ON THE WEIGHTED APPROXIMATION
OF CONTINUOUSLY DIFFERENTIABLE FUNCTIONS

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Abstract. This note is an improvement of the available methods for getting results on the weighted approximation of continuously differentiable functions.

We shall present what we believe to be a simplified version of the reasoning to establish some results concerning weighted approximation of continuously differentiable scalar functions on \( \mathbb{R}^n \) (see [3]). For references to weighted approximation of continuous scalar functions on \( \mathbb{R}^n \) (see [1] and [3]). Lemma 1 reduces the search for sufficient conditions in order that a weight on \( \mathbb{R} \) be \( C^m \)-fundamental to the finding of sufficient conditions for a weight on \( \mathbb{R} \) to be \( C \)-fundamental. Similarly, Lemma 2 reduces the finding of sufficient conditions for a weight on \( \mathbb{R}^n \) to be \( C^m \)-fundamental to the search of sufficient conditions for a weight on \( \mathbb{R} \) to be \( C^m \)-fundamental. We then apply Lemmas 1 and 2 together to obtain Propositions 2, 4, and 5.

Fix integers \( n \in \mathbb{N}, n \geq 1 \), and \( m \in \mathbb{N} \) (the case \( m = \infty \) will be excluded since it follows easily from all \( m \in \mathbb{N} \)). Let \( K \) denote either \( \mathbb{R} \) or \( \mathbb{C} \). Consider the algebras \( \mathcal{P}(\mathbb{R}^n) \) of all \( K \)-valued polynomials on \( \mathbb{R}^n \) and \( C^m(\mathbb{R}^n) \) of all continuously \( m \)-differentiable \( K \)-valued functions on \( \mathbb{R}^n \). Write \( N^m_n = \{ \alpha \in \mathbb{N}^n : |\alpha| \leq m \} \) where \( |\alpha| = \alpha_1 + \cdots + \alpha_n \) if \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \). Let \( D^\alpha f \) be the \( \alpha \)th partial derivative of \( f \in C^m(\mathbb{R}^n) \) for \( \alpha \in N^m_n \). Denote by \( \mathcal{D}^m(\mathbb{R}^n) \) the subalgebra of \( C^m(\mathbb{R}^n) \) of all functions with compact support.

A \( C^m \)-weight on \( \mathbb{R}^n \) is a family \( v = \{v_\alpha : \alpha \in N^m_n \} \) of upper semicontinuous functions \( v_\alpha \geq 0 \) on \( \mathbb{R}^n \). Such a weight \( v \) defines the vector space \( C^m v_\infty(\mathbb{R}^n) \) of all \( f \in C^m(\mathbb{R}^n) \) such that \( v_\alpha D^\alpha f \) tends to zero at infinity for every \( \alpha \in N^m_n \). Set \( \|f\|_{v_\alpha} = \sup\{v_\alpha(t) \cdot |D^\alpha f(t)| : t \in \mathbb{R}^n \} \) to get a seminorm \( f \in C^m v_\infty(\mathbb{R}^n) \mapsto \|f\|_{v_\alpha} \in \mathbb{R}_+ \) for every \( \alpha \in N^m_n \). The finite family of such seminorms makes \( C^m v_\infty(\mathbb{R}^n) \) into a seminormable space, actually a

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semimormed space if for instance we use the seminorm \( f \in C^m v_{\infty} (\mathbb{R}^n) \mapsto \| f \|_v = \sup \{ \| f \|_{\alpha} : \alpha \in \mathbb{N}^n_m \} = \sup \{ \alpha (t) \cdot |D^n f (t)| : t \in \mathbb{R}^n, \alpha \in \mathbb{N}^n_m \} \in \mathbb{R}_+ \).

We now have that \( \mathscr{D}^m (\mathbb{R}^n) \subset C^m v_{\infty} (\mathbb{R}^n) \). It is known that \( \mathscr{D}^m (\mathbb{R}^n) \) is dense in \( C^m v_{\infty} (\mathbb{R}^n) \) if the weight \( v \) is decreasing in the sense that, for every \( \alpha, \beta \in \mathbb{N}^n_m \) with \( \beta \leq \alpha \), there exists \( C_{\alpha \beta} \geq 0 \) such that \( v_\alpha \leq C_{\alpha \beta} v_\beta \) ([3, Lemma 1]). \( \mathscr{D}^m (\mathbb{R}^n) \) may be dense in \( C^m v_{\infty} (\mathbb{R}^n) \) even if \( v \) is not decreasing, and it may fail to be dense as well. The weight \( v \) is said to be rapidly decreasing if \( \mathscr{D} (\mathbb{R}^n) \subset C^m v_{\infty} (\mathbb{R}^n) \). If, moreover, \( \mathscr{D} (\mathbb{R}^n) \) is dense in \( C^m v_{\infty} (\mathbb{R}^n) \), then \( v \) is called \( C^m \)-fundamental.

