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ON THE ACTION OF STEENROD SQUARES ON POLYNOMIAL ALGEBRAS

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ABSTRACT. Let P_s be the mod-2 cohomology of the elementary abelian group $(Z/2Z) \times \cdots \times (Z/2Z)$ (s factors). The mod-2 Steenrod algebra A acts on P_s according to well-known rules. If $\mathbf{A} \subset A$ denotes the augmentation ideal, then we are interested in determining the image of the action $\mathbf{A} \otimes P_s \to P_s$: the space of elements in P_s that are hit by positive dimensional Steenrod squares. The problem is motivated by applications to cobordism theory [P1] and the homology of the Steenrod algebra [S]. Our main result, which generalizes work of Wood [W], identifies a new class of hit monomials.

1. INTRODUCTION

Let P_s be the mod-2 cohomology of the elementary abelian group $(Z/2Z) \times \cdots \times (Z/2Z)$ (s factors). Then P_s is a polynomial algebra:

$$P_s = F_2[t_1, t_2, \ldots, t_s] \tag{1.1}$$

on s generators, each of degree 1. The mod-2 Steenrod algebra A acts on P_s according to well-known rules. We write $\mathbf{A}P_s \subset P_s$ for the subvectorspace of "hit" elements; the subspace of those elements expressible as a finite sum $\sum_{j>0} Sq^j x_j$ for appropriate $x_j \in P_s$. We are concerned with the problem of determining this subspace. The papers [P1, S] provide motivation for this problem and suggest it is an important one. Our main result, Theorem 1.2, identifies a new class of monomials in $\mathbf{A}P_s$ and generalizes a result of R. Wood [W]. In its proof we use some of Wood's ideas.

To put our work in context, we begin by quoting one of Wood's results, a theorem that was originally conjectured by Peterson [P2]. Write $\alpha(n)$ for the number of ones in the binary expansion of n. Then:

Theorem 1.1 (R. Wood, [W]). *Suppose $x \in P_s$ is a monomial of degree δ, and suppose $\alpha[\delta + s] > s$. Then x is hit.*

So we ask: in those degrees δ for which $\alpha[\delta + s] \leq s$, which elements are

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hit? Theorem 1.2 identifies a class of such elements. First we introduce some language.

It is easy to show that $\alpha[\delta + s] \leq s$ if and only if δ can be written in the form $\delta = \sum_{i=1}^{s}(2^{p_i} - 1)$ for appropriate $p_i \geq 0$. Degrees of this form contain monomials we call *spikes*. z is a spike if $z = t_1^{2^{p_1}-1} \cdots t_s^{2^{p_s}-1}$ for appropriate $p_i \geq 0$. A given degree δ of P_s may contain several spikes. For example, in degree 27, P_5 contains $t_1^{15}t_2^7t_3^3t_4t_5$, $t_1^{15}t_2^3t_3^3t_4^3t_5^3$, $t_1^7t_2^7t_3^7t_4^3t_5^3$, and others obtained from these by permuting the variables. However, for each degree in which there are spikes, we will define in the next section a p· ticular one called the *minimal* spike. We need one more notation: if $n \geq 0$ is an integer, write $n = \sum_{i\geq 0}\alpha_i(n)2^i$ for its binary expansion. If $x \in P_s$ is a monomial, i.e., $x = t_1^{a_1} \cdots t_s^{a_s}$, define $\alpha_i(x)$ to be the integer $\alpha_i(x) = \sum_{j=1}^{s}\alpha_i(a_j)$. Our main result is:

Theorem 1.2. *Suppose $x \in P_s$ is a monomial of degree δ, where $\alpha[\delta + s] \leq s$. Let z be the minimal spike of degree δ. Suppose there is an integer $k \geq 0$ for which*

$$\begin{cases} \alpha_i(x) = \alpha_i(z) & \text{all } i < k, \\ \alpha_k(x) < \alpha_k(z). \end{cases} \tag{1.2}$$

Then x is hit.

For example in degree 27 of P_5, for which we have listed the spikes above, the minimal spike is $z = t_1^{15}t_2^7t_3^3t_4t_5$. The monomial $x = t_1^9t_2^{11}t_3^3t_4^3t_5$ satisfies (1.2) if $k = 2$, and so it is hit.

