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COMPARISON THEOREMS FOR THE ν-ZEROES OF LEGENDRE FUNCTIONS $P_\nu^m(z_0)$ WHEN $-1 < z_0 < 1$

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ABSTRACT. We consider the problem of ordering the elements of $\{\nu_j^m(z_0)\}$, the set of ν-zeroes of Legendre functions $P_\nu^m(z_0)$ for $m = 0, 1, \ldots$ and $z_0 \in (-1, 1)$. In general, we seek to determine conditions on (m, j) and (n, i) under which we can assert that $\nu_j^m < \nu_i^n$. A number of such results were established in [2] for $z_0 \in [0, 1)$, and in the work that we present here we extend a number of these to the case $z_0 \in (-1, 1)$. In addition, we prove $\nu_{j+1}^m < \nu_j^{m+2}$ for $z_0 \in (-1, 0)$ and $\nu_2^3 < \nu_1^6$ for $z_0 \in (-1, 1)$. Using the results established here and in [2], we are able to determine the ordering of the first eleven ν-zeroes of $P_\nu^m(z_0)$ for $0 < z_0 < 1$ and show that the twelfth ν-zero is not necessarily distinct.

1. INTRODUCTION

For a fixed $z_0 \in (-1, 1)$ and $m = 0, 1, \ldots$, we will let $\{\nu_j^m(z_0)\}$ denote the set of positive ν-zeroes of Legendre functions $P_\nu^m(z_0)$. The principal goal is determine conditions on (m, j) and (n, i) under which we can assert that $\nu_j^m < \nu_i^n$. One such result which follows from the Sturm-Liouville theory is that

$$\nu_j^m(z_0) < \nu_j^{m+1}(z_0) < \nu_{j+1}^m(z_0), \qquad -1 < z_0 < 1 \tag{1.1}$$

(See [10].) The problem of ordering the ν_j^m's when $0 \le z_0 < 1$ was first considered in [2], where the following results were established:

$$\nu_j^{m+2}(z_0) < \nu_{j+1}^m(z_0), \qquad 0 < z_0 < 1, \tag{1.2}$$

$$\nu_j^{m+2}(0) = \nu_{j+1}^m(0), \tag{1.3}$$

(1.4)
$$\nu_2^0(z_0) < \nu_1^3(z_0), \qquad \nu_2^1(z_0) < \nu_1^4(z_0), \qquad \nu_3^0(z_0) < \nu_1^5(z_0), \qquad 0 < z_0 < 1.$$

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Combining (1.1)–(1.4), it follows that the first ten ν-zeroes are

$$(1.5)\qquad \nu_1^0 < \nu_1^1 < \nu_1^2 < \nu_2^0 < \nu_1^3 < \nu_2^1 < \nu_1^4 < \nu_2^2 < \nu_3^0 < \nu_1^5, \qquad 0 < z_0 < 1.$$

The ordering in (1.5) is unique.

The above results lead to several additional questions: Which are the next ν_j^m's in the chain of inequalities in (1.5)? How do ν_{j+1}^m and ν_j^{m+2} compare for $z_0 \in (-1, 1)$? In the following, we will consider these and related questions. The main results of this paper are contained in §3.

On the basis of numerical calculations, it was conjectured in [2] that the ν-zero which followed ν_1^5 in (1.5) was ν_2^3. An analytical proof of this follows from the inequality,

$$(1.6)\qquad \nu_2^3(z_0) < \nu_1^6(z_0), \qquad -1 < z_0 < 1,$$

which is established in Theorem 1. In Lemma 2, we show that

$$(1.7)\qquad \nu_{j+1}^m(z_0) < \nu_j^{m+2}(z_0), \qquad -1 < z_0 < 0.$$

Theorem 3 combines (1.1)–(1.3) with Lemma 2 and gives the relative ordering of $\nu_i^m(z_0)$, $\nu_j^{m+1}(z_0)$, and $\nu_k^{m+2}(z_0)$ for $-1 < z_0 < 1$.

