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ASYMPTOTIC BEHAVIOR OF STABLE MANIFOLDS

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ABSTRACT. The relation between local stable manifolds of an ordinary differential equation and its discretization is studied. We show that a local stable manifold of a hyperbolic fixed point of an ordinary differential equation is the limit of local stable manifolds of the same fixed point of its discretizations as the discretization parameter $h > 0$ approaches 0.

The main purpose of this paper is to investigate the following problem. Let us consider a mapping $s_\varepsilon: R^n \to R^n$,

$$s_\varepsilon(z) = z + \varepsilon \cdot f(z)$$

where $f \in C^2$, $\varepsilon \in R\backslash\{0\}$, $f(0) = 0$, $Df(0)$ is hyperbolic, i.e., the eigenvalues of $Df(0)$ have nonzero real parts. We can write the mapping s_ε in the form $(-)$ with $g(x, y)$, $r(x, y) = O(|x| + |y|)$

$$(-) \qquad s_\varepsilon(x, y) = (x + \varepsilon(Ax + g(x, y), y + \varepsilon(By + r(x, y))$$

where $A \in \mathscr{L}(R^k)$, $B \in \mathscr{L}(R^m)$ have positive, negative real parts of their eigenvalues, respectively. It is easy to see that $I+\varepsilon B$, $I+\varepsilon A$ have no eigenvalues on the unit circle for all small $\varepsilon \neq 0$. Hence the mapping s_ε has the hyperbolic fixed point 0. For this point there exists a local stable manifold W_ε^s for each small ε [2]. The following theorem gives us information about their limit as $\varepsilon > 0$ approaches 0.

Theorem 1. *There is $\delta > 0$ and a continuous mapping*

$$h : \langle 0, \delta) \to C^1(B_\delta, R^k)$$

such that

(i) *the graph of $h(\varepsilon, \cdot) \in C^2(B_\delta, R^k)$ is a local stable manifold W_ε^s of 0 for s_ε and $\varepsilon > 0$*

(ii) *the graph of $h(0, \cdot)$ is a local stable manifold of 0 for the differential equation $z' = f(z)$*

$$(B_\delta = \{x \in R^m, |x| \leq \delta\}).$$

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We remark that this problem is directly related to the method of averaging (c.f. [6] for further details).

The plan of our paper is as follows. In the first part, we present the proof of Theorem 1. In the second part, we give a simple proof of a result from [3], where the author of that paper has considered a similar problem for periodic ordinary differential equations. Finally, we investigate a discrete version of this problem.

Proof of Theorem 1. By the well-known arguments used in [1] and [5], we can suppose the existence of $\varepsilon_0 > 0$, $K > 0$ with the properties

$$1 + \varepsilon \cdot K \leq |(I + \varepsilon \cdot A)|, \qquad |I + \varepsilon \cdot B| \leq 1 - \varepsilon \cdot K$$

for each $0 \leq \varepsilon < \varepsilon_0$.

Let

$$Z = \{\{(x_n, y_n)\}_1^\infty,\ x_n \in R^k,\ y_n \in R^m,\ \sup(|x_n| + |y_n|) < \infty\}.$$

Z is a Banach space with the norm

$$|\{(x_n, y_n)\}_1^\infty| = \sup(|x_n| + |y_n|).$$

Let us consider the linear mappings

$$\overline{A} : X \to X, \qquad (\overline{A}(\{x_n\}_1^\infty)_n = x_{n+1} - (I + \varepsilon A)x_n)$$
$$\overline{B} : \overline{X} \to \overline{X}, \qquad (\overline{B}(\{y_n\}_1^\infty)_n = y_{n+1} - (I + \varepsilon B)y_n)$$

where

$$X = \{\{x_n\}_1^\infty,\ x_n \in R^k,\ \sup|x_n| < \infty\},$$
$$\overline{X} = \{\{y_n\}_1^\infty,\ y_n \in R^m,\ \sup|y_n| < \infty\}.$$

We solve the equation

$$x_{n+1} - (I + \varepsilon A)x_n = h_n$$

in the space X. This equation has a unique bounded solution for all small $\varepsilon > 0$ [7, p. 272] (i.e., $1 \gg \varepsilon > 0$), namely,

$$x_n = -((I + \varepsilon A)^{-1} h_n + (I + \varepsilon A)^{-2} h_{n+1} + \cdots), \qquad n \geq 1.$$

