ON THE WEIGHTED APPROXIMATION
OF CONTINUOUSLY DIFFERENTIABLE FUNCTIONS

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Abstract. This note is an improvement of the available methods for getting results on the weighted approximation of continuously differentiable functions.

We shall present what we believe to be a simplified version of the reasoning to establish some results concerning weighted approximation of continuously differentiable scalar functions on $\mathbb{R}^n$ (see [3]). For references to weighted approximation of continuous scalar functions on $\mathbb{R}^n$ (see [1] and [3]). Lemma 1 reduces the search for sufficient conditions in order that a weight on $\mathbb{R}$ be $C^m$-fundamental to the finding of sufficient conditions for a weight on $\mathbb{R}$ to be $C$-fundamental. Similarly, Lemma 2 reduces the finding of sufficient conditions for a weight on $\mathbb{R}^n$ to be $C^m$-fundamental to the search of sufficient conditions for a weight on $\mathbb{R}$ to be $C^m$-fundamental. We then apply Lemmas 1 and 2 together to obtain Propositions 2, 4, and 5.

Fix integers $n \in \mathbb{N}$, $n \geq 1$, and $m \in \mathbb{N}$ (the case $m = \infty$ will be excluded since it follows easily from all $m \in \mathbb{N}$). Let $K$ denote either $\mathbb{R}$ or $\mathbb{C}$. Consider the algebras $\mathcal{P}(\mathbb{R}^n)$ of all $K$-valued polynomials on $\mathbb{R}^n$ and $C^m(\mathbb{R}^n)$ of all continuously $m$-differentiable $K$-valued functions on $\mathbb{R}^n$. Write $N^m_n = \{\alpha \in N^n : |\alpha| \leq m\}$ where $|\alpha| = \alpha_1 + \cdots + \alpha_n$ if $\alpha = (\alpha_1, \ldots, \alpha_n) \in N^n$. Let $D^\alpha f$ be the $\alpha$th partial derivative of $f \in C^m(\mathbb{R}^n)$ for $\alpha \in N^m_n$. Denote by $\mathcal{D}^m(\mathbb{R}^n)$ the subalgebra of $C^m(\mathbb{R}^n)$ of all functions with compact support.

A $C^m$-weight on $\mathbb{R}^n$ is a family $v = \{v_\alpha : \alpha \in N^m_n\}$ of upper semicontinuous functions $v_\alpha \geq 0$ on $\mathbb{R}^n$. Such a weight $v$ defines the vector space $C^m v_\infty(\mathbb{R}^n)$ of all $f \in C^m(\mathbb{R}^n)$ such that $v_\alpha D^\alpha f$ tends to zero at infinity for every $\alpha \in N^m_n$. Set $\|f\|_{v_\alpha} = \sup \{v_\alpha(t) \cdot |D^\alpha f(t)| : t \in \mathbb{R}^n\}$ to get a seminorm $f \in C^m v_\infty(\mathbb{R}^n) \mapsto \|f\|_{v_\alpha} \in \mathbb{R}_+$ for every $\alpha \in N^m_n$. The finite family of such seminorms makes $C^m v_\infty(\mathbb{R}^n)$ into a seminormable space, actually a
semimormed space if for instance we use the seminorm \( f \in C^m v^\infty(\mathbb{R}^n) \mapsto \|f\|_v = \sup \{\|f\|_{v_\alpha} : \alpha \in \mathbb{N}_m^\alpha\} = \sup \{v_\alpha(\alpha) \cdot |D^\alpha f(t)| : t \in \mathbb{R}, \alpha \in \mathbb{N}_m^\alpha\} \in \mathbb{R}_+ \).

We now have that \( \mathcal{D}^m(\mathbb{R}^n) \subset C^m v^\infty(\mathbb{R}^n) \). It is known that \( \mathcal{D}^m(\mathbb{R}^n) \) is dense in \( C^m v^\infty(\mathbb{R}^n) \) if the weight \( v \) is decreasing in the sense that, for every \( \alpha, \beta \in \mathbb{N}_m^\alpha \) with \( \beta \leq \alpha \), there exists \( C_{\alpha\beta} \geq 0 \) such that \( v_\alpha \leq C_{\alpha\beta} v_\beta \) ([3, Lemma 1]). \( \mathcal{D}^m(\mathbb{R}^n) \) may be dense in \( C^m v^\infty(\mathbb{R}^n) \) even if \( v \) is not decreasing, and it may fail to be dense as well. The weight \( v \) is said to be rapidly decreasing if \( \mathcal{P}(\mathbb{R}^n) \subset C^m v^\infty(\mathbb{R}^n) \). If, moreover, \( \mathcal{P}(\mathbb{R}^n) \) is dense in \( C^m v^\infty(\mathbb{R}^n) \), then \( v \) is called \( C^m \)-fundamental.

