COMPARISON THEOREMS FOR THE \( \nu \)-ZEROES OF LEGENDRE FUNCTIONS \( P^m_{\nu}(z_0) \) WHEN \(-1 < z_0 < 1\)

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Abstract. We consider the problem of ordering the elements of \( \{\nu_j^m(z_0)\} \), the set of \( \nu \)-zeroes of Legendre functions \( P^m_{\nu}(z_0) \) for \( m = 0, 1, \ldots \) and \( z_0 \in (-1, 1) \). In general, we seek to determine conditions on \((m, j)\) and \((n, i)\) under which we can assert that \( \nu_j^m < \nu_i^n \). A number of such results were established in [2] for \( z_0 \in [0, 1) \), and in the work that we present here we extend a number of these to the case \( z_0 \in (-1, 1) \). In addition, we prove \( \nu_{j+1}^m < \nu_j^{m+2} \) for \( z_0 \in (-1, 0) \) and \( \nu_2^4 < \nu_6^1 \) for \( z_0 \in (-1, 1) \). Using the results established here and in [2], we are able to determine the ordering of the first eleven \( \nu \)-zeroes of \( P^m_{\nu}(z_0) \) for \( 0 < z_0 < 1 \) and show that the twelfth \( \nu \)-zero is not necessarily distinct.

1. Introduction

For a fixed \( z_0 \in (-1, 1) \) and \( m = 0, 1, \ldots \), we will let \( \{\nu_j^m(z_0)\} \) denote the set of positive \( \nu \)-zeroes of Legendre functions \( P^m_{\nu}(z_0) \). The principal goal is determine conditions on \((m, j)\) and \((n, i)\) under which we can assert that \( \nu_j^m < \nu_i^n \). One such result which follows from the Sturm-Liouville theory is that

\[
\nu_j^m(z_0) < \nu_j^{m+1}(z_0) < \nu_{j+1}^m(z_0), \quad -1 < z_0 < 1
\]  

(See [10].) The problem of ordering the \( \nu_j^m \)'s when \( 0 \leq z_0 < 1 \) was first considered in [2], where the following results were established:

\[
\nu_j^{m+2}(z_0) < \nu_j^m(z_0), \quad 0 < z_0 < 1,
\]

(1.2)

\[
\nu_j^{m+2}(0) = \nu_j^m(0),
\]

(1.3)

\[
\nu_j^m(0) < \nu_{j+1}^m(z_0), \quad 0 < z_0 < 1.
\]

(1.4)

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Combining (1.1)-(1.4), it follows that the first ten \( \nu \)-zeroes are
\[
(1.5) \quad \nu_1^0 < \nu_1^1 < \nu_2^0 < \nu_2^1 < \nu_1^2 < \nu_1^3 < \nu_2^0 < \nu_2^1 < \nu_3^0 < \nu_1^5, \quad 0 < z_0 < 1.
\]
The ordering in (1.5) is unique.

The above results lead to several additional questions: Which are the next \( \nu_j^m \)'s in the chain of inequalities in (1.5)? How do \( \nu_{j+1}^m \) and \( \nu_{j+2}^m \) compare for \( z_0 \in (-1, 1) \)? In the following, we will consider these and related questions. The main results of this paper are contained in §3.

On the basis of numerical calculations, it was conjectured in [2] that the \( \nu \)-zero which followed \( \nu_1^1 \) in (1.5) was \( \nu_2^2 \). An analytical proof of this follows from the inequality,
\[
(1.6) \quad \nu_2^2(z_0) < \nu_1^6(z_0), \quad -1 < z_0 < 1,
\]
which is established in Theorem 1. In Lemma 2, we show that
\[
(1.7) \quad \nu_{j+1}^m(z_0) < \nu_j^{m+2}(z_0), \quad -1 < z_0 < 0.
\]

Theorem 3 combines (1.1)-(1.3) with Lemma 2 and gives the relative ordering of \( \nu_i^m(z_0) \), \( \nu_{j+1}^m(z_0) \), and \( \nu_{k+2}^m(z_0) \) for \(-1 < z_0 < 1\).

