ASYMPTOTIC BEHAVIOR OF STABLE MANIFOLDS

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Abstract. The relation between local stable manifolds of an ordinary differential equation and its discretization is studied. We show that a local stable manifold of a hyperbolic fixed point of an ordinary differential equation is the limit of local stable manifolds of the same fixed point of its discretizations as the discretization parameter \( h > 0 \) approaches 0.

The main purpose of this paper is to investigate the following problem. Let us consider a mapping \( s_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n \),
\[
s_\varepsilon(z) = z + \varepsilon \cdot f(z)
\]
where \( f \in C^2, \varepsilon \in \mathbb{R} \setminus \{0\}, f(0) = 0, Df(0) \) is hyperbolic, i.e., the eigenvalues of \( Df(0) \) have nonzero real parts. We can write the mapping \( s_\varepsilon \) in the form
\[
\begin{aligned}
(-) \quad s_\varepsilon(x, y) &= (x + \varepsilon (Ax + g(x, y)), y + \varepsilon (By + r(x, y)))
\end{aligned}
\]
where \( A \in \mathcal{L}(\mathbb{R}^k), B \in \mathcal{L}(\mathbb{R}^m) \) have positive, negative real parts of their eigenvalues, respectively. It is easy to see that \( I + \varepsilon B, I + \varepsilon A \) have no eigenvalues on the unit circle for all small \( \varepsilon \neq 0 \). Hence the mapping \( s_\varepsilon \) has the hyperbolic fixed point 0. For this point there exists a local stable manifold \( W^s_\varepsilon \), for each small \( \varepsilon \) [2]. The following theorem gives us information about their limit as \( \varepsilon > 0 \) approaches 0.

Theorem 1. There is \( \delta > 0 \) and a continuous mapping
\[
h : (0, \delta) \to C^1(B_\delta, \mathbb{R}^k)
\]
such that

(i) the graph of \( h(\varepsilon, \cdot) \in C^2(B_\delta, \mathbb{R}^k) \) is a local stable manifold \( W^s_\varepsilon \) of 0 for \( s_\varepsilon \) and \( \varepsilon > 0 \)

(ii) the graph of \( h(0, \cdot) \) is a local stable manifold of 0 for the differential equation \( z' = f(z) \)
\[
(B_\delta = \{ x \in \mathbb{R}^m, |x| \leq \delta \}).
\]
We remark that this problem is directly related to the method of averaging (c.f. [6] for further details).

The plan of our paper is as follows. In the first part, we present the proof of Theorem 1. In the second part, we give a simple proof of a result from [3], where the author of that paper has considered a similar problem for periodic ordinary differential equations. Finally, we investigate a discrete version of this problem.

Proof of Theorem 1. By the well-known arguments used in [1] and [5], we can suppose the existence of $\varepsilon_0 > 0$, $K > 0$ with the properties

$$1 + \varepsilon \cdot K \leq |(I + \varepsilon \cdot A)|, \quad |I + \varepsilon \cdot B| \leq 1 - \varepsilon \cdot K$$

for each $0 \leq \varepsilon < \varepsilon_0$.

Let

$$Z = \{(x_n, y_n)\}^{\infty}_{n=1}, \quad x_n \in \mathbb{R}^k, \quad y_n \in \mathbb{R}^m, \quad \sup(|x_n| + |y_n|) < \infty\}.$$

$Z$ is a Banach space with the norm

$$|\{(x_n, y_n)\}^{\infty}_{n=1}| = \sup(|x_n| + |y_n|).$$

Let us consider the linear mappings

$$\bar{A} : X \rightarrow X, \quad \bar{A}((x_n)_{n=1}^{\infty}) = x_{n+1} - (I + \varepsilon A)x_n,$$

$$\bar{B} : X \rightarrow X, \quad \bar{B}((y_n)_{n=1}^{\infty}) = y_{n+1} - (I + \varepsilon B)y_n$$

where

$$X = \{(x_n)_{n=1}^{\infty}, \quad x_n \in \mathbb{R}^k, \quad \sup|x_n| < \infty\},$$

$$\bar{X} = \{(y_n)_{n=1}^{\infty}, \quad y_n \in \mathbb{R}^m, \quad \sup|y_n| < \infty\}.$$