Hence

$$|x_n| \leq \frac{|h_n|}{(1 + \varepsilon \cdot K)} + \frac{|h_{n+1}|}{(1 + \varepsilon \cdot K)^2} + \cdots$$
$$\leq |h| \cdot \left(\frac{1}{1 + K\varepsilon} + \cdots\right) = |h| \cdot \frac{1}{\varepsilon \cdot K}.$$

Thus

$$|x| \leq |h| \cdot \frac{1}{\varepsilon \cdot K}, \qquad x = \{x_n\}_1^\infty, \qquad h = \{h_n\}_1^\infty.$$

In the same way we solve in $\overline{X}$

$$y_{n+1} - (I + \varepsilon B)y_n = h_n.$$

This equation has a unique bounded solution for $1 \gg \varepsilon > 0$ with the initial condition $y_1 = c$:

$$y_n = (I + \varepsilon B)^{n-1} c + (I + \varepsilon B)^{n-2} h_1 + \cdots + h_{n-1}, \qquad n \geq 2.$$

Hence

$$\begin{aligned} |y_n| &\leq |c| + |h| \cdot ((1 - \varepsilon \cdot K)^{n-2} + \cdots + 1) \\ &< |c| + |h| \frac{1}{\varepsilon \cdot K}. \end{aligned}$$

Finally, we solve the equation

$$\begin{aligned} x_{n+1} &= (I + \varepsilon A) x_n + \varepsilon g(x_n, y_n) \\ y_{n+1} &= (I + \varepsilon B) y_n + \varepsilon r(x_n, y_n) \end{aligned} \tag{1}$$

in Z near $(0, 0)$. By the above results (1) can be written in the form

$$\begin{aligned} \{x_n\}_1^\infty &= \varepsilon(\overline{A})^{-1}(\{g(x_n, y_n)\}_1^\infty) \\ \{y_n\}_2^\infty &= \varepsilon(\overline{B})^{-1}(\{r(x_n, y_n)\}_1^\infty), \qquad y_1 = c. \end{aligned} \tag{+}$$

But $\varepsilon(\overline{A})^{-1}$, $\varepsilon(\overline{B})^{-1}$ are uniformly bounded for $1 \gg \varepsilon > 0$. Hence we can uniformly apply the implicit function theorem to obtain a solution of $(+)$:

$$\{x_n(\varepsilon, c)\}_1^\infty, \qquad \{y_n(\varepsilon, c)\}_2^\infty.$$

From this it follows that the graph W_ε^s of $x_1(\varepsilon, \cdot)$ in $B_\delta \times R^k$ contains all points from which bounded orbits of s_ε near $0 \in R^k \times R^m$ start. Obviously $x_1(\varepsilon, \cdot) \in C^2(B_\delta, R^k)$ and the set

$$\{x_1(\varepsilon, \cdot)\}_{\varepsilon \in (0, \varepsilon_0)}$$

is bounded in $C^2(B_\delta, R^k)$.

Now we show that W_ε^s is the local stable manifold of 0 for s_ε. By (1) we have

$$|x_n| \leq |(I + \varepsilon A)^{-1}| \cdot |h_n| + |(I + \varepsilon A)^{-2}| \cdot |h_{n+1}| + \cdots, \qquad h_n = \varepsilon \cdot g(x_n, y_n)$$
$$|y_n| \leq |(I + \varepsilon B)^{n-1}| \cdot |c| + |(I + \varepsilon B)^{n-2}| \cdot |g_1| + \cdots + |g_{n-1}|, \qquad g_n = \varepsilon \cdot r(x_n, y_n).$$

If $\overline{\lim}(|x_n| + |y_n|) = b > 0$ for a small $\{(x_n, y_n)\}_1^\infty$ then

$$|h_n| \leq \frac{K}{3} \cdot b \cdot \varepsilon, \quad |g_n| \leq \frac{K}{3} \cdot b \cdot \varepsilon, \quad n \gg 1 \text{ (i.e., } n \text{ is large).}$$

Hence

$$\begin{aligned} |x_n| &\leq \frac{1}{1 + \varepsilon \cdot K} \cdot \frac{K}{3} \cdot b \cdot \varepsilon + \frac{1}{(1 + \varepsilon \cdot K)^2} \cdot \frac{K}{3} \cdot b \cdot \varepsilon + \cdots \\ &= \frac{1}{3} \cdot b, \qquad n \gg 1 \end{aligned}$$

and for a fixed large p

$$|y_n| \le (1-\varepsilon K)^{n-1}\cdot|c| + \cdots + (1-\varepsilon K)^{n-p}\cdot|g_{p-1}| + (1-\varepsilon K)^{n-p-1}\cdot\frac{K\varepsilon}{3}\cdot b + \cdots + \frac{K\varepsilon}{3}\cdot b.$$

We see that

$$\overline{\lim}(|x_n| + |y_n|) \le \frac{1}{3}\cdot b + \frac{1}{3}\cdot b < b.$$

This proves that $\lim(|x_n| + |y_n|) = 0$.