In addition, we will show that the inequalities in (1.4) hold for all \( z_0 \in (-1, 1) \). (See Theorem 2.) Although (1.1)-(1.6) imply that the first eleven \( \nu \)-zeros of \( P_\nu^m(z_0) \) are distinct for all \( 0 < z_0 < 1 \), in §4 we show that the twelfth \( \nu \)-zero is not necessarily distinct. Moreover, Theorem 3 shows that the ordering in (1.5) is not preserved for \(-1 < z_0 < 0\).

2. Preliminaries

In this section, we present some properties of the Legendre functions and their zeroes which will be needed in §3. For convenience, here and in the following sections \( m \) will denote a nonnegative integer and \( n, i, j, k \) will denote positive integers, unless otherwise stated.

The solution \( y = P_\nu^m(z) \) that satisfies
\[
(2.1a) \quad \frac{d}{dz} \left[ (1 - z^2) \frac{d}{dz} y \right] + \left( \nu(\nu + 1) - \frac{m^2}{1 - z^2} \right) y = 0, \quad -1 < z < 1,
\]
(2.1b) \( y(1) \) is bounded,

is called a Legendre function of the first kind of degree \( \nu \) and order \( m \). (See [4] for a general discussion of the Legendre functions and their properties.) If \(-1 < z \leq 1 \) and \( \nu > 0 \), then \( P_\nu^m(z) \) can be expressed [3, p. 148] as
\[
(2.2) \quad P_\nu^m(z) = \frac{(-1)^m \Gamma(\nu + m + 1)}{2^m m! \Gamma(\nu - m + 1)} (1 - z^2)^{m/2} \sum_{n=0}^{\infty} \frac{(1 + m + \nu)_n (m - \nu)_n}{(m + 1)_n n! 2^n} (1 - z)^n,
\]
where \( \Gamma(z) \) is the gamma function and \( (a)_n \) denotes the Pochhammer symbol,
\[
(a)_0 = 1,
(a)_n = a(a + 1) \ldots (a + n - 1).
\]
Later, we will need the following identity:
\begin{equation}
(2.3) \quad P_{\nu}^{m+2}(z) + 2(m+1)z(1-z^2)^{-1/2}P_{\nu}^{m+1}(z) + (\nu - m)(\nu + m + 1)P_{\nu}^m(z) = 0
\end{equation}
(See [3, p. 161].) For a fixed $z_0 \in (-1, 1)$, the pairs $(\nu(\nu+1), y(z))$ satisfying (2.1) and $y(z_0) = 0$ will be denoted by
\begin{equation}
(2.4) \quad (\nu_j^m(\nu_j^m + 1), P_{\nu_j^m}^m(z)) ,
\end{equation}
where $P_{\nu_j^m}^m(z_0) = 0$, $m = 0, 1, \ldots$, and $j = 1, 2, \ldots$.

The next result summarizes a few of the important properties of the $\nu_j^m$’s. Its proof follows the arguments in [2, Lemma 1] with some minor modifications.

**Lemma 1.** Let $m$ be a nonnegative integer. There exists a unique sequence $\{\nu_j^m(\tau)| j = 1, 2, \ldots\}$ such that for every $j$, the function $\nu = \nu_j^m(\tau)$ satisfies
\begin{equation}
(*) \quad P_{\nu_j^m}^m(\tau) = 0, \quad \text{for all } \tau \in (-1, 1).
\end{equation}
Moreover, each $\nu_j^m(\tau)$ is analytic and strictly increasing as a function of $\tau$ for $\tau \in (-1, 1)$.