We solve the equation

$$x_{n+1} - (I + \varepsilon A)x_n = h_n$$

in the space $X$. This equation has a unique bounded solution for all small $\varepsilon > 0$ [7, p. 272] (i.e., $1 \gg \varepsilon > 0$), namely,

$$x_n = -((I + \varepsilon A)^{-1}h_n + (I + \varepsilon A)^{-2}h_{n+1} + \cdots), \quad n \geq 1.$$

Hence

$$|x_n| \leq \frac{|h_n|}{1 + \varepsilon \cdot K} + \frac{|h_{n+1}|}{(1 + \varepsilon \cdot K)^2} + \cdots$$

$$\leq |h| \cdot \left(\frac{1}{1 + K\varepsilon} + \cdots\right) = |h| \cdot \frac{1}{\varepsilon \cdot K}.$$

Thus

$$|x| \leq |h| \cdot \frac{1}{\varepsilon \cdot K}, \quad x = \{x_n\}_{n=1}^{\infty}, \quad h = \{h_n\}_{n=1}^{\infty}.$$

In the same way we solve in $\bar{X}$

$$y_{n+1} - (I + \varepsilon B)y_n = h_n.$$
This equation has a unique bounded solution for \( 1 \gg \epsilon > 0 \) with the initial condition \( y_1 = c \):

\[
y_n = (I + \epsilon B)^{n-1} c + (I + \epsilon B)^{n-2} h_1 + \cdots + h_{n-1}, \quad n \geq 2.
\]

Hence

\[
|y_n| \leq |c| + |h| \cdot ((1 - \epsilon \cdot K)^{n-2} + \cdots + 1) < |c| + |h| \cdot \frac{1}{\epsilon \cdot K}.
\]

Finally, we solve the equation

\[
\begin{align*}
x_{n+1} &= (I + \epsilon A)x_n + \epsilon g(x_n, y_n) \\
y_{n+1} &= (I + \epsilon B)y_n + \epsilon r(x_n, y_n)
\end{align*}
\]

in \( Z \) near \((0,0)\). By the above results (1) can be written in the form

\[
\begin{align*}
\{x_n\}_1^\infty &= \epsilon(A)^{-1} \{g(x_n, y_n)\}_1^\infty \\
\{y_n\}_2^\infty &= \epsilon(B)^{-1} \{r(x_n, y_n)\}_1^\infty, \quad y_1 = c.
\end{align*}
\]

But \( \epsilon(A)^{-1} \), \( \epsilon(B)^{-1} \) are uniformly bounded for \( 1 \gg \epsilon > 0 \). Hence we can uniformly apply the implicit function theorem to obtain a solution of (1):

\[
\{x_n(\epsilon, c)\}_1^\infty, \quad \{y_n(\epsilon, c)\}_2^\infty.
\]

From this it follows that the graph \( W_{\epsilon}^s \) of \( x_1(\epsilon, \cdot) \) in \( B_\delta \times R^k \) contains all points from which bounded orbits of \( s_\epsilon \) near \( 0 \in R^k \times R^m \) start. Obviously \( x_1(\epsilon, \cdot) \in C^2(B_\delta, R^k) \) and the set

\[
\{x_1(\epsilon, \cdot)\}_{\epsilon \in (0,\epsilon_0)}
\]

is bounded in \( C^2(B_\delta, R^k) \).