Now we take a sequence $\{x_1(\varepsilon_n, \cdot)\}_{n=1}^{\infty} \in C^2(B_\delta, R^k)$ such that $\varepsilon_n > 0$, $\varepsilon_n \to 0$. This sequence is bounded in $C^2(B_\delta, R^k)$ and hence there exists a subsequence $\{x_1(\varepsilon_{n_i}, \cdot)\}_{i=1}^{\infty}$ which has a limit point $h \in C^1(B_\delta, R^k)$ in the space $C^1(B_\delta, R^k)$.

On the other hand, since W_ε^s is the invariant manifold of s_ε, $x_1(\varepsilon, \cdot)$ must satisfy

$$(I+\varepsilon A)x_1(\varepsilon, y) + \varepsilon g(x_1(\varepsilon, y), y) = x_1(\varepsilon, (I+\varepsilon B)y + \varepsilon r(x_1(\varepsilon, y), y)).$$

Hence

$$\begin{aligned}\varepsilon_{n_i}&(Ax_1(\varepsilon_{n_i}, y) + g(x_1(\varepsilon_{n_i}, y), y))\\ &= x_1(\varepsilon_{n_i}, (I+\varepsilon_{n_i}B)y + \varepsilon_{n_i} r(x_1(\varepsilon_{n_i}, y), y)) - x_1(\varepsilon_{n_i}, y)\end{aligned}$$

and

$$\begin{aligned}Ax_1&(\varepsilon_{n_i}, y) + g(x_1(\varepsilon_{n_i}, y), y)\\ &= (D_y x_1(\varepsilon_{n_i}, y) + O(\varepsilon_{n_i}))(By + r(x_1(\varepsilon_{n_i}, y), y))\end{aligned}$$

where we used the mean value theorem. Hence

$$\begin{aligned}Ah(y) + g(h(y), y) &= D_y h(y)(By + r(h(y), y))\\ h(0) &= 0.\end{aligned} \tag{4}$$

If δ is sufficiently small, then (4) has a unique solution and the graph of h is a local stable manifold of 0 for the equation

$$z' = f(z).$$

Indeed, the point 0 is hyperbolic for $z' = f(z)$. By the assumptions on f the point 0 has local stable and local unstable C^2-manifolds. Using a C^2-change of coordinates we can consider that the x-axis is the local unstable manifold and the y-axis is the local stable manifold. In these new coordinates the equation $z' = f(z)$ has the form $z' = f_1(z)$, i.e.,

$$\begin{aligned}x' &= A_1 x + g_1(x, y)\\ y' &= B_1 y + r_1(x, y)\end{aligned} \tag{5}$$

where A_1, B_1 have the above properties, $g_1(x, y) = O(|x|) \cdot O(|x| + |y|)$, $r_1(x, y) = O(|y|) \cdot O(|x| + |y|)$. We can assume that

$$(B_1 y, y) \leq -a \cdot |y|^2, \qquad a > 0$$

where $(\cdot, \cdot)$ is a scalar product. The graph of h in the new coordinates is the graph of some mapping $h_1 : B_\delta \to R^k$. We have for a small $|x| + |y|$:

$$(B_1 y + r_1(x, y), y) \leq -a_1 \cdot |y|^2, \qquad 0 < a_1 < a.$$

Using this property and the fact that the graph of h_1 is invariant by $z' = f_1(z)$, we see that the graph of h is the local stable manifold of 0 for $z' = f(z)$.

From the above results we obtain the proof of Theorem 1.

We note that using the above method we can give a simple proof of a result from the paper [3]. The author of that paper considers an ordinary differential equation

$$z' = \varepsilon f(z, t, \varepsilon) \tag{6}$$

where by [3, Proposition 2.2] we can suppose that

$$f(z, t, \varepsilon) = (Ax + g(x, y, t, \varepsilon), By + r(x, y, t, \varepsilon))$$

where A, B have the above properties, r, g are 2π-periodic in t, $r, g \in C^2$, $g(0, 0, \cdot, \cdot) = 0$, $D_{x,y} g(0, 0, \cdot, 0) = 0$, $r(0, 0, \cdot, \cdot) = 0$, $D_{x,y} r(0, 0, \cdot, 0) = 0$.