It will be convenient at times to consider $P_{\nu}^m(\cos \phi)$ for $0 < \phi \leq \phi_0 < \pi$ where $z = \cos \phi$ and $z_0 = \cos \phi_0$. As a function of $\phi_0$, we see from Lemma 1 that $\nu_j^m(\phi_0)$ is decreasing for $\phi_0 \in (0, \pi)$. It will be clear from the context whether $\nu_j^m$ is to be considered as a function of $z_0$ or as a function of $\phi_0 = \arccos(z_0)$.

A straightforward calculation shows that if $P_{\nu}^m(z)$ is a solution of (2.1), then $u = \frac{1}{\sin \phi}P_{\nu}^m(\cos \phi)$ satisfies
\begin{equation}
(2.5) \quad u'' + \left(\left(\nu + \frac{1}{2}\right)^2 + \frac{1 - 4m^2}{4 \sin^2 \phi}\right)u = 0.
\end{equation}
From [7, p. 17], we see that $v = \sqrt{\phi}J_m((\nu + \frac{1}{2})\phi)$ is a solution of
\begin{equation}
\nu'' + \left(\left(\nu + \frac{1}{2}\right)^2 + \frac{1 - 4m^2}{4 \phi^2}\right)v = 0 ,
\end{equation}
where $J_m$ is the Bessel function of order $m$. From the Sturm-Liouville theory (See [10, Chapter 7]), $P_{\nu_j^m}^m(z)$ has $j - 1$ $z$-zeroes on $(z_0, 1)$. Moreover, for a general $\nu$ (not necessarily one of the $\nu_j^m$’s), we see from (2.5) that there are $[\nu - m]$ $z$-zeroes of $P_{\nu}^m(z)$ on $(-1, 1)$, where $[x] = n$ for $n \leq x < n + 1$. $P_n^m(z)$ has exactly $n - m$ $z$-zeroes on $(-1, 1)$ and $P_{n+1}^m$ has exactly $n - m + 1$ $z$-zeroes on $(-1, 1)$. (See [6, p. 246].) To see this, suppose $\nu = \nu^*$ and $n < \nu^* < n + 1$. By applying the Sturm Comparison Theorem [5] to the solutions of (2.5) for $\nu = n$, $\nu^*$ and $n + 1$, respectively, we see that $P_{\nu_j^m}^m(z)$ must have at least $n - m$ $z$-zeroes on $(-1, 1)$ and at most $n - m + 1$ $z$-zeroes on $(-1, 1)$. We conclude that $P_{\nu_j^m}^m(z)$ has exactly $\lfloor \nu^* - m \rfloor$ $z$-zeroes on $(-1, 1)$. 
For fixed $\nu$, we will denote the $z$-zeroes of $P^m_{\nu}(z)$ that are between $-1$ and $1$ by $z^m_{\nu,i}$, where $z^m_{\nu,i} > z^m_{\nu,i+1}$. By definition of the $\nu^m_{j}(z_0)$'s, it follows that

$$\nu^m_{j}(z^m_{\nu,j}) = \nu.$$  \hfill (2.6)

We define $\phi^m_{\nu,j}$ to be the solution of

$$z^m_{\nu,j} = \cos(\phi^m_{\nu,j})$$

such that $0 < \phi^m_{\nu,j} < \pi$. It follows that $\phi^m_{\nu,j} < \phi^m_{\nu,j+1}$.

From (2.2), we see that

$$P^m_{2m}(z_0) = (-1)^m \Gamma(2m + 1) (1 - z_0^2)^{m/2}.$$  \hfill (2.7)

From Rodrigue's formula [6, p. 174] and Rolle's Theorem, it follows that $P^m_{m+2j-1}(z)$ has $j - 1$ $z$-zeroes on $(0, 1)$ and $\nu^m_{j} = m + 2j - 1$ when $z_0 = 0$. Similarly, $P^m_{m+j-1}(z)$ has $j - 1$ $z$-zeroes on $(-1, 1)$ and

$$\lim_{z_0 \to 1^{-}} \nu^m_{j}(z_0) = m + j - 1.$$  \hfill (2.8)