**Lemma 1.** Let \( m \in \mathbb{N}, \ v_i \geq 0, \ i = 0, \ldots, m, \) and \( u \geq 0 \) be upper semicontinuous on \( \mathbb{R} \). Consider the \( C^m \)-weight \( v = (v_0, \ldots, v_m) \) on \( \mathbb{R} \) which is assumed to be decreasing. Let \( u \) be \( C \)-fundamental on \( \mathbb{R} \) such that \( s, t \in \mathbb{R}, |s| \leq |t| \) imply \( u(t) \leq u(s) \) and \( |t|^{m-i} v_i(t) \leq u(t), \ v_i(t) \leq u(t) \) for \( t \in \mathbb{R}, \ i = 0, \ldots, m \). Then \( v \) is a \( C^m \)-fundamental weight on \( \mathbb{R} \).

**Proof.** The lemma is true if \( m = 0 \) because then \( v_0 \leq u \) and hence \( v_0 \) is \( C \)-fundamental along with \( u \). Assume now that \( m \geq 1 \) and that the lemma is true for \( m - 1 \). Notice that \( \mathcal{P}(\mathbb{R}) \subset C^m v^\infty(\mathbb{R}) \) because \( \mathcal{P}(\mathbb{R}) \subset C^m u^\infty(\mathbb{R}) \) and \( v_i \leq u, \ i = 0, \ldots, m \). Hence \( v \) is rapidly decreasing. Fix any \( f \in \mathcal{D}(\mathbb{R}) \) and \( \varepsilon > 0 \). Then \( f^{(m)} \in \mathcal{D}(\mathbb{R}) \subset C u^\infty(\mathbb{R}) \) and there is a \( P^{(m)}(t) \in \mathcal{D}(\mathbb{R}) \) so that

\[
(1) \quad u(t) \cdot |P^{(m)}(t) - f^{(m)}(t)| \leq \varepsilon, \quad t \in \mathbb{R}.
\]

Choose \( P \in \mathcal{P}(\mathbb{R}) \) whose \( m \)th derivative is \( P^{(m)} \) such that \( P^{(i)}(0) = f^{(i)}(0), \ i = 0, \ldots, m - 1 \). We claim that

\[
(2) \quad v_i(t) \cdot |P^{(i)}(t) - f^{(i)}(t)| \leq \varepsilon, \quad t \in \mathbb{R}, \ i = 0, \ldots, m.
\]

In fact, for \( i = m \) this follows from (1) and \( v_m \leq u \). For every \( t \in \mathbb{R}, \ i = 0, \ldots, m - 1 \), there is an \( s \in \mathbb{R}, \ |s| \leq |t| \) such that \( |P^{(i)}(t) - f^{(i)}(t)| \leq |t|^{m-i} \cdot |P^{(m)}(s) - f^{(m)}(s)| \) by the mean value theorem applied \( m - i \) times.

Notice that \( |t|^{m-i} v_i(t) \leq u(t) \leq u(s) \). Hence (1) with \( s \) in place of \( t \) shows that (2) is true for \( i = 0, \ldots, m - 1 \) as well as for \( i = m \). It follows that \( \mathcal{D}^m(\mathbb{R}) \) is contained in the closure of \( \mathcal{D}(\mathbb{R}) \) in \( C^m v^\infty(\mathbb{R}) \). Therefore, \( \mathcal{P}(\mathbb{R}) \) is dense in \( C^m v^\infty(\mathbb{R}) \) along with \( \mathcal{D}^m(\mathbb{R}) \). Thus \( v \) is \( C^m \)-fundamental. \( \square \)

Let \( v \geq 0 \) be upper semicontinuous on \( \mathbb{R} \). We say that \( v \) is an analytic weight on \( \mathbb{R} \) when there exist \( C > 0, \ c > 0 \) such that \( v(t) \leq C e^{-c|t|}, \ t \in \mathbb{R} \).