In addition, we will show that the inequalities in (1.4) hold for all $z_0 \in (-1, 1)$. (See Theorem 2.) Although (1.1)–(1.6) imply that the first eleven ν-zeroes of $P_\nu^m(z_0)$ are distinct for all $0 < z_0 < 1$, in §4 we show that the twelfth ν-zero is not necessarily distinct. Moreover, Theorem 3 shows that the ordering in (1.5) is not preserved for $-1 < z_0 \le 0$.

2. Preliminaries

In this section, we present some properties of the Legendre functions and their zeroes which will be needed in §3. For convenience, here and in the following sections m will denote a nonnegative integer and n, i, j, k will denote positive integers, unless otherwise stated.

The solution $y = P_\nu^m(z)$ that satisfies

$$(2.1\text{a})\qquad \frac{d}{dz}\left((1-z^2)\frac{d}{dz}y\right) + \left(\nu(\nu+1) - \frac{m^2}{1-z^2}\right)y = 0, \qquad -1 < z < 1,$$

$$(2.1\text{b})\qquad y(1) \text{ is bounded},$$

is called a Legendre function of the first kind of degree ν and order m. (See [4] for a general discussion of the Legendre functions and their properties.) If $-1 < z \le 1$ and $\nu > 0$, then $P_\nu^m(z)$ can be expressed [3, p. 148] as

(2.2)

$$P_\nu^m(z) = \frac{(-1)^m\Gamma(\nu+m+1)}{2^m m!\Gamma(\nu-m+1)}(1-z^2)^{m/2}\sum_{n=0}^{\infty}\frac{(1+m+\nu)_n(m-\nu)_n}{(m+1)_n n!2^n}(1-z)^n,$$

where $\Gamma(z)$ is the gamma function and $(a)_n$ denotes the Pochhammer symbol,

$$(a)_0 = 1,$$
$$(a)_n = a(a+1)\dots(a+n-1).$$

Later, we will need the following identity:

$$(2.3)\quad P_\nu^{m+2}(z)+2(m+1)z(1-z^2)^{-1/2}P_\nu^{m+1}(z)+(\nu-m)(\nu+m+1)P_\nu^m(z)=0$$

(See [3, p. 161].) For a fixed $z_0 \in (-1, 1)$, the pairs $(\nu(\nu+1), y(z))$ satisfying (2.1) and $y(z_0)=0$ will be denoted by

$$(2.4)\qquad (\nu_j^m(\nu_j^m+1),\ P_{\nu_j^m}^m(z)),$$

where $P_{\nu_j^m}^m(z_0)=0$, $m=0, 1, \dots$, and $j=1, 2, \dots$.

The next result summarizes a few of the important properties of the ν_j^m's. Its proof follows the arguments in [2, Lemma 1] with some minor modifications.

Lemma 1. *Let m be a nonnegative integer. There exists a unique sequence $\{\nu_j^m(\tau) | j=1, 2, \dots\}$ such that for every j, the function $\nu=\nu_j^m(\tau)$ satisfies*

$$(*)\qquad P_\nu^m(\tau)=0, \quad \textit{for all } \tau \in (-1, 1).$$

Moreover, each $\nu_j^m(\tau)$ is analytic and strictly increasing as a function of τ for $\tau \in (-1, 1)$.

It will be convenient at times to consider $P_\nu^m(\cos\phi)$ for $0<\phi\le\phi_0<\pi$ where $z=\cos\phi$ and $z_0=\cos\phi_0$. As a function of ϕ_0, we see from Lemma 1 that $\nu_j^m(\phi_0)$ is decreasing for $\phi_0 \in (0, \pi)$. It will be clear from the context whether ν_j^m is to be considered as a function of z_0 or as a function of $\phi_0 = \arccos(z_0)$.