We discuss a little further the relationship of our theorem to Wood's work, for Wood's main result is actually more general than Theorem 1.1. In [W] he shows:

Theorem 1.3 (R. Wood, [W]). *Suppose $x \in P_s$ is a monomial of degree δ, and suppose $\alpha[\delta + \alpha_0(x)] > \alpha_0(x)$. Then x is hit.*

We will show that Theorem 1.3 implies the case $k = 0$ of our Theorem 1.2. But in the cases $k > 0$, Theorem 1.2 is new.

Finally we remark that these results do not completely solve the problem of determining $\mathbf{A}P_s$. In the first place, it is not enough to determine hit monomials; for there are hit elements which are not sums of hit monomials. The simplest example is $Sq^1(t_1t_2)$. In the second place, we have not even determined all hit monomials. For example, it can be shown by ad hoc arguments that $t_1^5t_2^5t_3^5t_4^5t_5^5$ is hit; but this fact is not implied by any of our general theorems. Hence there is still much work to be done.

2. Some tools

In this section we list some number theoretic lemmas and some properties of the Steenrod algebra that we need to prove Theorem 1.2. All the lemmas in

this section are given without proof; as they are either elementary or proven in the literature (in which case we give a reference).

We begin with properties of the functions α_i as defined in the introduction. Given monomials $x = t_1^{a_1} \cdots t_s^{a_s}$, $y = t_1^{b_1} \cdots t_s^{b_s}$, we say the pair $\{x, y\}$ has index m if $\alpha_i(a_j) + \alpha_i(b_j) \leq 1$ for all $j \leq s$ and all $i < m$, and if $\alpha_m(a_j) = \alpha_m(b_j) = 1$ for at least one j. If we allow $m = \infty$, then each pair of monomials has a unique index $m \geq 0$. If m is the index of $\{x, y\}$ then

$$\begin{cases} \alpha_i(xy) = \alpha_i(x) + \alpha_i(y) & 0 \leq i < m, \\ \alpha_m(xy) < \alpha_m(x) + \alpha_m(y) & \text{if } m < \infty. \end{cases} \tag{2.1}$$

If $\delta \geq 0$, $s \geq 1$ are integers, we say "δ is s-sharp" if δ can be written $\delta = \sum_{i=1}^{s}(2^{p_i} - 1)$ for appropriate $p_i \geq 0$.

Lemma 2.1. *An integer $\delta \geq 0$ is s-sharp if and only if $\alpha(\delta + s) \leq s$.*

By a "representation of δ as an s-sharp" we mean a finite ordered sequence $\{p_1, \ldots, p_s\}$ satisfying $\delta = \sum_{i=1}^{s}(2^{p_i} - 1)$. If in such a sum we have $p_{i_1} = p_{i_2}$ for some $i_1 \neq i_2$, then the representation of δ as an s-sharp need not be unique, not even up to permutation of the p_i's. However, we single out a "minimal" representation.

Definition 2.2. The sequence $\{p_1, \ldots, p_s\}$ is called the minimal representation of δ as an s-sharp if, in addition to $\delta = \sum_{i=1}^{s}(2^{p_i} - 1)$ we have both

$$p_1 \geq p_2 \geq \cdots \geq p_s \geq 0 \tag{2.2}$$

and

$$p_{i-1} = p_i \qquad \text{only if } i = s \text{ or } p_{i+1} = 0. \tag{2.3}$$

For example, the minimal representation of 17 as a 5-sharp is $\{4, 1, 1, 0, 0\}$.

It is easy to see that if δ is s-sharp it has a unique minimal representation. The term "minimal" is justified by:

Lemma 2.3. *Let δ be s-sharp, with minimal representation $\{p_1, \ldots, p_s\}$. Let $\{q_1, \ldots, q_s\}$ also be a representation of δ as an s-sharp, ordered so that $q_1 \geq q_2 \geq \cdots \geq q_s$. Let q_{i_0} be the smallest of the numbers $\{q_i\}$ for which $q_i \neq p_i$ (if there is such a number). Then $q_{i_0} > p_{i_0}$.*

We now give a paraphrase of Lemma 2.3 that does not assume any special ordering of the $\{q_i\}$.