Now we show that \( W_{\epsilon}^s \) is the local stable manifold of 0 for \( s_\epsilon \). By (1) we have

\[
\begin{align*}
|x_n| &\leq |(I + \epsilon A)^{-1}| \cdot |h_n| + |(I + \epsilon A)^{-2}| \cdot |h_{n+1}| + \cdots, \quad h_n = \epsilon \cdot g(x_n, y_n) \\
|y_n| &\leq |(I + \epsilon B)^{n-1}| \cdot |c| + |(I + \epsilon B)^{n-2}| \cdot |g_1| + \cdots + |g_{n-1}|, \quad g_n = \epsilon \cdot r(x_n, y_n).
\end{align*}
\]

If \( \lim (|x_n| + |y_n|) = b > 0 \) for a small \( ((x_n, y_n))_1^\infty \) then

\[
|h_n| \leq \frac{K}{3} \cdot b \cdot \epsilon, \quad |g_n| \leq \frac{K}{3} \cdot b \cdot \epsilon, \quad n \gg 1 \text{ (i.e., } n \text{ is large)}.
\]

Hence

\[
|x_n| \leq \frac{1}{1 + \epsilon \cdot K} \cdot \frac{K}{3} \cdot b \cdot \epsilon + \frac{1}{(1 + \epsilon \cdot K)^2} \cdot \frac{K}{3} \cdot b \cdot \epsilon + \cdots = \frac{1}{3} \cdot b, \quad n \gg 1
\]
and for a fixed large \( p \)

\[
|y_n| \leq (1 - \varepsilon K)^{n-1} \cdot |c| + \cdots + (1 - \varepsilon K)^{n-p} \cdot |g_{p-1}|
\]

\[
+ (1 - \varepsilon K)^{n-p-1} \cdot \frac{K\varepsilon}{3} \cdot b + \cdots + \frac{K\varepsilon}{3} \cdot b.
\]

We see that

\[
\lim(|x_n| + |y_n|) \leq \frac{1}{3} \cdot b + \frac{1}{3} \cdot b < b.
\]

This proves that \( \lim(|x_n| + |y_n|) = 0 \).

Now we take a sequence \( \{x_1(\varepsilon, \cdot)\}_{i=1}^{\infty} \in C^2(B_\delta, R^k) \) such that \( \varepsilon_n > 0 \), \( \varepsilon_n \to 0 \). This sequence is bounded in \( C^2(B_\delta, R^k) \) and hence there exists a subsequence \( \{x_1(\varepsilon_n, \cdot)\}_{i=1}^{\infty} \) which has a limit point \( h \in C^1(B_\delta, R^k) \) in the space \( C^1(B_\delta, R^k) \).

On the other hand, since \( W^s_\varepsilon \) is the invariant manifold of \( s_\varepsilon \), \( x_1(\varepsilon, \cdot) \) must satisfy

\[
(I + \varepsilon A)x_1(\varepsilon, y) + \varepsilon g(x_1(\varepsilon, y), y) = x_1(\varepsilon, (I + \varepsilon B)y + \varepsilon r(x_1(\varepsilon, y), y)).
\]

Hence

\[
\varepsilon_n(Ax_1(\varepsilon_n, y) + g(x_1(\varepsilon_n, y), y)) = x_1(\varepsilon_n, (I + \varepsilon_n B)y + \varepsilon_n r(x_1(\varepsilon_n, y), y)) - x_1(\varepsilon_n, y)
\]

and

\[
Ax_1(\varepsilon_n, y) + g(x_1(\varepsilon_n, y), y)
\]

\[
= (D_yx_1(\varepsilon_n, y) + O(\varepsilon_n))(By + r(x_1(\varepsilon_n, y), y))
\]

where we used the mean value theorem. Hence

\[
Ah(y) + g(h(y), y) = D_yh(y)(By + r(h(y), y))
\]

\[
h(0) = 0.
\]

If \( \delta \) is sufficiently small, then (4) has a unique solution and the graph of \( h \) is a local stable manifold of 0 for the equation

\[
z' = f(z).
\]