A straightforward calculation shows that if $P_\nu^m(z)$ is a solution of (2.1), then $u=\sqrt{\sin\phi}P_\nu^m(\cos\phi)$ satisfies

$$(2.5)\qquad u''+\left(\left(\nu+\frac{1}{2}\right)^2+\frac{1-4m^2}{4\sin^2\phi}\right)u=0.$$

From [7, p. 17], we see that $v=\sqrt{\phi}J_m((\nu+\frac{1}{2})\phi)$ is a solution of

$$v''+\left(\left(\nu+\frac{1}{2}\right)^2+\frac{1-4m^2}{4\phi^2}\right)v=0,$$

where J_m is the Bessel function of order m. From the Sturm-Liouville theory (See [10, Chapter 7]), $P_{\nu_j^m}^m(z)$ has $j-1$ z-zeroes on $(z_0, 1)$. Moreover, for a general ν (not necessarily one of the ν_j^m's), we see from (2.5) that there are $[\nu-m]$ z-zeroes of $P_\nu^m(z)$ on $(-1, 1)$, where $[x]=n$ for $n\le x<n+1$. $P_n^m(z)$ has exactly $n-m$ z-zeroes on $(-1, 1)$ and P_{n+1}^m has exactly $n-m+1$ z-zeroes on $(-1, 1)$. (See [6, p. 246].) To see this, suppose $\nu=\nu^*$ and $n<\nu^*<n+1$. By applying the Sturm Comparison Theorem [5] to the solutions of (2.5) for $\nu=n$, ν^* and $n+1$, respectively, we see that $P_{\nu^*}^m(z)$ must have *at least* $n-m$ z-zeroes on $(-1, 1)$ and *at most* $n-m+1$ z-zeroes on $(-1, 1)$. We conclude that $P_{\nu^*}^m(z)$ has exactly $[\nu^*-m]$ z-zeroes on $(-1, 1)$.

For fixed ν, we will denote the z-zeroes of $P_\nu^m(z)$ that are between -1 and 1 by $z_{\nu,i}^m$, where $z_{\nu,i}^m > z_{\nu,i+1}^m$. By definition of the $\nu_j^m(z_0)$'s, it follows that

$$\nu_j^m(z_{\nu,j}^m) = \nu. \tag{2.6}$$

We define $\phi_{\nu,j}^m$ to be the solution of

$$z_{\nu,j}^m = \cos(\phi_{\nu,j}^m)$$

such that $0 < \phi_{\nu,j}^m < \pi$. It follows that $\phi_{\nu,i}^m < \phi_{\nu,i+1}^m$.

From (2.2), we see that

$$P_m^m(z_0) = \frac{(-1)^m\Gamma(2m+1)}{2^m m!}(1-z_0^2)^{m/2}. \tag{2.7}$$

From Rodrigue's formula [6, p. 174] and Rolle's Theorem, it follows that $P_{m+2j-1}^m(z)$ has $j-1$ z-zeroes on $(0, 1)$ and $\nu_j^m = m+2j-1$ when $z_0 = 0$. Similarly, $P_{m+j-1}^m(z)$ has $j-1$ z-zeroes on $(-1, 1)$ and

$$\lim_{z_0 \to -1+} \nu_j^m(z_0) = m+j-1.$$

Since $\nu_j^m(z_0)$ is increasing on $(-1, 1)$, we see that

$$m+j-1 < \nu_j^m(z_0) < m+2j-1, \qquad -1 < z_0 < 0. \tag{2.8}$$

From [8], we have

$$\frac{1}{3} < \frac{1}{\sin^2\phi} - \frac{1}{\phi^2} < \alpha(\overline{\phi}), \qquad 0 < \phi < \overline{\phi} \le \pi/2, \tag{2.9}$$

where $\alpha(\phi) = \sin^{-2}\phi - \phi^{-2}$. Note that $\lim_{\phi\to 0^+}\alpha(\phi) = 1/3$. From (2.9), we see that

$$\frac{\phi^2}{1+\alpha(\overline{\phi})\phi^2} < \sin^2\phi < \frac{\phi^2}{1+\frac{1}{3}\phi^2}, \qquad 0 < \phi < \overline{\phi} \le \pi/2. \tag{2.10}$$