Lemma 2.4. *Let δ be s-sharp, with minimal representation $\{p_1, \ldots, p_s\}$. Let $\{q_1, \ldots, q_s\}$ also be a representation of δ as an s-sharp. Then there is a unique integer d, $0 \leq d \leq \infty$, such that*

$$\alpha_i(t_1^{2^{q_1}} \cdots t_s^{2^{q_s}}) = \alpha_i(t_1^{2^{p_1}} \cdots t_s^{2^{p_s}}) \quad \text{if } i < d,$$
$$\alpha_d(t_1^{2^{q_1}} \cdots t_s^{2^{q_s}}) < \alpha_d(t_1^{2^{p_1}} \cdots t_s^{2^{p_s}}) \quad \text{if } d < \infty.$$

In fact, if we arrange the q_i's in descending order, as in Lemma 2.3, then d is the integer p_{i_0} of that lemma.

Suppose δ is r-sharp for some $r < s$. Since terms of the form $2^0 - 1$ can be added to any equation of the form $\delta = \sum_{i=1}^{r}(2^{p_i} - 1)$, we conclude δ is s-sharp as well. However:

Definition 2.5. An integer $\delta \geq 0$ is called strictly s-sharp if, in its minimal representation $\{p_1, \dots, p_s\}$, we have $p_i > 0$ for all $i \leq s$.

Lemma 2.6. *If δ is strictly s-sharp, then δ is not r-sharp for any $r < s$.*

This follows from Lemma 2.3.

To every representation $\{p_1, \dots, p_s\}$ of δ as an s-sharp there corresponds a spike:

$$z = t_1^{2^{p_1}-1} \cdots t_s^{2^{p_s}-1}. \tag{2.4}$$

Definition 2.7. z in (2.4) is called the *minimal spike* of degree δ if $\{p_1, \dots, p_s\}$ is the minimal representation of δ as an s-sharp.

The term minimal spike is justified by:

Lemma 2.8. *Let $\{p_1, \dots, p_s\}$ be the minimal representation of δ as an s-sharp, and let z in (2.4) be the corresponding minimal spike. Let $\{q_1, \dots, q_s\}$ also be a representation of δ as an s-sharp, and let z' be the corresponding spike. If there is an integer k for which $\alpha_k(z') \neq \alpha_k(z)$, let k_0 be the least such integer. Then $\alpha_{k_0}(z') > \alpha_{k_0}(z)$.*

An immediate consequence is:

Remark 2.9. If $x \in P_s$ is a monomial of degree δ which satisfies (1.2) for the minimal spike z of that degree, then x satisfies (1.2) for all spikes z' of that degree (although the integer k may depend on the spike).

Remark 2.10. We can now see that Wood's Theorem 1.3 implies the case $k = 0$ of our Theorem 1.2. In fact, suppose $x \in P_s$ satisfies the hypotheses of Theorem 1.2 in the case $k = 0$. Let x have degree δ, and let z be the minimal spike of this degree. Then δ is strictly r-sharp for some $r \leq s$, and $\alpha_0(x) < \alpha_0(z) = r$. If we had $\alpha(\delta + \alpha_0(x)) \leq \alpha_0(x)$, then by Lemma 2.1, δ would be $\alpha_0(x)$-sharp, contradicting Lemma 2.6. So we must have $\alpha(\delta + \alpha_0(x)) > \alpha_0(x)$, and Wood's Theorem 1.3 implies x is hit. So Theorem 1.2 is proved in the case $k = 0$.

Finally we quote two results about the Steenrod algebra that we will use often. We denote by $\chi : A \to A$ the canonical antiautomorphism.

Lemma 2.11. *Let* M *be any A-algebra, and suppose $x, y \in$* M. *Then for any $k \geq 0$,*

$$(\chi Sq^k x) \cdot y \equiv x \cdot (Sq^k y) \mod \mathbf{AM}.$$

Wood points this out on the last page of [W]. A special case of this formula is given by Adams in [A], and the general case was known to Brown and Peterson in the mid-sixties.

Finally we need some information on how χSq^j acts on P_1. The key formula is given by Brown and Peterson in [BP]:

$$\chi Sq^j(t_1) = \begin{cases} (t_1)^{2^p} & \text{if } j = 2^p - 1 \text{ for some } p \geq 0, \\ 0 & \text{otherwise}. \end{cases} \tag{2.5}$$