Indeed, the point 0 is hyperbolic for \( z' = f(z) \). By the assumptions on \( f \) the point 0 has local stable and local unstable \( C^2 \)-manifolds. Using a \( C^2 \)-change of coordinates we can consider that the \( x \)-axis is the local unstable manifold and the \( y \)-axis is the local stable manifold. In these new coordinates the equation \( z' = f(z) \) has the form \( z' = f_1(z) \), i.e.,

\[
\begin{align}
 x' &= A_1 x + g_1(x, y) \\
y' &= B_1 y + r_1(x, y)
\end{align}
\]
where $A_1, B_1$ have the above properties, $g_1(x, y) = O(|x|) \cdot O(|x| + |y|)$, $r_1(x, y) = O(|y|) \cdot O(|x| + |y|)$. We can assume that

$$(B_1 y, y) \leq -a \cdot |y|^2, \quad a > 0$$

where $(\cdot, \cdot)$ is a scalar product. The graph of $h$ in the new coordinates is the graph of some mapping $h_1 : B_{\delta} \to R^k$. We have for a small $|x| + |y|:

$$(B_1 y + r_1(x, y), y) \leq -a_1 \cdot |y|^2, \quad 0 < a_1 < a.$$  

Using this property and the fact that the graph of $h_1$ is invariant by $z' = f_1(z)$, we see that the graph of $h$ is the local stable manifold of 0 for $z' = f(z)$.

From the above results we obtain the proof of Theorem 1.

We note that using the above method we can give a simple proof of a result from the paper [3]. The author of that paper considers an ordinary differential equation

$$(6) \quad z' = \varepsilon f(z, t, \varepsilon)$$

where by [3, Proposition 2.2] we can suppose that

$$f(z, t, \varepsilon) = (Ax + g(x, y, t, \varepsilon), By + r(x, y, t, \varepsilon))$$

where $A, B$ have the above properties, $r, g$ are $2\pi$-periodic in $t$, $r, g \in C^2$, $g(0, 0, \cdot, \cdot) = 0$, $D_x g(0, 0, \cdot, \cdot) = 0$, $r(0, 0, \cdot, \cdot) = 0$, $D_x r(0, 0, \cdot, \cdot) = 0$.

The appropriate spaces are the following:

$$Z = \{(x, y), x \in C^0(R_+, R^k), y \in C^0(R_+, R^m), \sup(|x| + |y|) < \infty\}$$

$$X = \{x \in C^0(R_+, R^k), \sup |x| < \infty\}$$

$$\bar{X} = \{y \in C^0(R_+, R^m), \sup |y| < \infty\}.$$  

By the variation of the constants formula we can obtain an equation similar to $(+).$ Solving this equation we have a mapping

$$h(\varepsilon, \cdot, \cdot) \in C^2(B_{\delta} \times R^k)$$

where $(h(\varepsilon, y, t), y)$ is the initial value from which a bounded orbit of $(6)$ near $0 \in R^k \times R^m$ starts. Moreover, the set

$$\{h(\varepsilon, \cdot, \cdot)\}_{\varepsilon \in (0, \varepsilon_0)}$$

is bounded in $C^2(B_{\delta} \times R^k)$ and $h(\varepsilon, \cdot, \cdot)$ is $2\pi$-periodic in $t$. A result of Hale [4, pp. 166-167] gives us that the graph $W^s_\varepsilon$ of $h(\varepsilon, \cdot, \cdot)$ is a local stable manifold of 0 for $(6)$. Hence $W^s_\varepsilon$ is invariant by $(6)$ and this implies:

$$D_x h(\varepsilon, \cdot, \cdot) + D_y h(\varepsilon, \cdot, \cdot)(\varepsilon B \cdot + \varepsilon r(h, \cdot, \cdot, \varepsilon)) = \varepsilon Ah + \varepsilon g(h, \cdot, \cdot, \varepsilon).$$

