g(x, i, 0).
\]

Suppose \( \bar{g}(0) = 0 \) and \( D\bar{g}(0) \) is hyperbolic, i.e., has no eigenvalues on the imaginary axis.
Lemma 3. There exists \( \varepsilon_1 > 0 \) and a \( C^2 \)-mapping
\[ x : \{1, \ldots, p\} \times (-\varepsilon_1, \varepsilon_1) \to \mathbb{R}^n \]
such that for each \( \varepsilon \in (-\varepsilon_1, \varepsilon_1), \varepsilon \neq 0 \) (8) has a unique small \( p \)-periodic orbit \( \{x(i, \varepsilon)\}_{i=1}^p \) and \( x(\cdot, 0) = 0 \).

Proof. We solve the equation
\[ x_2 = x_1 + \varepsilon g(x_1, 1, \varepsilon) \]
\[ \vdots \]
\[ x_p = x_{p-1} + \varepsilon g(x_{p-1}, p-1, \varepsilon) \]
\[ x_1 = x_p + \varepsilon g(x_p, p, \varepsilon). \]

Setting
\[ w = (x_1, \ldots, x_p), \quad Hw = (x_2 - x_1, \ldots, x_1 - x_p) \]
\[ F_\varepsilon(w) = (g(x_1, 1, \varepsilon), \ldots, g(x_p, p, \varepsilon)) \]
\[ Pw = \left(\frac{x_1 + \cdots + x_p}{p}, \ldots, \frac{x_1 + \cdots + x_p}{p}\right) \]
this equation can be rewritten in the form
\[ Hu = (I - P) \cdot \varepsilon F_\varepsilon(u + v), \quad u \in \text{Im} \, H \]
\[ 0 = \varepsilon PF_\varepsilon(u + v), \quad v \in \text{Ker} \, H, \]
since \((\mathbb{R}^n)^p = \text{Ker} \, H \oplus \text{Im} \, H\) and \( \text{Ker} \, P = \text{Im} \, H, \text{Im} \, P = \text{Ker} \, H \). Now using the implicit function theorem the first equation has a solution \( u(\varepsilon, v) \) for \( \varepsilon, v \) small such that \( u(0, \cdot) = 0 \). Hence
\[ 0 = PF_\varepsilon(u(\varepsilon, v) + v). \]

We can also apply the implicit function theorem for this equation since \( v \in \text{Ker} \, H = \{x \in (\mathbb{R}^n)^p, x_1 = \cdots = x_p = z\} \), \( PF_\varepsilon(u(0, v) + v) = \left(\overline{g}(z), \ldots, \overline{g}(z)\right)\) and \( \overline{g}(0) = 0, \, D\overline{g}(0) \) is invertible.

From Lemma 3 we make the \( p \)-periodic coordinate change
\[ z_i = x_i - x(i, \varepsilon) \]
and (8) becomes
\[ z_{i+1} = z_i + \varepsilon f(z_i, i, \varepsilon) \]
where \( f \in C^2, f(0, i, \varepsilon) = 0 \) and \( \overline{g}(y) = \overline{f}(y) = \frac{1}{p} \sum_{i=1}^p f(y, i, 0) \). Next we use averaging to make (9) autonomous to order 1.
Lemma 4. Let $f$ be as in (9). Then by the change of coordinates

$$y_i = z_i + \varepsilon \sum_{j=1}^{i-1} (f(z_j, j, 0) - \bar{g}(z_j)), \quad i > 1$$

$$y_1 = z_1$$

(9) becomes

(10) \[ y_{i+1} = y_i + \varepsilon h(y_i, i, \varepsilon) \]

where $h \in C^2$, $h(x, i, 0) = \bar{f}(x)$.

Proof. Using the above change of coordinates we immediately obtain the assertion of Lemma 4. We note that this change of coordinates is $p$-periodic in the second variable.