Multiplying (2.9) by $1-4m^2$ with $m \ge 1$, we obtain

$$\frac{1-4m^2}{4\phi^2} - \frac{h^2}{4} > \frac{1-4m^2}{4\sin^2\phi} > \frac{1-4m^2}{4\phi^2} - \frac{k^2(\overline{\phi})}{4}, \qquad 0 < \phi < \overline{\phi} \le \pi/2, \tag{2.11}$$

where $h^2 = (4m^2-1)/3$ and $k^2(\overline{\phi}) = (4m^2-1)\alpha(\overline{\phi})$.

Next, we consider the following pair of differential equations and their re-

spective solutions which are related to Bessel's equation [7, p. 17],

$$U'' + \left(\left(\nu + \frac{1}{2}\right)^2 - \frac{h^2}{4} + \frac{1-4m^2}{4\phi^2}\right) U = 0, \qquad 0 < \phi < \overline{\phi},$$

$$U = \sqrt{\phi} J_m \left(\sqrt{\left(\nu + \frac{1}{2}\right)^2 - \frac{h^2}{4}}\,\phi\right),$$

$$V'' + \left(\left(\nu + \frac{1}{2}\right)^2 - \frac{k^2(\overline{\phi})}{4} + \frac{1-4m^2}{4\phi^2}\right) V = 0, \qquad 0 < \phi < \overline{\phi},$$

$$V = \sqrt{\phi} J_m \left(\sqrt{\left(\nu + \frac{1}{2}\right)^2 - \frac{k^2}{4}}\,\phi\right).$$

Let u be a solution of (2.5). From (2.11) and the Sturm Comparison Theorem, it follows that the kth zero of U occurs before the kth zero of u and the kth zero of u occurs before the kth zero of V. In particular, we have for $m \geq 1$,

$$\text{(2.12a)} \qquad \frac{j_k^m}{\sqrt{\left(\nu + \frac{1}{2}\right)^2 - \frac{h^2}{4}}} < \phi_{\nu,k}^m < \frac{j_k^m}{\sqrt{\left(\nu + \frac{1}{2}\right)^2 - \frac{k^2}{4}}},$$

where j_k^m is the kth positive zero of $J_m(z)$. If $m = 0$, we find that

$$\text{(2.12b)} \qquad \frac{j_k^0}{\sqrt{\left(\nu + \frac{1}{2}\right)^2 + \frac{\alpha}{4}}} < \phi_{\nu,k}^0 < \frac{j_k^0}{\sqrt{\left(\nu + \frac{1}{2}\right)^2 + \frac{1}{12}}}.$$

3. Ordering the ν-zeroes of Legendre functions

This section contains the principle results of this paper. We begin with a comparison of ν_2^3 and ν_1^6:

Theorem 1. $\nu_2^3(z_0) < \nu_1^6(z_0)$ *for all* $-1 < z_0 < 1$.

Proof. Here, it will be convenient to let $z_0 = \cos\phi_0$ and to consider ν_1^6 and ν_2^3 as functions of ϕ_0. First, we will show that if $\nu = \nu_1^6 = \nu_2^3$, then $\nu > 7$. Then, we will show that $\nu = \nu_1^6 = \nu_2^3$ is impossible if $\nu > 7$.