Since

$$\frac{1}{2\pi} \int_0^{2\pi} D_t h(\cdot, \cdot, s) \, ds = 0$$
we obtain
\[ \frac{1}{2\pi} \int_0^{2\pi} D_y h(e, \cdot, s)(B \cdot + r(h, \cdot, s, e)) \, ds = \frac{1}{2\pi} \int_0^{2\pi} (Ah(e, \cdot, s) + g(h, \cdot, s, e)) \, ds. \]

We take a sequence \( \{h(e_n, \cdot, \cdot)\}_1^{\infty}, e_n \to 0 \). Then there is a subsequence \( \{h(e_n, \cdot, \cdot)\}_1^{\infty} \) which tends to \( \tilde{h} \in C^1(B_\delta \times R, R^k) \) in this space. Hence
\[ \frac{1}{2\pi} \int_0^{2\pi} D_y \tilde{h}(\cdot, s)(B \cdot + r(\tilde{h}, \cdot, s, 0)) \, ds = \frac{1}{2\pi} \int_0^{2\pi} (A\tilde{h}(\cdot, s) + g(\tilde{h}, \cdot, s, 0)) \, ds, \]
\[ D_y \tilde{h}(\cdot, s) = 0. \]

We see that \( \tilde{h} \) is independent of \( t \), \( \tilde{h}(y, t) = \tilde{h}(y) \) and
\[ D_y \tilde{h}(B, + \frac{1}{2\pi} \int_0^{2\pi} r(\tilde{h}, \cdot, s, 0) \, ds) = A\tilde{h} + \frac{1}{2\pi} \int_0^{2\pi} g(\tilde{h}, \cdot, s, 0) \, ds \]
\[ \tilde{h}(0) = 0. \]

This gives us that the graph of \( \tilde{h} \) is a local stable manifold of 0 of the equation
(7) \[ z' = \frac{1}{2\pi} \int_0^{2\pi} f(z, s, 0) \, ds. \]

Summing up we have [3]:

**Theorem 2.** There exists \( \delta > 0 \) and a \( C^0 \)-mapping
\[ h : (0, \delta) \to C^1(B_\delta \times R, R^k) \]
such that
(i) \( h \) is \( 2\pi \)-periodic in \( t \),
(ii) the graph of \( h(e, \cdot, \cdot) \in C^2(B_\delta \times R, R^k) \) in \( R^n \times R \) is a local stable manifold of 0 of (6) for \( e > 0 \),
(iii) the graph of \( h(0, \cdot, \cdot) \) in \( R^n \times R \) is \( W \times R \), where \( W \) is a local stable manifold of 0 for (7).

Finally, we shall investigate a discrete version of (6). For each initial point \( (x, m) \in R^n \times Z \), where the set \( Z \) is the set of integers, we define an orbit \( \{x_i\}_m^{\infty} \) in the following way
(8) \[ x_{i+1} = x_i + \varepsilon g(x_i, i, \varepsilon), \quad i \geq m, \quad x_m = x \]
where \( g \in C^2, \ g : R^n \times Z \times R \to R^n \) and \( g \) is \( p \)-periodic in the second variable, i.e., \( g(\cdot, i + p, \cdot) = g(\cdot, i, \cdot) \).

Consider also the averaged mapping
\[ \overline{g}(x) = \frac{1}{p} \sum_{i=1}^{p} g(x, i, 0). \]

Suppose \( \overline{g}(0) = 0 \) and \( D\overline{g}(0) \) is hyperbolic, i.e., has no eigenvalues on the imaginary axis.
Lemma 3. There exists \( \varepsilon_1 > 0 \) and a \( C^2 \)-mapping

\[
x : \{1, \ldots, p\} \times (-\varepsilon_1, \varepsilon_1) \rightarrow \mathbb{R}^n
\]

such that for each \( \varepsilon \in (-\varepsilon_1, \varepsilon_1), \varepsilon \neq 0 \) (8) has a unique small \( p \)-periodic orbit \( \{x(i, \varepsilon)\}_{i=1}^{p} \) and \( x(\cdot, 0) = 0 \).