**Lemma 1.** Let \( m \in \mathbb{N} \), \( v_i \geq 0 \), \( i = 0, \ldots, m \), and \( u \geq 0 \) be upper semi-continuous on \( \mathbb{R} \). Consider the \( C^m \)-weight \( v = (v_0, \ldots, v_m) \) on \( \mathbb{R} \) which is assumed to be decreasing. Let \( u \) be \( C \)-fundamental on \( \mathbb{R} \) such that \( s, t \in \mathbb{R} \), \( |s| \leq |t| \) imply \( u(t) \leq u(s) \) and \( |t|^{m-i} v_i (t) \leq u(t) \), \( v_i (t) \leq u(t) \) for \( t \in \mathbb{R} \), \( i = 0, \ldots, m \). Then \( v \) is a \( C^m \)-fundamental weight on \( \mathbb{R} \).

**Proof.** The lemma is true if \( m = 0 \) because then \( v_0 \leq u \) and hence \( v_0 \) is \( C \)-fundamental along with \( u \). Assume now that \( m \geq 1 \) and that the lemma is true for \( m - 1 \). Notice that \( \mathscr{D} (\mathbb{R}) \subset C^m v_{\infty} (\mathbb{R}) \) because \( \mathscr{D} (\mathbb{R}) \subset C^u_{\infty} (\mathbb{R}) \) and \( v_i \leq u \), \( i = 0, \ldots, m \). Hence \( v \) is rapidly decreasing. Fix any \( f \in \mathscr{D}^m (\mathbb{R}) \) and \( \varepsilon > 0 \). Then \( f^{(m)} \in \mathscr{D} (\mathbb{R}) \subset C^u_{\infty} (\mathbb{R}) \) and there is a \( P^{(m)} \in \mathscr{D} (\mathbb{R}) \) so that

\[
(1) \quad u(t) \cdot |P^{(m)}(t) - f^{(m)}(t)| \leq \varepsilon, \quad t \in \mathbb{R}.
\]

Choose \( P \in \mathscr{D} (\mathbb{R}) \) whose \( m \)th derivative is \( P^{(m)} \) such that \( P^{(i)}(0) = f^{(i)}(0), \quad i = 0, \ldots, m - 1 \). We claim that

\[
(2) \quad v_i (t) \cdot |P^{(i)}(t) - f^{(i)}(t)| \leq \varepsilon, \quad t \in \mathbb{R}, \quad i = 0, \ldots, m.
\]

In fact, for \( i = m \) this follows from (1) and \( v_m \leq u \). For every \( t \in \mathbb{R}, \quad i = 0, \ldots, m - 1 \), there is an \( s \in \mathbb{R}, \quad |s| \leq |t| \) such that \( |t|^{m-i} \cdot |P^{(m)}(s) - f^{(m)}(s)| \) by the mean value theorem applied \( m - i \) times. Notice that \( |t|^{m-i} v_i (t) \leq u(t) \leq u(s) \). Hence (1) with \( s \) in place of \( t \) shows that (2) is true for \( i = 0, \ldots, m - 1 \) as well as for \( i = m \). It follows that \( \mathscr{D}^m (\mathbb{R}) \) is contained in the closure of \( \mathscr{D} (\mathbb{R}) \) in \( C^m v_{\infty} (\mathbb{R}) \). Therefore, \( \mathscr{D} (\mathbb{R}) \) is dense in \( C^m v_{\infty} (\mathbb{R}) \) along with \( \mathscr{D}^m (\mathbb{R}) \). Thus \( v \) is \( C^m \)-fundamental. \( \square \)

Let \( v \geq 0 \) be upper semicontinuous on \( \mathbb{R} \). We say that \( v \) is an analytic weight on \( \mathbb{R} \) when there exist \( C > 0, \quad c > 0 \) such that \( v(t) \leq Ce^{-ct} |t| \), \( t \in \mathbb{R} \).