3. Proof of Theorem 1.2

As we have shown in Remark 2.10, the case $k = 0$ of Theorem 1.2 is implied by Wood's Theorem 1.3. So we will continue our proof by induction, taking $k \geq 1$ and assuming Theorem 1.2 already proved in the cases $0, 1, \dots, k-1$. Let $x \in P_s$ be a monomial of degree δ. Suppose $\alpha[\delta + s] \leq s$. Let z be the minimal spike of degree of δ and suppose (1.2) holds. We must show x is hit. Since $\alpha[\delta + s] \leq s$, δ is strictly r-sharp for some $r \leq s$, so the minimal representation of δ as an s-sharp is $\{p_1, \dots, p_s\}$, with $\delta = \sum_{i=1}^{s}(2^{p_i} - 1) = \sum_{i=1}^{r}(2^{p_i} - 1)$ and

$$p_1 > p_2 > \cdots > p_{r-1} \geq p_r > 0 \tag{3.1}$$

and $p_{r+1} = \cdots = p_s = 0$. Then the minimal spike is:

$$z = t_1^{2^{p_1}-1} \cdots t_r^{2^{p_r}-1}. \tag{3.2}$$

Then $\alpha_0(x) = \alpha_0(z) = r$, so without loss of generality we may suppose:

$$x = t_1^{a_1} \cdots t_s^{a_s} = t_1^{2b_1+1} \cdots t_r^{2b_r+1} (t_{r+1})^{2b_{r+1}} \cdots (t_s)^{2b_s}. \tag{3.3}$$

By Lemma 2.11 we can write:

$$\begin{aligned} x &= y^2(t_1 \cdots t_r) = (Sq^f y)(t_1 \cdots t_r) \\ &\equiv y(\chi Sq^f)(t_1 \cdots t_r) \mod \mathbf{A}P_s, \end{aligned} \tag{3.4}$$

where

$$y = t_1^{b_1} \cdots t_s^{b_s} \tag{3.5}$$

and

$$f = \deg y = \frac{\delta - r}{2} = \sum_{i=1}^{r}(2^{p_i - 1} - 1). \tag{3.6}$$

It is clear from (3.1) that $\{p_1 - 1, \dots, p_r - 1\}$ is the minimal representation of f as an r-sharp. We also note

$$\alpha_i(y) = \alpha_{i+1}(x) \qquad (\forall i \geq 0). \tag{3.7}$$

Since $\chi : A \to A$ is a morphism of coalgebras we have from (2.5) and (3.4):

$$x \equiv \sum_{\{q_1, \dots, q_r\}} y(\chi Sq^{2^{q_1}-1} t_1) \cdots (\chi Sq^{2^{q_r}-1} t_r) \mod \mathbf{A}P_s \tag{3.8}$$

or finally:

$$x \equiv \sum_{\{q_1, \dots, q_r\}} y(t_1^{2^{q_1}} \cdots t_r^{2^{q_r}}) \mod \mathbf{A}P_s, \tag{3.9}$$

where the sum is over all representations $\{q_1, \dots, q_r\}$ of f as an r-sharp. We write:

$$w = w(q_1, \dots, q_r) = y(t_1^{2^{q_1}} \cdots t_r^{2^{q_r}}) \tag{3.10}$$

for a typical term in (3.9). We will assume $\{q_1, \dots, q_r\}$ given and show that $w(q_1, \dots, q_r)$ is hit; this will complete our proof.

Since $\{p_1 - 1, \dots, p_r - 1\}$ is the minimal representation of f as an r-sharp, we know from Lemma 2.4 that there exists a unique integer d, with $0 \le d \le \infty$, such that

$$\begin{cases} \alpha_i(t_1^{2^{q_1}} \cdots t_r^{2^{q_r}}) = \alpha_i(t_1^{2^{p_1-1}} \cdots t_r^{2^{p_r-1}}) & \text{if } i < d, \\ \alpha_d(t_1^{2^{q_1}} \cdots t_r^{2^{q_r}}) < \alpha_d(t_1^{2^{p_1-1}} \cdots t_r^{2^{p_r-1}}) & \text{if } d < \infty. \end{cases} \tag{3.11}$$

Let m be the index of the pair of monomials $(y, t_1^{2^{q_1}} \cdots t_r^{2^{q_r}})$, as defined in the previous section. Our proof breaks into three cases, depending on which of the integers m, d, or $k-1$ is smallest.

Case 1. $m = \min\{m, d, k-1\}$.