Since $\nu^m_{j}(z_0)$ is increasing on $(-1, 1)$, we see that

$$m + j - 1 < \nu^m_{j}(z_0) < m + 2j - 1, \quad -1 < z_0 < 0.$$  \hfill (2.9)

From [8], we have

$$\frac{1}{3} < \frac{1}{\sin^2 \phi} - \frac{1}{\phi^2} < \alpha(\phi), \quad 0 < \phi < \phi \leq \pi/2,$$  \hfill (2.10)

where $\alpha(\phi) = \sin^{-2} \phi - \phi^{-2}$. Note that $\lim_{\phi \to 0^+} \alpha(\phi) = 1/3$. From (2.9), we see that

$$\frac{\phi^2}{1 + \alpha(\phi)\phi^2} < \sin^2 \phi < \frac{\phi^2}{1 + 3\phi^2}, \quad 0 < \phi < \phi \leq \pi/2.$$  \hfill (2.11)

Multiplying (2.9) by $1 - 4m^2$ with $m \geq 1$, we obtain

$$\frac{1 - 4m^2}{4\phi^2} - \frac{h^2}{4} > \frac{1 - 4m^2}{4\sin^2 \phi} > \frac{1 - 4m^2}{4\phi^2} - \frac{k^2(\phi)}{4}, \quad 0 < \phi < \phi \leq \pi/2,$$

where $h^2 = (4m^2 - 1)/3$ and $k^2(\phi) = (4m^2 - 1)\alpha(\phi)$.

Next, we consider the following pair of differential equations and their re-
Let \( u \) be a solution of (2.5). From (2.11) and the Sturm Comparison Theorem, it follows that the \( k \)th zero of \( U \) occurs before the \( k \)th zero of \( u \) and the \( k \)th zero of \( u \) occurs before the \( k \)th zero of \( V \). In particular, we have for \( m \geq 1 \),

\[
\begin{align*}
\frac{j_k^m}{\sqrt{(\nu + \frac{1}{2})^2 - \frac{k^2}{4}} - \frac{h^2}{4}} < \phi_{\nu,k}^m < \frac{j_k^m}{\sqrt{(\nu + \frac{1}{2})^2 - \frac{k^2}{4}}} ,
\end{align*}
\]

where \( j_k^m \) is the \( k \)th positive zero of \( J_m(z) \). If \( m = 0 \), we find that

\[
\begin{align*}
\frac{j_k^0}{\sqrt{(\nu + \frac{1}{2})^2 + \frac{q}{4}}} < \phi_{\nu,k}^0 < \frac{j_k^0}{\sqrt{(\nu + \frac{1}{2})^2 + \frac{1}{4}}} .
\end{align*}
\]

### 3. Ordering the \( \nu \)-zeros of Legendre functions

This section contains the principle results of this paper. We begin with a comparison of \( \nu_2^3 \) and \( \nu_1^6 \):

**Theorem 1.** \( \nu_2^3(z_0) < \nu_1^6(z_0) \) for all \(-1 < z_0 < 1\).

**Proof.** Here, it will be convenient to let \( z_0 = \cos \phi_0 \) and to consider \( \nu_1^6 \) and \( \nu_2^3 \) as functions of \( \phi_0 \). First, we will show that if \( \nu = \nu_1^6 = \nu_2^3 \), then \( \nu > 7 \). Then, we will show that \( \nu = \nu_1^6 = \nu_2^3 \) is impossible if \( \nu > 7 \).

**Part 1.** Since \( \nu_2^3(\phi_0) \) and \( \nu_1^6(\phi_0) \) are decreasing in \( \phi_0 \), by (2.8), we have

\[
\lim_{\phi_0 \to \pi^{-}} \nu_3^3(\phi_0) = 4, \quad \lim_{\phi_0 \to \pi^{-}} \nu_1^6(\phi_0) = 6.
\]

Since \( \nu_1^6(\phi_0) > 6 \) for \( \phi_0 \in (0, \pi) \) and \( \nu_2^3(\phi_0) < 6 \) for \( \phi_0 \in (\pi/2, \pi) \), we see that if \( \nu = \nu_2^3(\phi_0) = \nu_1^6(\phi_0) \) for some \( \phi_0 \), then \( \nu \geq 6 \) and \( 0 < \phi_0 < \pi/2 \).