It is then known that \( v \) is a \( C \)-fundamental weight on \( \mathbb{R} \) ([2, §28, Lemma 2]). More generally, if \( v \geq 0 \) is upper semicontinuous on \( \mathbb{R}^n \), we say that \( v \) is an analytic weight on \( \mathbb{R}^n \) when there exist \( C > 0, \quad c > 0 \) such that \( v(t) \leq Ce^{-c(|t_1|+\cdots+|t_n|)} \), \( t \in \mathbb{R}^n \). It is then known that \( v \) is \( C \)-fundamental on \( \mathbb{R}^n \).

**Proposition 2.** Let \( m \in \mathbb{N} \), \( v_i \geq 0 \), \( i = 0, \ldots, m \), be upper semicontinuous on \( \mathbb{R} \). Consider the \( C^m \)-weight \( v = (v_0, \ldots, v_m) \) on \( \mathbb{R} \) which is assumed to be decreasing. Let each \( v_i, \quad i = 0, \ldots, m \), be an analytic weight. Then \( v \) is a \( C^m \)-fundamental weight on \( \mathbb{R} \).
Proof. Assume that \( v_i(t) \leq Ce^{-c|t|} \), \( t \in \mathbb{R} \), \( i = 0, \ldots, m \), for some \( C > 0 \), \( c > 0 \). Choose \( D > 0 \), \( 0 < d < c \) so that, if \( u(t) = De^{-d|t|} \), \( t \in \mathbb{R} \), all assumptions in Lemma 1 are satisfied. \( \square \)

**Lemma 3.** Let \( n \in \mathbb{N} \), \( n \geq 1 \), \( m \in \mathbb{N} \), \( v_\alpha \geq 0 \), \( \alpha \in \mathbb{N}_m^n \), be upper semicontinuous on \( \mathbb{R} \). Consider the \( C^m \)-weight \( v = (v_\alpha ; \alpha \in \mathbb{N}_m^n) \) on \( \mathbb{R}^n \) which is assumed to be decreasing. Let \( u_{i,j} \geq 0 \), \( i = 1, \ldots, n \), \( j = 0, \ldots, m \), be upper semicontinuous on \( \mathbb{R} \). Consider the \( C^m \)-weights \( u_i = (u_{i,0}, \ldots, u_{i,m}) \), \( i = 1, \ldots, n \), on \( \mathbb{R} \) which are supposed to be decreasing and \( C^m \)-fundamental. Assume

\[
v_\alpha(t) \leq u_{1,\alpha_1}(t_1) \cdots u_{n,\alpha_n}(t_n), \quad t \in \mathbb{R}^n, \quad \alpha \in \mathbb{N}_m^n.
\]

Then \( v \) is \( C^m \)-fundamental on \( \mathbb{R}^n \).

**Proof.** Consider the \( n \)-linear mapping \( \pi \) that, with every

\[
(f_1, \ldots, f_n) \in C^m(u_1)_00(\mathbb{R}) \times \cdots \times C^m(u_n)_00(\mathbb{R})
\]

associates \( f_1 \otimes \cdots \otimes f_n \in C^m_{v_\infty}(\mathbb{R}^n) \) where \( (f_1 \otimes \cdots \otimes f_n)(t) = f_1(t_1) \cdots f_n(t_n) \) for \( t \in \mathbb{R}^n \). The assumptions make it sure that \( \pi \) is well defined and continuous because

\[
\|f_1 \otimes \cdots \otimes f_n\|_{v_\alpha} \leq \|f_1\|_{u_{1,\alpha_1}} \cdots \|f_n\|_{u_{n,\alpha_n}}.
\]