The appropriate spaces are the following:

$$Z = \{(x, y), x \in C^0(R_+, R^k), \ y \in C^0(R_+, R^m), \sup(|x| + |y|) < \infty\}$$
$$X = \{x \in C^0(R_+, R^k), \ \sup |x| < \infty\}$$
$$\overline{X} = \{y \in C^0(R_+, R^m), \ \sup |y| < \infty\}.$$

By the variation of the constants formula we can obtain an equation similar to $(+)$. Solving this equation we have a mapping

$$h(\varepsilon, \cdot, \cdot) \in C^2(B_\delta \times R, R^k)$$

where $(h(\varepsilon, y, t), y)$ is the initial value from which a bounded orbit of (6) near $0 \in R^k \times R^m$ starts. Moreover, the set

$$\{h(\varepsilon, \cdot, \cdot)\}_{\varepsilon \in (0, \varepsilon_0)}$$

is bounded in $C^2(B_\delta \times R, R^k)$ and $h(\varepsilon, \cdot, \cdot)$ is 2π-periodic in t. A result of Hale [4, pp. 166–167] gives us that the graph W_ε^s of $h(\varepsilon, \cdot, \cdot)$ is a local stable manifold of 0 for (6). Hence W_ε^s is invariant by (6) and this implies:

$$D_t h(\varepsilon, \cdot, \cdot) + D_y h(\varepsilon, \cdot, \cdot)(\varepsilon B \cdot + \varepsilon r(h, \cdot, \cdot, \varepsilon)) = \varepsilon A h + \varepsilon g(h, \cdot, \cdot, \varepsilon).$$

Since

$$\frac{1}{2\pi} \int_0^{2\pi} D_t h(\cdot, \cdot, s)\, ds = 0$$

we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} D_y h(\varepsilon, \cdot, s)(B\cdot + r(h, \cdot, s, \varepsilon))\, ds$$
$$= \frac{1}{2\pi}\int_0^{2\pi} (Ah(\varepsilon, \cdot, s) + g(h, \cdot, s, \varepsilon)\, ds.$$

We take a sequence $\{h(\varepsilon_n, \cdot, \cdot)\}_1^\infty$, $\varepsilon_n \to 0$. Then there is a subsequence $\{h(\varepsilon_{n_i}, \cdot, \cdot)\}_1^\infty$ which tends to $\overline{h} \in C^1(B_\delta \times R, R^k)$ in this space. Hence

$$\frac{1}{2\pi}\int_0^{2\pi} D_y\overline{h}(\cdot, s)(B\cdot + r(\overline{h}, \cdot, s, 0)\, ds = \frac{1}{2\pi}\int_0^{2\pi} (A\overline{h}(\cdot, s) + g(\overline{h}, \cdot, s, 0))\, ds$$
$$D_t\overline{h}(\cdot, s) = 0.$$

We see that $\overline{h}$ is independent of t, $\overline{h}(y, t) = \overline{h}(y)$ and

$$D_y\overline{h}.(B. + \frac{1}{2\pi}\int_0^{2\pi} r(\overline{h}, \cdot, s, 0)\, ds) = A\overline{h} + \frac{1}{2\pi}\int_0^{2\pi} g(\overline{h}, \cdot, s, 0)\, ds$$
$$\overline{h}(0) = 0.$$

This gives us that the graph of $\overline{h}$ is a local stable manifold of 0 of the equation

$$z' = \frac{1}{2\pi}\int_0^{2\pi} f(z, s, 0)\, ds. \tag{7}$$

Summing up we have [3]:

Theorem 2. *There exists $\delta > 0$ and a C^0-mapping*

$$h : \langle 0, \delta) \to C^1(B_\delta \times R, R^k)$$

such that

(i) *h is 2π-periodic in t,*

(ii) *the graph of $h(\varepsilon, \cdot, \cdot) \in C^2(B_\delta \times R, R^k)$ in $R^n \times R$ is a local stable manifold of 0 of (6) for $\varepsilon > 0$,*

(iii) *the graph of $h(0, \cdot, \cdot)$ in $R^n \times R$ is $W \times R$, where W is a local stable manifold of 0 for (7).*

Finally, we shall investigate a discrete version of (6). For each initial point $(x, m) \in R^n \times Z$, where the set Z is the set of integers, we define an orbit $\{x_i\}_m^\infty$ in the following way

$$x_{i+1} = x_i + \varepsilon g(x_i, i, \varepsilon), \qquad i \geq m, \ x_m = x \tag{8}$$

where $g \in C^2$, $g : R^n \times Z \times R \to R^n$ and g is p-periodic in the second variable, i.e., $g(\cdot, i+p, \cdot) = g(\cdot, i, \cdot)$.