Part 1. Since $\nu_2^3(\phi_0)$ and $\nu_1^6(\phi_0)$ are decreasing in ϕ_0, by (2.8), we have

$$\lim_{\phi_0 \to \pi^-} \nu_2^3(\phi_0) = 4, \qquad \lim_{\phi_0 \to \pi^-} \nu_1^6(\phi_0) = 6.$$

Since $\nu_1^6(\phi_0) > 6$ for $\phi_0 \in (0, \pi)$ and $\nu_2^3(\phi_0) < 6$ for $\phi_0 \in (\pi/2, \pi)$, we see that if $\nu = \nu_2^3(\phi_0) = \nu_1^6(\phi_0)$ for some ϕ_0, then $\nu \geq 6$ and $0 < \phi_0 < \pi/2$.

Next, suppose that $\nu = \nu_2^3 = \nu_1^6$ for some $z_0 = \cos\phi_0$ and $0 < \phi_0 < \pi/2$. From (2.3) we see that

$$\text{(A)} \qquad \begin{pmatrix} 10z_0(1-z_0^2)^{-1/2} & (\nu-4)(\nu+5) \\ 1 & 8z_0(1-z_0^2)^{-1/2} \end{pmatrix} \begin{pmatrix} P_\nu^5(z_0) \\ P_\nu^4(z_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since by (1.1), $P_\nu^5(z_0)$ and $P_\nu^4(z_0)$ cannot vanish simultaneously, we see that the determinant of the 2×2 matrix in (A) must be zero, and we are led to the condition that

$$\text{(B)} \qquad z_0^2 = \frac{\nu(\nu+1) - 20}{\nu(\nu+1) + 60}.$$

From (B), we see that for $6 \le \nu < 7$, $\sqrt{11/51} \le z_0 < \sqrt{9/29}$. On the other hand, if we set $\overline{\phi} = \arccos\sqrt{11/51}$, $m = 6$, $k = 1$ and $j_1^6 \doteq 9.9361$ in (2.12a), we find that for such a ϕ_0 and ν,

$$z_0 = \cos\phi_0 = \cos(\phi_{\nu,1}^6) < \cos\left(\frac{j_1^6}{\sqrt{(7.5)^2 - 143/12}}\right) \doteq 0.0784,$$

which is a contradiction. Thus, we have shown that if $\nu = \nu_1^6 = \nu_2^3$, then $\nu \ge 7$. Moreover, we have also shown that if $\nu_1^6 = \nu_2^3$ for some ϕ_0, then $\phi_0 \le \arccos\sqrt{9/29}$.

Part 2. By the relationship in (B), we are motivated to define a function $z_0(\nu)$ for $\nu \ge 7$ as follows:

$$\text{(3.1)} \qquad z_0(\nu) = \left(\frac{\nu(\nu+1) - 20}{\nu(\nu+1) + 60}\right)^{1/2}.$$

For such a $z_0(\nu)$, we also define $\phi_0(\nu) = \arccos(z_0(\nu))$ and observe that

$$\text{(3.2)} \qquad \sin^2(\phi_0(\nu)) = \frac{80}{\nu(\nu+1) + 60}.$$

From (2.12a), (2.10), and (3.2), with $m = 6$, $k = 1$ and $\overline{\phi} = \arccos\sqrt{9/29}$, we see that for *all* $\nu \ge 7$,

$$\text{(3.3)} \qquad \begin{aligned} \sin^2(\phi_{\nu,1}^6) &> \sin^2\left(\frac{j_1^6}{\sqrt{(\nu+\frac{1}{2})^2 - \frac{h^2}{4}}}\right) \\ &\ge \frac{(j_1^6)^2}{\nu(\nu+1) + \frac{1}{4} - \frac{h^2}{4} + \alpha(\overline{\phi})(j_1^6)^2} \\ &> \frac{80}{\nu(\nu+1) + 60} \\ &= \sin^2(\phi_0(\nu)), \end{aligned}$$