It is then known that \( v \) is a \( C \)-fundamental weight on \( \mathbb{R} \) ([2, §28, Lemma 2]). More generally, if \( v \geq 0 \) is upper semicontinuous on \( \mathbb{R}^n \), we say that \( v \) is an analytic weight on \( \mathbb{R}^n \) when there exist \( C > 0, \ c > 0 \) such that \( v(t) \leq C e^{-c(|t_1| + \cdots + |t_n|)}, \ t \in \mathbb{R}^n \). It is then known that \( v \) is \( C \)-fundamental on \( \mathbb{R}^n \).

**Proposition 2.** Let \( m \in \mathbb{N}, \ v_i \geq 0, \ i = 0, \ldots, m, \) be upper semicontinuous on \( \mathbb{R} \). Consider the \( C^m \)-weight \( v = (v_0, \ldots, v_m) \) on \( \mathbb{R} \) which is assumed to be decreasing. Let each \( v_i, \ i = 0, \ldots, m \), be an analytic weight. Then \( v \) is a \( C^m \)-fundamental weight on \( \mathbb{R} \).
Proof. Assume that \( v_\alpha(t) \leq Ce^{-c|t|}, \ t \in \mathbb{R}, \ i = 0, \ldots, m, \) for some \( C > 0, c > 0. \) Choose \( D > 0, 0 < d < c \) so that, if \( u(t) = De^{-d|t|}, \ t \in \mathbb{R}, \) all assumptions in Lemma 1 are satisfied. \( \square \)

Lemma 3. Let \( n \in \mathbb{N}, \ n \geq 1, \ m \in \mathbb{N}, \ v_\alpha \geq 0, \alpha \in \mathbb{N}_m^n, \) be upper semicontinuous on \( \mathbb{R}. \) Consider the \( C^m \)-weight \( v = (v_\alpha; \alpha \in \mathbb{N}_m^n) \) on \( \mathbb{R}^n \) which is assumed to be decreasing. Let \( u_{ij} \geq 0, \ i = 1, \ldots, n, \ j = 0, \ldots, m, \) be upper semicontinuous on \( \mathbb{R}. \) Consider the \( C^m \)-weights \( u_i = (u_{i0}, \ldots, u_{im}), \ i = 1, \ldots, n, \) on \( \mathbb{R} \) which are supposed to be decreasing and \( C^m \)-fundamental. Assume \( v_\alpha(t) \leq u_{1\alpha_1}(t_1) \cdots u_{n\alpha_n}(t_n), \ t \in \mathbb{R}^n, \alpha \in \mathbb{N}_m^n. \)

Then \( v \) is \( C^m \)-fundamental on \( \mathbb{R}^n. \)

Proof. Consider the \( n \)-linear mapping \( \pi \) that, with every \( (f_1, \ldots, f_n) \in C^m(u_1)_{\infty}(\mathbb{R}) \times \cdots \times C^m(u_n)_{\infty}(\mathbb{R}) \)

associates \( f_1 \otimes \cdots \otimes f_n \in C^m(v_\alpha)_{\infty}(\mathbb{R}^n) \) where \( (f_1 \otimes \cdots \otimes f_n)(t) = f_1(t_1) \cdots f_n(t_n) \) for \( t \in \mathbb{R}^n. \) The assumptions make it sure that \( \pi \) is well defined and continuous because

\[
\|f_1 \otimes \cdots \otimes f_n\|_{v_\alpha} \leq \|f_1\|_{u_{1\alpha_1}} \cdots \|f_n\|_{u_{n\alpha_n}}.
\]