Proof. We solve the equation

\[
x_2 = x_1 + \varepsilon g(x_1, 1, \varepsilon)
\]

\[
\vdots
\]

\[
x_p = x_{p-1} + \varepsilon g(x_{p-1}, p-1, \varepsilon)
\]

\[
x_1 = x_p + \varepsilon g(x_p, p, \varepsilon).
\]

Setting

\[
w = (x_1, \ldots, x_p), \quad Hw = (x_2 - x_1, \ldots, x_1 - x_p)
\]

\[
F_\varepsilon(w) = (g(x_1, 1, \varepsilon), \ldots, g(x_p, p, \varepsilon))
\]

\[
Pw = \left( \frac{x_1 + \cdots + x_p}{p}, \ldots, \frac{x_1 + \cdots + x_p}{p} \right)
\]

this equation can be rewritten in the form

\[
Hu = (I - P) \cdot \varepsilon F_\varepsilon(u + v), \quad u \in \text{Im } H
\]

\[
0 = \varepsilon PF_\varepsilon(u + v), \quad v \in \text{Ker } H,
\]

since \( (\mathbb{R}^n)^p = \text{Ker } H \oplus \text{Im } H \) and \( \text{Ker } P = \text{Im } H, \text{Im } P = \text{Ker } H \). Now using the implicit function theorem the first equation has a solution \( u(\varepsilon, v) \) for \( \varepsilon, v \) small such that \( u(0, \cdot) = 0 \). Hence

\[
0 = PF_\varepsilon(u(\varepsilon, v) + v).
\]

We can also apply the implicit function theorem for this equation since \( v \in \text{Ker } H = \{x \in (\mathbb{R}^n)^p, x_1 = \cdots = x_p = z\}, PF_0(u(0, v) + v) = (\overline{g}(z), \ldots, \overline{g}(z)) \) and \( \overline{g}(0) = 0, D\overline{g}(0) \) is invertible.

From Lemma 3 we make the \( p \)-periodic coordinate change

\[
z_i = x_i - x(i, \varepsilon)
\]

and (8) becomes

\[
z_{i+1} = z_i + \varepsilon f(z_i, i, \varepsilon)
\]

where \( f \in C^2, f(0, i, \varepsilon) = 0 \) and \( \overline{g}(y) = \overline{f}(y) = \frac{1}{p} \sum_1^p f(y, i, 0) \). Next we use averaging to make (9) autonomous to order 1.
Lemma 4. Let \( f \) be as in (9). Then by the change of coordinates
\[
y_i = z_i + \varepsilon \sum_{j=1}^{i-1} (f(z_j, j, 0) - \overline{g}(z_j)), \quad i > 1
\]
\[
y_1 = z_1
\]
(9) becomes
\[
y_{i+1} = y_i + \varepsilon h(y_i, i, \varepsilon)
\]
where \( h \in C^2, h(x, i, 0) = \overline{f}(x) \).

Proof. Using the above change of coordinates we immediately obtain the assertion of Lemma 4. We note that this change of coordinates is \( p \)-periodic in the second variable.