Next, suppose that \( \nu = \nu_2^3 = \nu_1^6 \) for some \( z_0 = \cos \phi_0 \) and \( 0 < \phi_0 < \pi/2 \). From (2.3) we see that

\[
\begin{align*}
(A) & \left( \frac{10z_0(1 - z_0^2)^{-1/2}}{1} \right) \left( \frac{(\nu - 4)(\nu + 5)}{8z_0(1 - z_0^2)^{-1/2}} \right) \left( \frac{P_\nu^5(z_0)}{P_\nu^4(z_0)} \right) = (0).
\end{align*}
\]
Since by (1.1), $P^5_\nu(z_0)$ and $P^4_\nu(z_0)$ cannot vanish simultaneously, we see that the determinant of the $2 \times 2$ matrix in (A) must be zero, and we are led to the condition that

$$(B) \quad z_0^2 = \frac{\nu(\nu + 1) - 20}{\nu(\nu + 1) + 60}.$$ 

From (B), we see that for $6 \leq \nu < 7$, $\sqrt{11/51} \leq z_0 < \sqrt{9/29}$. On the other hand, if we set $\bar{\phi} = \arccos \sqrt{11/51}$, $m = 6$, $k = 1$ and $j_1^6 = 9.9361$ in (2.12a), we find that for such a $\phi_0$ and $\nu$,

$$z_0 = \cos \phi_0 = \cos(\phi_{\nu,1}^6) < \cos \left( \frac{j_1^6}{\sqrt{(7.5)^2 - 143/12}} \right) \approx 0.0784,$$

which is a contradiction. Thus, we have shown that if $\nu = \nu_1^6 = \nu_2^3$, then $\nu \geq 7$. Moreover, we have also shown that if $\nu_1^6 = \nu_2^3$ for some $\phi_0$, then $\phi_0 \leq \arccos \sqrt{9/29}$.

**Part 2.** By the relationship in (B), we are motivated to define a function $z_0(\nu)$ for $\nu \geq 7$ as follows:

$$(3.1) \quad z_0(\nu) = \left( \frac{\nu(\nu + 1) - 20}{\nu(\nu + 1) + 60} \right)^{1/2}.$$ 

For such a $z_0(\nu)$, we also define $\phi_0(\nu) = \arccos(z_0(\nu))$ and observe that

$$(3.2) \quad \sin^2(\phi_0(\nu)) = \frac{80}{\nu(\nu + 1) + 60}.$$ 

From (2.12a), (2.10), and (3.2), with $m = 6$, $k = 1$ and $\bar{\phi} = \arccos \sqrt{9/29}$, we see that for all $\nu \geq 7$,

$$\sin^2(\phi_{\nu,1}^{6}) > \sin^2 \left( \frac{j_1^6}{\sqrt{(\nu + \frac{1}{2})^2 - \frac{h^2}{4}}} \right) \geq \frac{(j_1^6)^2}{\nu(\nu + 1) + \frac{1}{4} - \frac{h^2}{4} + \alpha(\bar{\phi})(j_1^6)^2} \geq \frac{80}{\nu(\nu + 1) + 60} = \sin^2(\phi_0(\nu)),$$