By hypothesis, \( \mathcal{P}(\mathbb{R}) \) is dense in \( C^m(u_i)_{\infty}(\mathbb{R}) \) so \( \mathcal{D}^m(\mathbb{R}) \) is contained in the closure of \( \mathcal{P}(\mathbb{R}) \) in \( C^m(u_i)_{\infty}(\mathbb{R}), \ i = 1, \ldots, n. \) Thus \( \pi[\mathcal{D}^m(\mathbb{R}) \times \cdots \times \mathcal{D}^m(\mathbb{R})] \) is contained in the closure of \( \pi[\mathcal{P}(\mathbb{R}) \times \cdots \times \mathcal{P}(\mathbb{R})] \) in \( C^m(u_\alpha)_{\infty}(\mathbb{R}^n). \) Hence the vector subspace \( \mathcal{D}^m(\mathbb{R}) \otimes \cdots \otimes \mathcal{D}^m(\mathbb{R}) \) generated by \( \pi[\mathcal{D}^m(\mathbb{R}) \times \cdots \times \mathcal{D}^m(\mathbb{R})] \) is contained in the closure in \( C^m(u_\alpha)_{\infty}(\mathbb{R}^n) \) of the vector subspace \( \mathcal{P}(\mathbb{R}^n) = \mathcal{P}(\mathbb{R}) \otimes \cdots \otimes \mathcal{P}(\mathbb{R}) \) generated by \( \pi[\mathcal{P}(\mathbb{R}) \times \cdots \times \mathcal{P}(\mathbb{R})] \). It is known that \( \mathcal{D}^m(\mathbb{R}) \otimes \cdots \otimes \mathcal{D}^m(\mathbb{R}) \) is dense in \( \mathcal{D}^m(\mathbb{R}) \) in the natural inductive limit topology of \( \mathcal{D}^m(\mathbb{R}), \) hence in the coarser topology on \( \mathcal{D}^m(\mathbb{R}) \) defined by the norm \( f \in \mathcal{D}^m(\mathbb{R}) \mapsto \sup\{|D^{|a|}f(t)|: t \in \mathbb{R}^n, |a| \leq m \} \in \mathbb{R}_+, \) hence (because all \( v_\alpha \) are upper bounded along with all \( u_{ij} \)) in the even coarser topology that the natural topology of \( C^m(u_\alpha)_{\infty}(\mathbb{R}^n) \) induces on \( \mathcal{D}^m(\mathbb{R}). \) Since \( \mathcal{D}^m(\mathbb{R}) \) is dense in \( C^m(u_\alpha)_{\infty}(\mathbb{R}^n), \) then \( \mathcal{D}^m(\mathbb{R}) \otimes \cdots \otimes \mathcal{D}^m(\mathbb{R}) \) is dense in \( C^m(u_\alpha)_{\infty}(\mathbb{R}^n). \) The fact that \( \mathcal{D}^m(\mathbb{R}) \otimes \cdots \otimes \mathcal{D}^m(\mathbb{R}) \) is contained in the closure of \( \mathcal{P}(\mathbb{R}^n) \) in \( C^m(u_\alpha)_{\infty}(\mathbb{R}^n) \) implies then that \( \mathcal{P}(\mathbb{R}^n) \) is dense in \( C^m(u_\alpha)_{\infty}(\mathbb{R}^n). \) Thus \( v \) is \( C^m \)-fundamental. \( \square \)

Proposition 4. Let \( n \in \mathbb{N}, \ n \geq 1, \ m \in \mathbb{N}, \ v_\alpha \geq 0, \alpha \in \mathbb{N}_m^n, \) be upper semicontinuous on \( \mathbb{R}^n. \) Consider the \( C^m \)-weight \( v = (v_\alpha; \alpha \in \mathbb{N}_m^n) \) on \( \mathbb{R}^n \) which is assumed to be decreasing. Assume that each \( v_\alpha, \alpha \in \mathbb{N}_m^n, \) is an analytic weight. Then \( v \) is \( C^m \)-fundamental on \( \mathbb{R}^n. \)

Proof. Let \( v_\alpha(t) \leq Ce^{-c(|t_1|+\cdots+|t_n|)} \), \( t \in \mathbb{R}^n, \alpha \in \mathbb{N}_m^n, \) for suitable \( C > 0, c > 0. \) Choose \( u_{ij}(t_i) = C^{1/n}e^{-c|t_i|}, \ i = 1, \ldots, n, \ j = 1, \ldots, m, \) so that all assumptions in Lemma 3 are satisfied. \( \square \)
Let \( v \geq 0 \) be upper semicontinuous on \( R \) and rapidly decreasing. Put \( M_m = \sup\{ |r^n| \cdot v(t) : t \in R \} \in R_+ \), \( m = 0, 1, \ldots \). We say that \( v \) is a quasi-analytic weight on \( R \) when \( \sum_{m=1}^{\infty} \left( \frac{1}{\sqrt{M_m}} \right) = +\infty \). It is then known that \( v \) is a \( C \)-fundamental weight on \( R \) and that every analytic weight on \( R \) is quasi-analytic (see [2, §29, Lemma 2]). More generally, if \( v \geq 0 \) is upper semicontinuous on \( R^n \), we say that \( v \) is a quasi-analytic weight on \( R^n \) if there are quasi-analytic weights \( v_1, \ldots, v_n \) on \( R \) such that \( v(t) \leq v_1(t_1) \cdots v_n(t_n) \), \( t \in R^n \). It is then known that \( v \) is \( C \)-fundamental on \( R^n \), and that every analytic weight on \( R^n \) is quasi-analytic.