Hence our problem (8) can be reduced to (10), where \( \overline{f}(x) = \overline{g}(x) \) has the hyperbolic fixed point \( x = 0 \). Next we write (10) in the following form
\[
x_{i+1} = x_i + \varepsilon (Ax_i + s(x_i, y_i, i, \varepsilon)), \quad i \geq m
\]
\[
y_{i+1} = y_i + \varepsilon (By_i + r(x_i, y_i, i, \varepsilon)), \quad i \geq m
\]
where \( A, B \) have the properties from Theorem 1: \( s, r, s(0, 0, i, \varepsilon) = 0, \ D_x,s(0, 0, i, \varepsilon) = 0, \ D_x,r(0, 0, i, \varepsilon) = 0, \ D_x, y r(0, 0, i, \varepsilon) = 0 \) and \( s, r \) are \( p \)-periodic in \( i \). Now we can use the above procedure to obtain \( C^2 \)-mappings \( h_m(\cdot, \varepsilon) \in C^2(B_{\delta}, R^k) \) such that the graph \( W^m_{\varepsilon} \) of \( h_m \) is a local stable manifold of \( 0 \in R^p \) of (11). We note that iterations in (11) start from \( i = m \). By the periodic condition for \( r, s \) we have
\[
h_{p+i} = h_i.
\]
In this case the set \( \{h_m(\cdot, \varepsilon)\}_{\varepsilon \in (0, \varepsilon_0)} \) is also bounded in \( C^2(B_{\delta}, R^k) \). We take a sequence \( \{(h_1(\cdot, \varepsilon_1), \ldots, h_p(\cdot, \varepsilon_i))\}_{i=1}^{\infty}, \varepsilon_i \to 0 \). Then there exists a subsequence \( \{(h_1(\cdot, \varepsilon_{n_1}), \ldots, h_p(\cdot, \varepsilon_{n_i}))\}_{i=1}^{\infty} \) which has a limit point \( (h_1, \ldots, h_p) \) in the space
\[
C^1(B_{\delta}, R^k) \times \cdots \times C^1(B_{\delta}, R^k).
\]
(This follows from the well-known theorem of Arzela-Ascoli and we also used this theorem in the above proofs of our theorems.) Since the sequence \( \{W^1_{\varepsilon}, \ldots, W^p_{\varepsilon}\} \) is invariant for (11) we have
\[
h_{i+1}(y + \varepsilon(By + r(h_i(x, \varepsilon), y, i, \varepsilon)), \varepsilon) = h_i(y, \varepsilon) + \varepsilon(Ah_i(y, \varepsilon) + s(h_i(y, \varepsilon), y, i, \varepsilon)).
\]
Thus by the mean value theorem
\[
(D_y h_{i+1}(y, \varepsilon) + O(\varepsilon)) \cdot (By + r(h_i(y, \varepsilon), y, i, \varepsilon)) \]
\[
= \varepsilon(Ah_i(y, \varepsilon) + s(h_i(y, \varepsilon), y, i, \varepsilon)) + h_i(y, \varepsilon) - h_{i+1}(y, \varepsilon)
\]
and
\[ \frac{1}{p} \sum_{i=1}^{p} (D_y h_{i+1}(y, \varepsilon) + O(\varepsilon))(By + r(h_i(y, \varepsilon), y, i, \varepsilon)) \]
\[ = \frac{1}{p} \sum_{i=1}^{p} (Ah_i(y, \varepsilon) + s(h_i(y, \varepsilon), y, i, \varepsilon)). \]

Hence we obtain
\[ h_1 = h_2 = \cdots = h_p = h \]
and
\[ \frac{1}{p} \sum_{i=1}^{p} D_y h(y)(By + r(h(y), y, i, 0)) \]
\[ = \frac{1}{p} \sum_{i=1}^{p} (Ah(y) + s(h(y), y, i, 0)) \]
\[ = h(0) = 0. \]

Thus the graph of \( h \) is a local stable manifold for the averaged equation
\[ (12) \quad z' = \mathcal{G}(z). \]

From the above results we obtain the following theorem.

**Theorem 5.** There exists \( \delta > 0 \) and a \( C^1 \)-mapping
\[ h : \{1, 2, \ldots, p\} \times (0, \delta) \to C^1(B_\delta, R^k) \]

such that

1. The graph of \( h(\cdot, \cdot, \varepsilon), \varepsilon > 0 \) is a local stable manifold of a unique small \( p \)-periodic orbit of (8).
2. There is a local stable manifold \( W \) of 0 of the averaged equation (12) such that the graph of \( h(\cdot, \cdot, 0) \) is \( W \times \{1, 2, \ldots, p\} \). (Note that the stable manifolds are subsets of \( R^k \times \{1, 2, \ldots, p\} \).)

**REFERENCES**


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