By hypothesis, \( \mathcal{P}(\mathbb{R}) \) is dense in \( C^m(u_i)_00(\mathbb{R}) \) so \( \mathcal{D}^m(\mathbb{R}) \) is contained in the closure of \( \mathcal{P}(\mathbb{R}) \) in \( C^m(u_i)_00(\mathbb{R}) \), \( i = 1, \ldots, n \). Thus \( \pi[\mathcal{D}^m(\mathbb{R}) \times \cdots \times \mathcal{D}^m(\mathbb{R})] \) is contained in the closure of \( \pi[\mathcal{P}(\mathbb{R}) \times \cdots \times \mathcal{P}(\mathbb{R})] \) in \( C^m_{v_\infty}(\mathbb{R}^n) \). Hence the vector subspace \( \mathcal{D}^m(\mathbb{R}) \otimes \cdots \otimes \mathcal{D}^m(\mathbb{R}) \) generated by \( \pi[\mathcal{D}^m(\mathbb{R}) \times \cdots \times \mathcal{D}^m(\mathbb{R})] \) is contained in the closure in \( C^m_{v_\infty}(\mathbb{R}^n) \) of the vector subspace \( \mathcal{P}(\mathbb{R}^n) = \mathcal{P}(\mathbb{R}) \otimes \cdots \otimes \mathcal{P}(\mathbb{R}) \) generated by \( \pi[\mathcal{P}(\mathbb{R}) \times \cdots \times \mathcal{P}(\mathbb{R})] \). It is known that \( \mathcal{D}^m(\mathbb{R}) \otimes \cdots \otimes \mathcal{D}^m(\mathbb{R}) \) is dense in \( \mathcal{D}^m(\mathbb{R}^n) \) in the natural inductive limit topology of \( \mathcal{D}^m(\mathbb{R}) \), hence in the coarser topology on \( \mathcal{D}^m(\mathbb{R}^n) \) defined by the norm \( f \in \mathcal{D}^m(\mathbb{R}^n) \mapsto \sup\{|D^m f(t)| : t \in \mathbb{R}^n, |a| \leq m \} \in \mathbb{R}_+ \), hence (because all \( v_\alpha \) are upper bounded along with all \( u_{i,j} \) in the even coarser topology that the natural topology of \( C^m_{v_\infty}(\mathbb{R}^n) \) induces on \( \mathcal{D}^m(\mathbb{R}^n) \). Since \( \mathcal{D}^m(\mathbb{R}^n) \) is dense in \( C^m_{v_\infty}(\mathbb{R}^n) \), then \( \mathcal{D}^m(\mathbb{R}) \otimes \cdots \otimes \mathcal{D}^m(\mathbb{R}) \) is dense in \( C^m_{v_\infty}(\mathbb{R}^n) \). The fact that \( \mathcal{D}^m(\mathbb{R}) \otimes \cdots \otimes \mathcal{D}^m(\mathbb{R}) \) is contained in the closure of \( \mathcal{P}(\mathbb{R}^n) \) in \( C^m_{v_\infty}(\mathbb{R}^n) \) implies then that \( \mathcal{P}(\mathbb{R}^n) \) is dense in \( C^m_{v_\infty}(\mathbb{R}^n) \). Thus \( v \) is \( C^m \)-fundamental. \( \square \)

**Proposition 4.** Let \( n \in \mathbb{N} \), \( n \geq 1 \), \( m \in \mathbb{N} \), \( v_\alpha \geq 0 \), \( \alpha \in \mathbb{N}_m^n \), be upper semicontinuous on \( \mathbb{R}^n \). Consider the \( C^m \)-weight \( v = (v_\alpha ; \alpha \in \mathbb{N}_m^n) \) on \( \mathbb{R}^n \) which is assumed to be decreasing. Assume that each \( v_\alpha \), \( \alpha \in \mathbb{N}_m^n \), is an analytic weight. Then \( v \) is \( C^m \)-fundamental on \( \mathbb{R}^n \).

**Proof.** Let \( v_\alpha(t) \leq Ce^{-c(|t_1| + \cdots + |t_n|)} \), \( t \in \mathbb{R}^n \), \( \alpha \in \mathbb{N}_m^n \), for suitable \( C > 0 \), \( c > 0 \). Choose \( u_{i,j}(t_i) = C^{1/n}e^{-c|t_i|} \), \( i = 1, \ldots, n \), \( j = 1, \ldots, m \), so that all assumptions in Lemma 3 are satisfied. \( \square \)
Let $v \geq 0$ be upper semicontinuous on $\mathbb{R}$ and rapidly decreasing. Put $M_m = \sup \{|r^m| \cdot v(t) : t \in \mathbb{R}\} \in \mathbb{R}_+$, $m = 0, 1, \ldots$. We say that $v$ is a quasi-analytic weight on $\mathbb{R}$ when $\sum_{m=1}^{\infty} (1/\sqrt[M]{M_m}) = +\infty$. It is then known that $v$ is a $C$-fundamental weight on $\mathbb{R}$ and that every analytic weight on $\mathbb{R}$ is quasi-analytic (see [2, §29, Lemma 2]). More generally, if $v \geq 0$ is upper semicontinuous on $\mathbb{R}^n$, we say that $v$ is a quasi-analytic weight on $\mathbb{R}^n$ if there are quasi-analytic weights $v_1, \ldots, v_n$ on $\mathbb{R}$ such that $v(t) \leq v_1(t_1) \cdots v_n(t_n)$, $t \in \mathbb{R}^n$. It is then known that $v$ is $C$-fundamental on $\mathbb{R}^n$, and that every analytic weight on $\mathbb{R}^n$ is quasi-analytic.