where $j_1^6 \approx 9.9361$. To complete the proof, we observe that if $\nu^* = \nu_1^6 = \nu_2^3$ for some $\nu^* \geq 7$, then necessarily we must have $\phi_{\nu,1}^{6} = \phi_{\nu,2}^{3} = \phi_0(\nu^*)$. However, from (3.3) we see that $\phi_0(\nu) < \phi_{\nu,1}^{6}$ for all $\nu \geq 7$. It follows that $\nu_1^6 \neq \nu_2^3$ for $0 < \phi_0 < \pi$. Since $\nu_2^3 < \nu_1^6$ for $\pi/2 \leq \phi_0 < \pi$, we conclude that $\nu_2^3 < \nu_1^6$ for all $0 < \phi_0 < \pi$ (or equivalently, for all $z_0 \in (-1, 1)$).
As a consequence of (1.1)–(1.5) and Theorem 1, we see that the first eleven \( \nu \)-zeroes are
\[
\nu_0 < \nu_1 < \nu_2 < \nu_3 < \nu_4 < \nu_5 < \nu_6 < \nu_7 < \nu_8 < \nu_9, \quad 0 < z_0 < 1,
\]
and that this ordering is unique. The inequalities \( \nu_0 < \nu_1, \nu_2 < \nu_3, \) and \( \nu_3 < \nu_4 \) were established in [2] for \( 0 \leq z_0 < 1 \). By applying (2.8) and arguing as we did at the beginning of the proof of Theorem 1, these inequalities can be shown to hold for \(-1 \leq z_0 < 0\) as well. In particular, we have the following:

**Theorem 2.** \( \nu_2 < \nu_3, \nu_2 < \nu_3 \) for all \( z_0 \in (-1, 1) \).

The inequality \( \nu_j^{m+2} < \nu_j^m \) for \( 0 < z_0 < 1 \) was established in [2]. Next, we consider the case \(-1 < z_0 < 0\).

**Lemma 2.** If \(-1 < z_0 < 0\), then \( \nu_j^m < \nu_j^{m+2} \).

**Proof.** The \( \nu_j^m \)'s are simple zeroes of \( P^m_\nu(z_0) \). (See [2].) From (1.1), we have that \( \nu_{j+1}^m, \nu_{j+2}^m \in (\nu_{j+1}^{m+1}, \nu_{j+1}^{m+1}) \). Suppose that \( \nu_{j+2}^m(z_0) \leq \nu_{j+1}^m(z_0) \) for some \( z_0 \in (-1, 0) \). From (2.7)–(2.8), we see that
\[
\text{sign}(P^m_\nu(z_0)) = (-1)^m, \quad \nu_j^m < \nu < \nu_{j+1}^m, \quad z_0 \in (-1, 0).
\]
The signs of \( P_{\nu+1}^m(z_0) \) and \( P_{\nu+2}^m(z_0) \) can also be determined in this way. We are led to the results summarized in Table 1.

<table>
<thead>
<tr>
<th>( z_0 \in (-1, 0) )</th>
<th>( \nu \in (\nu_j^m, \nu_j^{m+2}) )</th>
<th>( \nu \in (\nu_j^{m+2}, \nu_j^{m+1}) )</th>
<th>( \nu \in (\nu_{j+1}^m, \nu_{j+1}^{m+1}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>sign(( P^m_\nu(z_0) ))</td>
<td>((-1)^{m+j})</td>
<td>((-1)^{m+j+1})</td>
<td>((-1)^{m+j+2})</td>
</tr>
<tr>
<td>sign(( P_{\nu+1}^m(z_0) ))</td>
<td>((-1)^{m+j+1})</td>
<td>((-1)^{m+j+2})</td>
<td>((-1)^{m+j+3})</td>
</tr>
<tr>
<td>sign(( P_{\nu+2}^m(z_0) ))</td>
<td>((-1)^{m+j+2})</td>
<td>((-1)^{m+j+3})</td>
<td>((-1)^{m+j+4})</td>
</tr>
</tbody>
</table>

The second column in Table 1 contradicts (2.3), and we conclude that \( \nu_j^{m+2} > \nu_j^m \) for all \( z_0 \in (-1, 0) \).