Consider also the averaged mapping

$$\overline{g}(x) = \frac{1}{p}\sum_1^p g(x, i, 0).$$

Suppose $\overline{g}(0) = 0$ and $D\overline{g}(0)$ is hyperbolic, i.e., has no eigenvalues on the imaginary axis.

Lemma 3. *There exists* $\varepsilon_1 > 0$ *and a* C^2*-mapping*

$$x : \{1, \dots, p\} \times (-\varepsilon_1, \varepsilon_1) \to R^n$$

such that for each $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$, $\varepsilon \neq 0$ (8) *has a unique small p-periodic orbit* $\{x(i, \varepsilon)\}_{i=1}^p$ *and* $x(\cdot, 0) = 0$.

Proof. We solve the equation

$$\begin{aligned} x_2 &= x_1 + \varepsilon g(x_1, 1, \varepsilon) \\ &\;\;\vdots \\ x_p &= x_{p-1} + \varepsilon g(x_{p-1}, p-1, \varepsilon) \\ x_1 &= x_p + \varepsilon g(x_p, p, \varepsilon). \end{aligned}$$

Setting

$$\begin{aligned} w &= (x_1, \dots, x_p), \qquad Hw = (x_2 - x_1, \dots, x_1 - x_p) \\ F_\varepsilon(w) &= (g(x_1, 1, \varepsilon), \dots, g(x_p, p, \varepsilon)) \\ Pw &= \left(\frac{x_1 + \cdots + x_p}{p}, \cdots, \frac{x_1 + \cdots + x_p}{p}\right) \end{aligned}$$

this equation can be rewritten in the form

$$\begin{aligned} Hu &= (I - P) \cdot \varepsilon F_\varepsilon(u + v), \quad u \in \operatorname{Im} H \\ 0 &= \varepsilon P F_\varepsilon(u + v), \quad v \in \operatorname{Ker} H, \end{aligned}$$

since $(R^n)^p = \operatorname{Ker} H \oplus \operatorname{Im} H$ and $\operatorname{Ker} P = \operatorname{Im} H$, $\operatorname{Im} P = \operatorname{Ker} H$. Now using the implicit function theorem the first equation has a solution $u(\varepsilon, v)$ for ε, v small such that $u(0, \cdot) = 0$. Hence

$$0 = PF_\varepsilon(u(\varepsilon, v) + v).$$

We can also apply the implicit function theorem for this equation since $v \in \operatorname{Ker} H = \{x \in (R^n)^p, x_1 = \cdots = x_p = z\}$, $PF_0(u(0, v) + v) = (\overline{g}(z), \cdots, \overline{g}(z))$ and $\overline{g}(0) = 0$, $D\overline{g}(0)$ is invertible.

From Lemma 3 we make the p-periodic coordinate change

$$z_i = x_i - x(i, \varepsilon)$$

and (8) becomes

$$z_{i+1} = z_i + \varepsilon f(z_i, i, \varepsilon) \tag{9}$$

where $f \in C^2$, $f(0, i, \varepsilon) = 0$ and $\overline{g}(y) = \overline{f}(y) = \frac{1}{p}\sum_1^p f(y, i, 0)$. Next we use averaging to make (9) autonomous to order 1.

Lemma 4. *Let f be as in* (9). *Then by the change of coordinates*

$$y_i = z_i + \varepsilon \sum_{j=1}^{i-1} (f(z_i, j, 0) - \overline{g}(z_i)), \qquad i > 1$$

$$y_1 = z_1$$

(9) *becomes*

$$y_{i+1} = y_i + \varepsilon h(y_i, i, \varepsilon) \tag{10}$$

where $h \in C^2$, $h(x, i, 0) = \overline{f}(x)$.

Proof. Using the above change of coordinates we immediately obtain the assertion of Lemma 4. We note that this change of coordinates is p-periodic in the second variable.