where $j_1^6 \doteq 9.9361$. To complete the proof, we observe that if $\nu^* = \nu_1^6 = \nu_2^3$ for some $\nu^* \ge 7$, then necessarily we must have $\phi_{\nu^*,1}^6 = \phi_{\nu^*,2}^3 = \phi_0(\nu^*)$. However, from (3.3) we see that $\phi_0(\nu) < \phi_{\nu,1}^6$ for *all* $\nu \ge 7$. It follows that $\nu_1^6 \ne \nu_2^3$ for $0 < \phi_0 < \pi$. Since $\nu_2^3 < \nu_1^6$ for $\pi/2 \le \phi_0 < \pi$, we conclude that $\nu_2^3 < \nu_1^6$ for all $0 < \phi_0 < \pi$ (or equivalently, for all $z_0 \in (-1, 1)$).

As a consequence of (1.1)–(1.5) and Theorem 1, we see that the first eleven ν-zeroes are

(3.4)
$$\nu_1^0 < \nu_1^1 < \nu_1^2 < \nu_2^0 < \nu_1^3 < \nu_2^1 < \nu_1^4 < \nu_2^2 < \nu_3^0 < \nu_1^5 < \nu_2^3, \qquad 0 < z_0 < 1,$$

and that this ordering is unique. The inequalities $\nu_2^0 < \nu_1^3$, $\nu_2^1 < \nu_1^4$, and $\nu_3^0 < \nu_1^5$ were established in [2] for $0 \le z_0 < 1$. By applying (2.8) and arguing as we did at the beginning of the proof of Theorem 1, these inequalities can be shown to hold for $-1 \le z_0 < 0$ as well. In particular, we have the following:

Theorem 2. $\nu_2^0 < \nu_1^3$, $\nu_2^1 < \nu_1^4$, $\nu_3^0 < \nu_1^5$ *for all* $z_0 \in (-1, 1)$.

The inequality $\nu_j^{m+2} < \nu_{j+1}^m$ for $0 < z_0 < 1$ was established in [2]. Next, we consider the case $-1 < z_0 < 0$.

Lemma 2. *If* $-1 < z_0 < 0$, *then* $\nu_{j+1}^m < \nu_j^{m+2}$.

Proof. The ν_j^m's are simple zeroes of $P_\nu^m(z_0)$. (See [2].) From (1.1), we have that $\nu_{j+1}^m, \nu_j^{m+2} \in (\nu_j^{m+1}, \nu_{j+1}^{m+1})$. Suppose that $\nu_j^{m+2}(z_0) \le \nu_{j+1}^m(z_0)$ for some $z_0 \in (-1, 0)$. From (2.7)–(2.8), we see that

$$\operatorname{sign}(P_\nu^m(z_0)) = (-1)^{m+j}, \qquad \nu_j^m < \nu < \nu_{j+1}^m, \qquad z_0 \in (-1, 0).$$

The signs of $P_\nu^{m+1}(z_0)$ and $P_\nu^{m+2}(z_0)$ can also be determined in this way. We are led to the results summarized in Table 1.

TABLE 1. Suppose $\nu_j^{m+2}(z_0) \le \nu_{j+1}^m(z_0)$ for $z_0 \in (-1, 0)$

$z_0 \in (-1, 0)$	$\nu \in (\nu_j^{m+1}, \nu_j^{m+2})$	$\nu \in (\nu_j^{m+2}, \nu_{j+1}^m)$	$\nu \in (\nu_{j+1}^m, \nu_{j+1}^{m+1})$
$\operatorname{sign}(P_\nu^m(z_0))$	$(-1)^{m+j}$	$(-1)^{m+j}$	$(-1)^{m+j+1}$
$\operatorname{sign}(P_\nu^{m+1}(z_0))$	$(-1)^{m+j+1}$	$(-1)^{m+j+1}$	$(-1)^{m+j+1}$
$\operatorname{sign}(P_\nu^{m+2}(z_0))$	$(-1)^{m+j+1}$	$(-1)^{m+j+2}$	$(-1)^{m+j+2}$

The second column in Table 1 contradicts (2.3), and we conclude that $\nu_j^{m+2} > \nu_{j+1}^m$ for all $z_0 \in (-1, 0)$.