**Proposition 5.** Let \( n \in N, n \geq 1, m \in N, \alpha \in N_m^n \), be upper semicontinuous on \( R^n \). Consider the \( C^m \)-weight \( v = (v_\alpha : \alpha \in N_m^n) \) which is supposed to be decreasing. Assume that there are quasi-analytic weights \( v_i, i = 1, \ldots, n \), on \( R \) such that \( v_\alpha(t) \leq v_1(t_1) \cdots v_n(t_n) \), \( t \in R^n \), \( \alpha \in N_m^n \). Then \( v \) is a \( C^m \)-fundamental weight on \( R^n \).

**Proof.** All assumptions of Lemma 3 apply. \( \square \)

**Lemma 6.** The linear mapping \( D: f \in D^1(R) \mapsto f' \in \mathcal{K}(R) = D^0(R) \) is injective. Its image is \( D^1(R) = \{ g \in \mathcal{K}(R) ; \int g = 0 \} \). If \( u \geq 0 \) is upper semicontinuous on \( R \), this image is dense in \( C_\infty u(R) \) if and only if \( \int 1/u = +\infty \).

**Proof.** Only the final part of the lemma requires a proof. Consider the linear form \( I: f \in \mathcal{K}(R) \mapsto \int f \in K \). Assume \( \int 1/u = +\infty \). We claim that \( I \) is not continuous on \( \mathcal{K}(R) \) for the seminorm induced by \( C_\infty u(R) \). In fact, given any \( c \geq 0 \), there is an \( f \in \mathcal{K}(R) \) such that \( 0 \leq f \leq 1/u \), \( \int f \geq c \). Therefore \( \|f\|_u \leq 1 \) and \( I(f) \geq c \) show that \( I \) is not continuous on \( \mathcal{K}(R) \) for the seminorm induced by \( C_\infty u(R) \). It follows that \( I^{-1}(0) = D^1(R) \) is dense in \( C_\infty u(R) \). Conversely let \( c = \int 1/u < +\infty \). Then the set where \( u \) vanishes has a void interior. It follows that the seminorm on \( C_\infty u(R) \) is actually a norm. We claim that \( |I(f)| \leq c \cdot \|f\|_u \) for \( f \in \mathcal{K}(R) \). This is clear if \( \|f\|_u = 0 \). If \( \|f\|_u > 0 \) then \( u(t)|f(t)| \leq \|f\|_u \) implies \( |f(t)| \leq \|f\|_u/u(t) \) for \( t \in R \), hence \( |I(f)| \leq \|f\|_u \) for \( f \in \mathcal{K}(R) \) as asserted. Thus \( I \) is continuous on \( \mathcal{K}(R) \) for the norm induced by \( C_\infty u(R) \) and \( I \) extends uniquely to a continuous linear form \( I \) on \( C_\infty u(R) \), since \( \mathcal{K}(R) \) is dense in \( C_\infty u(R) \). We know that \( I \neq 0 \) because \( I \) does not vanish on \( \mathcal{K}(R) \), but \( I \) does vanish on \( D^1(R) \). It follows that \( D^1(R) \) is not dense in \( C_\infty u(R) \). \( \square \)

**Example 7.** Consider \( v_0 = 0 \), \( v_1 \geq 0 \) upper semicontinuous on \( R \), and the \( C \)-weight \( v = (v_0, v_1) \) on \( R \). (Notice that \( v \) is not decreasing unless \( v_1 = 0 \).) Then \( D^1(R) \) is dense in \( C^1 v_\infty(R) \) if and only if \( \int 1/v_1 = +\infty \).

**Proof.** Since \( v_0 = 0 \) we see that \( D^1(R) \) is dense in \( C^1 v_\infty(R) \) if and only if \( D^1(R) \) is dense in \( C(v_1) \infty(R) \). It remains to apply Lemma 6 with \( v_1 \) in place of \( u \). \( \square \)
References


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