Hence our problem (8) can be reduced to (10), where $\overline{f}(x) = \overline{g}(x)$ has the hyperbolic fixed point $x = 0$. Next we write (10) in the following form

$$\begin{aligned} x_{i+1} &= x_i + \varepsilon(Ax_i + s(x_i, y_i, i, \varepsilon)), \\ y_{i+1} &= y_i + \varepsilon(By_i + r(x_i, y_i, i, \varepsilon)), \end{aligned} \qquad i \geq m \tag{11}$$

where A, B have the properties from Theorem 1: s, $r \in C^2$, $s(0, 0, i, \varepsilon) = 0$, $r(0, 0, i, \varepsilon) = 0$, $D_{x,y}s(0, 0, \cdot, 0) = 0$, $D_{x,y}r(0, 0, \cdot, 0) = 0$ and s, r are p-periodic in i. Now we can use the above procedure to obtain C^2-mappings $h_m(\cdot, \varepsilon) \in C^2(B_\delta, R^k)$ such that the graph W_ε^m of h_m is a local stable manifold of $0 \in R^n$ of (11). We note that iterations in (11) start from $i = m$. By the periodic condition for r, s we have

$$h_{p+i} = h_i.$$

In this case the set $\{h_m(\cdot, \varepsilon)\}_{\varepsilon \in (0, \varepsilon_0)}$ is also bounded in $C^2(B_\delta, R^k)$. We take a sequence $\{(h_1(\cdot, \varepsilon_i), \dots, h_p(\cdot, \varepsilon_i))\}_{i=1}^\infty$, $\varepsilon_i \to 0$. Then there exists a subsequence $\{(h_1(\cdot, \varepsilon_{n_i}), \dots, h_p(\cdot, \varepsilon_{n_i}))\}_{i=1}^\infty$ which has a limit point $(h_1, \dots, h_p)$ in the space

$$C^1(B_\delta, R^k) \times \cdots \times C^1(B_\delta, R^k).$$

(This follows from the well-known theorem of Arzela-Ascoli and we also used this theorem in the above proofs of our theorems.) Since the sequence $\{W_\varepsilon^1, \dots, W_\varepsilon^p\}$ is invariant for (11) we have

$$\begin{aligned} &h_{i+1}(y + \varepsilon(By + r(h_i(x, \varepsilon), y, i, \varepsilon)), \varepsilon) \\ &\quad = h_i(y, \varepsilon) + \varepsilon(Ah_i(y, \varepsilon) + s(h_i(y, \varepsilon), y, i, \varepsilon)). \end{aligned}$$

Thus by the mean value theorem

$$\begin{aligned} &(D_y h_{i+1}(y, \varepsilon) + O(\varepsilon)) \cdot (By + r(h_i(y, \varepsilon), y, i, \varepsilon)) \\ &\quad = \varepsilon(Ah_i(y, \varepsilon) + s(h_i(y, \varepsilon), y, i, \varepsilon)) + h_i(y, \varepsilon) - h_{i+1}(y, \varepsilon) \end{aligned}$$

and

$$\frac{1}{p}\sum_{i=1}^{p}(D_y h_{i+1}(y, \varepsilon) + O(\varepsilon))(By + r(h_i(y, \varepsilon), y, i, \varepsilon))$$
$$= \frac{1}{p}\sum_{i=1}^{p}(Ah_i(y, \varepsilon) + s(h_i(y, \varepsilon), y, i, \varepsilon)).$$

Hence we obtain

$$h_1 = h_2 = \cdots = h_p = h$$

and

$$\frac{1}{p}\sum_{i=1}^{p} D_y h(y)(By + r(h(y), y, i, 0))$$
$$= \frac{1}{p}\sum_{i=1}^{p}(Ah(y) + s(h(y), y, i, 0))$$
$$h(0) = 0.$$

Thus the graph of h is a local stable manifold for the averaged equation

$$z' = \overline{g}(z). \tag{12}$$

From the above results we obtain the following theorem.

Theorem 5. *There exists $\delta > 0$ and a C^0-mapping*

$$h : \{1, 2, \ldots, p\} \times \langle 0, \delta) \to C^1(B_\delta, R^k)$$

such that

(1) *The graph of $h(\cdot, \cdot, \varepsilon)$, $\varepsilon > 0$ is a local stable manifold of a unique small p-periodic orbit of* (8).

(2) *There is a local stable manifold W of 0 of the averaged equation* (12) *such that the graph of $h(\cdot, \cdot, 0)$ is $W \times \{1, 2, \ldots, p\}$. (Note that the stable manifolds are subsets of $R^n \times \{1, 2, \ldots, p\}$.)*

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