From (1.1)–(1.3) and Lemma 2, we are led to the following:

Theorem 3. *If* $\nu_j^m(z_0)$ *is a solution of* $(*)$, *then*

(i) $\nu_j^{m+1} < \nu_{j+1}^m < \nu_j^{m+2} < \nu_{j+1}^{m+1}$, $\quad -1 < z_0 < 0$,

(ii) $\nu_j^{m+2} = \nu_{j+1}^m = m + 2j - 1$, $\quad z_0 = 0$,

(iii) $\nu_j^{m+1} < \nu_j^{m+2} < \nu_{j+1}^m < \nu_{j+1}^{m+1}$, $\quad 0 < z_0 < 1$.

4. CONCLUDING REMARKS

Since the zeroes of the Bessel functions are distinct [9, p. 484] the elements of $\mathscr{J} = \{j_k^m\}$ can be arranged as an increasing sequence. In particular, we can

define integer-valued functions $m(i)$, $k(i)$ so that $j_{k(i)}^{m(i)}$ denotes the ith element in the sequence $\mathscr{J}$. Clearly, there is no such ordering of all the elements of $\mathscr{N}_{\phi_0} = \{\nu_j^m(\phi_0)\}$ that is independent of ϕ_0. On the other hand, if we let $\overline{\phi} = \phi_0 = \phi_{\nu,k}^m$ and $\nu = \nu_k^m(\phi_0)$ in (2.12), we see that

$$\lim_{\phi_0 \to 0^+} \phi_0 \left(\nu_k^m(\phi_0) + \frac{1}{2} \right) = j_k^m . \tag{4.1}$$

The limit in (4.1) is related to the well-known result, $\lim_{n\to\infty} \phi_{n,k}^0(n + \frac{1}{2}) = j_k^0$ (see [8]) and implies that for ϕ_0 sufficiently small, $\nu_{k(i)}^{m(i)}$ is the ith element in the sequence $\mathscr{N}_{\phi_0} = \{\nu_{j(i)}^{m(i)}(\phi_0)\}$.

In view of (1.6) and the first two inequalities in (1.4), it is natural to conjecture if there is an inequality that relates ν_{j+1}^m and ν_j^{m+3} for $\phi_0 \in (0, \pi/2)$. Such an inequality is not possible. From [1], we see that $j_{k(18)}^{m(18)} = j_1^8 = 12.225$, and $j_{k(19)}^{m(19)} = j_2^5 = 12.338$. Since $\nu_1^8(\pi/2) = 9$, $\nu_2^5(\pi/2) = 8$, and $j_1^8 < j_2^5$, we conclude that $\nu_1^8(\phi_0) = \nu_2^5(\phi_0)$ for some $\phi_0 \in (0, \pi/2)$. Numerical calculations indicate that $\nu_1^8 = \nu_2^5 \doteq 26.706$ when $\phi_0 \doteq 26.134°$.

Although (1.5) and Theorem 1 demonstrate that the first eleven ν-zeroes of $P_\nu^m(\cos\phi_0)$ are distinct for $0 < \phi_0 < \pi/2$, the twelfth ν-zero is not necessarily distinct. Since $\nu_3^1(\pi/2) = 6$, $\nu_1^6(\pi/2) = 7$, and $j_{k(12)}^{m(12)} = j_1^6 = 9.936$, and $j_{k(13)}^{m(13)} = j_3^1 = 10.173$, from (4.1), we see that $\nu_1^6(\phi_0) = \nu_3^1(\phi_0)$ for some $\phi_0 \in (0, \pi/2)$. Numerics indicate $\nu_1^6 = \nu_3^1 \doteq 15.780$ when $\phi_0 \doteq 35.821°$ (see [2]).

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