From (1.1)–(1.3) and Lemma 2, we are led to the following:

**Theorem 3.** If \( \nu_j^m(z_0) \) is a solution of \((*)\), then
(i) \( \nu_j^{m+1} < \nu_j^{m+2} < \nu_j^{m+1} \), \( -1 < z_0 < 0 \),
(ii) \( \nu_j^{m+2} = \nu_j^{m+1} = m + 2j - 1 \), \( z_0 = 0 \),
(iii) \( \nu_j^{m+1} < \nu_j^{m+2} < \nu_j^{m+1} \), \( 0 < z_0 < 1 \).

4. **Concluding remarks**

Since the zeroes of the Bessel functions are distinct [9, p. 484] the elements of \( \mathcal{J} = \{ j_k^m \} \) can be arranged as an increasing sequence. In particular, we can
define integer-valued functions \( m(i), k(i) \) so that \( j^{m(i)}_{k(i)} \) denotes the \( i \)th element in the sequence \( \mathcal{J} \). Clearly, there is no such ordering of all the elements of \( \mathcal{N}_{\phi_0} = \{ \nu^m_j(\phi_0) \} \) that is independent of \( \phi_0 \). On the other hand, if we let \( \phi = \phi_0 = \phi^m_{\nu,k} \) and \( \nu = \nu^m_k(\phi_0) \) in (2.12), we see that

\[
\lim_{\phi_0 \to 0^+} \phi_0 \left( \nu^m_k(\phi_0) + \frac{1}{2} \right) = j^m_k.
\]

The limit in (4.1) is related to the well-known result, \( \lim_{n \to \infty} \phi^0_{n,k}(n + \frac{1}{2}) = j^0_k \) (see [8]) and implies that for \( \phi_0 \) sufficiently small, \( \nu^m_{k(i)} \) is the \( i \)th element in the sequence \( \mathcal{N}_{\phi_0} = \{ \nu^m_{j(i)}(\phi_0) \} \).

In view of (1.6) and the first two inequalities in (1.4), it is natural to conjecture if there is an inequality that relates \( \nu^m_{j+1} \) and \( \nu^m_{j+3} \) for \( \phi_0 \in (0, \pi/2) \). Such an inequality is not possible. From [1], we see that \( j^{m(18)}_{k(18)} = j^8_1 = 12.225 \), and \( j^{m(19)}_{k(19)} = j^5_2 = 12.338 \). Since \( \nu^8_1(\pi/2) = 9 \), \( \nu^5_2(\pi/2) = 8 \), and \( j^8_1 < j^5_2 \), we conclude that \( \nu^8_1(\phi_0) = \nu^5_2(\phi_0) \) for some \( \phi_0 \in (0, \pi/2) \). Numerical calculations indicate that \( \nu^5_2 = 26.706 \) when \( \phi_0 = 26.134^\circ \).

Although (1.5) and Theorem 1 demonstrate that the first eleven \( \nu \)-zeros of \( P^m_{\nu}(\cos \phi_0) \) are distinct for \( 0 < \phi_0 < \pi/2 \), the twelfth \( \nu \)-zero is not necessarily distinct. Since \( \nu^3_1(\pi/2) = 6 \), \( \nu^3_2(\pi/2) = 7 \), and \( j^{m(12)}_{k(12)} = j^6_1 = 9.936 \), and \( j^{m(13)}_{k(13)} = j^1_3 = 10.173 \), from (4.1), we see that \( \nu^6_1(\phi_0) = \nu^3_3(\phi_0) \) for some \( \phi_0 \in (0, \pi/2) \). Numerics indicate \( \nu^6_1 = \nu^3_3 = 15.780 \) when \( \phi_0 = 35.821^\circ \) (see [2]).

REFERENCES

2. F. Baginski, *Ordering the zeroes of the Legendre functions \( P^m_{\nu}(z_0) \) when considered as a function of \( \nu \)*, J. Math. Anal. Appl. 147 (1990), 296–308.