Hence our problem (8) can be reduced to (10), where $\bar{f}(x) = \bar{g}(x)$ has the hyperbolic fixed point $x = 0$. Next we write (10) in the following form

(11) \[
x_{i+1} = x_i + \varepsilon (Ax_i + s(x_i, y_i, i, \varepsilon)),
\]

$$y_{i+1} = y_i + \varepsilon (By_i + r(x_i, y_i, i, \varepsilon)),$$

where $A$, $B$ have the properties from Theorem 1: $s$, $r \in C^2$, $s(0, 0, i, \varepsilon) = 0$, $r(0, 0, i, \varepsilon) = 0$, $D_x, s(0, 0, \cdot, 0) = 0$, $D_x, r(0, 0, \cdot, 0) = 0$ and $s$, $r$ are $p$-periodic in $i$. Now we can use the above procedure to obtain $C^2$-mappings $h_m(\cdot, \varepsilon) \in C^2(B_\delta, R^k)$ such that the graph $W^m_\varepsilon$ of $h_m$ is a local stable manifold of $0 \in R^p$ of (11). We note that iterations in (11) start from $i = m$. By the periodic condition for $r$, $s$ we have

$$h_{p+i} = h_i.$$ 

In this case the set \{h_m(\cdot, \varepsilon)\}_{\varepsilon \in (0, \varepsilon_0)} is also bounded in $C^2(B_\delta, R^k)$. We take a sequence \{(h_1(\cdot, \varepsilon_i), \ldots, h_p(\cdot, \varepsilon_i))\}_{i=1}^\infty, \varepsilon_i \to 0$. Then there exists a subsequence \{(h_1(\cdot, \varepsilon_{i_n}), \ldots, h_p(\cdot, \varepsilon_{i_n}))\}_{i=1}^\infty which has a limit point $(h_1, \ldots, h_p)$ in the space

$$C^1(B_\delta, R^k) \times \cdots \times C^1(B_\delta, R^k).$$

(This follows from the well-known theorem of Arzela-Ascoli and we also used this theorem in the above proofs of our theorems.) Since the sequence \{W^1_\varepsilon, \ldots, W^p_\varepsilon\} is invariant for (11) we have

$$h_{i+1}(y + \varepsilon (By + r(h_i(x, \varepsilon), y, i, \varepsilon)), \varepsilon) = h_i(y, \varepsilon) + \varepsilon (Ah_i(y, \varepsilon) + s(h_i(y, \varepsilon), y, i, \varepsilon)).$$

Thus by the mean value theorem

$$(D_y h_{i+1}(y, e) + O(e)) \cdot (By + r(h_i(y, e), y, i, \varepsilon)) = \varepsilon (Ah_i(y, \varepsilon) + s(h_i(y, \varepsilon), y, i, \varepsilon)) + h_i(y, e) - h_{i+1}(y, e)$$
and

\[ \frac{1}{p} \sum_{i=1}^{p} (D_{y} h_{i}^{y}(y, \varepsilon) + O(\varepsilon))(By + r(h_{i}(y, \varepsilon), y, i, \varepsilon)) \]

\[ = \frac{1}{p} \sum_{i=1}^{p} (A h_{i}(y, \varepsilon) + s(h_{i}(y, \varepsilon), y, i, \varepsilon)). \]

Hence we obtain

\[ h_{1} = h_{2} = \cdots = h_{p} = h \]

and

\[ \frac{1}{p} \sum_{i=1}^{p} D_{y} h(y)(By + r(h(y), y, i, 0)) \]

\[ = \frac{1}{p} \sum_{i=1}^{p} (Ah(y) + s(h(y), y, i, 0)) \]

\[ h(0) = 0. \]

Thus the graph of \( h \) is a local stable manifold for the averaged equation

\[ z' = \bar{g}(z). \]

From the above results we obtain the following theorem.

**Theorem 5.** There exists \( \delta > 0 \) and a \( C^{0} \)-mapping

\[ h : \{1, 2, \ldots, p\} \times (0, \delta) \to C^{1}(B_{\delta}, R^{k}) \]

such that

1. The graph of \( h(\cdot, \cdot, \varepsilon), \varepsilon > 0 \) is a local stable manifold of a unique small \( p \)-periodic orbit of (8).

2. There is a local stable manifold \( W \) of 0 of the averaged equation (12) such that the graph of \( h(\cdot, \cdot, 0) \) is \( W \times \{1, 2, \ldots, p\} \). (Note that the stable manifolds are subsets of \( R^{k} \times \{1, 2, \ldots, p\} \).)

**REFERENCES**


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