In this case, for each $i < m$ we have from (2.1), (3.10), (3.7), and (3.11):

$$\alpha_i(w) = \alpha_i(y) + \alpha_i(t_1^{2^{q_1}} \cdots t_r^{2^{q_r}}) = \alpha_{i+1}(x) + \alpha_i(t_1^{2^{p_1-1}} \cdots t_r^{2^{p_r-1}}). \tag{3.12}$$

But since $i < m$ we also have $i + 1 < k$, so by assumption (1.2), $\alpha_{i+1}(x) = \alpha_{i+1}(z)$. So with the aid of (3.2), (3.12) becomes

$$\alpha_i(w) = \alpha_{i+1}(z) + \alpha_{i+1}(t_1^{2^{p_1}} \cdots t_r^{2^{p_r}}) = \alpha_i(z) \qquad \forall i < m. \tag{3.13}$$

On the other hand when $i = m$, (2.1), (3.10), (3.7), and (3.11) give:

$$\alpha_m(w) < \alpha_m(y) + \alpha_m(t_1^{2^{q_1}} \cdots t_r^{2^{q_r}}) \le \alpha_{m+1}(x) + \alpha_m(t_1^{2^{p_1-1}} \cdots t_r^{2^{p_r-1}}). \tag{3.14}$$

In this case hypothesis (1.2) gives $\alpha_{m+1}(x) \le \alpha_{m+1}(z)$, so with the aid of (3.2), (3.14) becomes

$$\alpha_m(w) < \alpha_{m+1}(z) + \alpha_{m+1}(t_1^{2^{p_1}} \cdots t_r^{2^{p_r}}) = \alpha_m(z). \tag{3.15}$$

From (3.13) and (3.15) we see that the monomial w satisfies (1.2), with k replaced by m. Since $m < k$, our inductive hypothesis implies w is hit, and we are done.

Case 2. $d = \min\{m, d, k-1\}$.

In this case we again obtain (3.12) from (2.1), (3.10), (3.7), and (3.11); only this time, (3.12) is valid for all $i < d$. But if $i < d$ we have $i + 1 < k$; so

(3.13) follows from (3.12), (1.2), and (3.2) just as above, only this time for all $i < d$. On the other hand when $i = d$, (2.1), (3.10), (3.7), and (3.11) give:

$$\alpha_d(w) \leq \alpha_d(y) + \alpha_d(t_1^{2^{q_1}} \cdots t_r^{2^{q_r}}) < \alpha_{d+1}(x) + \alpha_d(t_1^{2^{p_1-1}} \cdots t_r^{2^{p_r-1}}). \tag{3.16}$$

Since $d + 1 \leq k$ the hypothesis (1.2) gives $\alpha_{d+1}(x) \leq \alpha_{d+1}(z)$, so with the aid of (3.2), (3.16) becomes

$$\alpha_d(w) < \alpha_{d+1}(z) + \alpha_{d+1}(t_1^{2^{p_1}} \cdots t_r^{2^{p_r}}) = \alpha_d(z). \tag{3.17}$$

In view of the validity of (3.13) for all $i < d$, and in view of (3.17), we conclude that the monomial w satisfies (1.2) with k replaced by d. Since $d < k$ our inductive assumption implies w is hit, and we are done.

Case 3. $k - 1 = \min\{m, d, k - 1\}$.

This time (3.12) follows from (2.1), (3.10), (3.7), and (3.11) for all $i < k-1$. (3.13) follows in turn; from (1.2) and (3.2), for all $i < k - 1$. On the other hand, when $i = k - 1$, (2.1), (3.10), (3.7), and (3.11) give:

$$\alpha_{k-1}(w) \leq \alpha_{k-1}(y) + \alpha_{k-1}(t_1^{2^{q_1}} \cdots t_r^{2^{q_r}}) \leq \alpha_k(x) + \alpha_{k-1}(t_1^{2^{p_1-1}} \cdots t_r^{2^{p_r-1}}). \tag{3.18}$$

But by hypothesis (1.2) we have $\alpha_k(x) < \alpha_k(z)$, so with the aid of (3.2), (3.18) becomes

$$\alpha_{k-1}(w) < \alpha_k(z) + \alpha_k(t_1^{2^{p_1}} \cdots t_r^{2^{p_r}}) = \alpha_{k-1}(z). \tag{3.19}$$

In view of the validity of (3.13) for all $i < k - 1$, and in view of (3.19), we see that the monomial w satisfies (1.2) with k replaced by $k - 1$. So our inductive hypothesis implies w is hit, and we are done.

This completes the proof of Theorem 1.2.

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