**Proposition 5.** Let $n \in \mathbb{N}$, $n \geq 1$, $m \in \mathbb{N}_0$, $v_\alpha \geq 0$, $\alpha \in \mathbb{N}^n_m$, be upper semicontinuous on $\mathbb{R}^n$. Consider the $C^m$-weight $v = (v_\alpha ; \alpha \in \mathbb{N}^n_m)$ which is supposed to be decreasing. Assume that there are quasi-analytic weights $v_i$, $i = 1, \ldots, n$, on $\mathbb{R}$ such that $v_\alpha(t) \leq v_1(t_1) \cdots v_n(t_n)$, $t \in \mathbb{R}^n$, $\alpha \in \mathbb{N}^n_m$. Then $v$ is a $C^m$-fundamental weight on $\mathbb{R}^n$.

**Proof.** All assumptions of Lemma 3 apply. \[ \square \]

**Lemma 6.** The linear mapping $D: f \in D^1(\mathbb{R}) \mapsto f' \in H(\mathbb{R}) = D^0(\mathbb{R})$ is injective. Its image is $D^1(\mathbb{R}) = \{ g \in H(\mathbb{R}) ; \int g = 0 \}$. If $u \geq 0$ is upper semicontinuous on $\mathbb{R}$, this image is dense in $C_{\infty}(\mathbb{R})$ if and only if $\int 1/u = +\infty$.

**Proof.** Only the final part of the lemma requires a proof. Consider the linear form $I: f \in H(\mathbb{R}) \mapsto \int f \in \mathbb{K}$. Assume $\int 1/u = +\infty$. We claim that $I$ is not continuous on $H(\mathbb{R})$ for the seminorm induced by $C_{\infty}(\mathbb{R})$. In fact, given any $c \geq 0$, there is an $f \in H(\mathbb{R})$ such that $0 \leq f \leq 1/u$, $\int f \geq c$. Therefore $\|f\|_u \leq 1$ and $I(f) \geq c$ show that $I$ is not continuous on $H(\mathbb{R})$ for the seminorm induced by $C_{\infty}(\mathbb{R})$. It follows that $I^{-1}(0) = D^1(\mathbb{R})$ is dense in $C_{\infty}(\mathbb{R})$. Conversely let $c = \int 1/u < +\infty$. Then the set where $u$ vanishes has a void interior. It follows that the seminorm on $C_{\infty}(\mathbb{R})$ is actually a norm. We claim that $|I(f)| \leq c \cdot \|f\|_u$ for $f \in H(\mathbb{R})$. This is clear if $\|f\|_u = 0$. If $\|f\|_u > 0$ then $\|f\|_u$ implies $|f(t)| \leq \|f\|_u/u(t)$ for $t \in \mathbb{R}$, hence $|I(f)| \leq c\|f\|_u$ for $f \in H(\mathbb{R})$ as asserted. Thus $I$ is continuous on $H(\mathbb{R})$ for the norm induced by $C_{\infty}(\mathbb{R})$ and $I$ extends uniquely to a continuous linear form $I$ on $C_{\infty}(\mathbb{R})$, since $H(\mathbb{R})$ is dense in $C_{\infty}(\mathbb{R})$. We know that $I \neq 0$ because $I$ does not vanish on $H(\mathbb{R})$, but $I$ does vanish on $D^1(\mathbb{R})$. It follows that $D^1(\mathbb{R})$ is not dense in $C_{\infty}(\mathbb{R})$. \[ \square \]

**Example 7.** Consider $v_0 = 0$, $v_1 \geq 0$ upper semicontinuous on $\mathbb{R}$, and the $C^1$-weight $v = (v_0, v_1)$ on $\mathbb{R}$. (Notice that $v$ is not decreasing unless $v_1 = 0$.) Then $D^1(\mathbb{R})$ is dense in $C^1v_{\infty}(\mathbb{R})$ if and only if $\int 1/v_1 = +\infty$.

**Proof.** Since $v_0 = 0$ we see that $D^1(\mathbb{R})$ is dense in $C^1v_{\infty}(\mathbb{R})$ if and only if $D^1(\mathbb{R})$ is dense in $C(v_1)_{\infty}(\mathbb{R})$. It remains to apply Lemma 6 with $v_1$ in place of $u$. \[ \square \]
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