CONVERGENCE OF ARGUMENTS
OF BLASCHKE PRODUCTS IN $L_p$-METRICS

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Abstract. It is shown that the naturally defined argument of a Blaschke product is a function which is the harmonic conjugate of an integrable function of constant sign. A direct construction of this function is obtained. This fact allows us to investigate the relation between conditions on the zeros of a Blaschke product and the convergence of the arguments of its partial finite subproducts in $L_p$-metrics, $0 < p \leq \infty$.

1. Introduction

Let $D \equiv \{z: |z| < 1\}$, $T \equiv \partial D$. Given $\{z_k\} \equiv \sigma \subset D$ (the points $\{z_k\}$ are enumerated taking into account their multiplicity), we construct the Blaschke product

\[ B(z) = \prod_{k} b_k(z), \quad b_k(z) = \frac{z_k - z}{1 - \overline{z_k} z}, \quad z \in D. \]

We suppose for the sake of simplicity that $0 \notin \sigma$. It is well known that the product (1.1) converges in $D$ iff the Blaschke condition

\[ \sum_{k} (1 - |z_k|) < \infty \quad \text{or} \quad \sum_{k} \log \frac{1}{|z_k|} < \infty \]

is satisfied.

When investigating the spectral shift function and the trace formula for non-selfadjoint operators [8] and in some problems of complex analysis, it is important to consider the log $B(z)$. It is natural to define $\log B(z)$ on the disc $D$ with radial cuts from $z_k$ to $\xi_k \equiv \frac{z_k}{|z_k|}$ in order to have

\[ \arg(B_1 B_2) = \arg B_1 + \arg B_2. \]

We are going to consider the behavior of $\log B(z)$ on $T$. This function can be represented in the form of a series (see (2.2)). The direct investigation of
this series is difficult. The basis for our analysis is that this series represents a function which is the harmonic conjugate of some integrable function and which can be exactly constructed from $\sigma$ (Theorem 2). This fact allows us to study the convergence of the series constructed in the different spaces $L_p$, $0 < p \leq \infty$, in terms of $\sigma$.

Let $f(\xi) = f(e^{i\phi}) \equiv f(\phi)$ be the boundary values of $f(z)$, $z \in \mathbb{D}$ on $\mathbb{T}$, and let $\tilde{f}$ be the harmonic conjugate of $f$. The symbol $C$ will be used to denote nonessential constants. In what follows we are going to use the standard notation of [3, 5].

2. The choice of the value of the argument: the basic theorem

Let $\nu_k(\phi) \equiv \arg b_k(\phi)$ be the branch of the argument of the Blaschke factor, which is fixed by the condition $\nu_k(\phi_k \pm 0) = \mp \pi$, where $\phi_k \equiv \arg z_k$. Then (1.1) leads to the formula

$$\tan \nu_k(\phi) = \frac{(1 - |z_k|^2) \sin(\phi - \phi_k)}{2|z_k| - (1 + |z_k|^2) \cos(\phi - \phi_k)}.$$ 

Hence,

$$(2.1) \quad \nu_k(\phi) = 2 \arctan \left\{ \frac{1 - |z_k|}{1 + |z_k|} \cot \frac{\phi - \phi_k}{2} \right\}.$$ 

Definition. The formal series

$$(2.2) \quad \nu(\phi) = \arg B(\phi) = \sum_k \nu_k(\phi)$$

is called the principal value $\nu(\phi)$ of the argument of the Blaschke product $B$ on $\mathbb{T}$.

If the series (2.2) converges in some sense, then the natural condition (1.3) is obviously satisfied.

We now consider the product

$$(2.3) \quad G(z) = \left( \prod_k g_k(z) \right)^2, \quad g_k(z) = \frac{\xi_k - z}{1 - \overline{z_k}z}$$

in $\mathbb{D}$.

Theorem 1. The product $G(z)$ converges in $\mathbb{D}$ iff the Blaschke condition is satisfied. The function $G(z)$ is an outer function and $\|G\|_\infty < 1.$

Proof. The first part of the theorem is a direct consequence of the equality

$$(2.4) \quad 1 - g_k(z) = 1 - \frac{\xi_k - z}{1 - \overline{z_k}z} = \frac{1 - |z_k|^2}{1 - \overline{z_k}z}.$$ 

Henceforth, $\|f\|_p = \left( \frac{1}{2\pi} \int_0^{2\pi} |f|^p \, d\phi \right)^{1/p}, \quad 0 < p < \infty, \quad \|f\|_\infty = \sup |f(\phi)|.$
The factors $g_k^2(z)$ are obviously outer functions, and consequently

$$g_k^2(z) = e^{u_k(z) + i\tilde{u}_k(z)}, \quad z \in \mathbb{D},$$

where $u_k(z)$ is the Poisson integral of the function

$$u_k(\varphi) \equiv \log |g_k^2(z)| = -\log \left\{ \frac{1}{|z_k|} \left( 1 + \frac{(1 - |z_k|^2)^2}{4|z_k|^2 \sin^2 \frac{\varphi \cdot \varphi_k}{2}} \right) \right\} < 0.$$  

By the Cauchy theorem,

$$\frac{1}{2\pi i} \int_T \{u_k(\xi) + i\tilde{u_k}(\xi)\} \frac{d\xi}{\xi} = \log g_k^2(0) = -2 \log \frac{1}{|z_k|}.$$  

But the left-hand side of (2.6) is equal to $\int_0^{2\pi} u_k(\varphi) \frac{d\varphi}{2\pi} = -\|u_k\|_1$ i.e. $\|u_k\|_1 = 2\log 1/|z_k|$. Consequently, the series $\sum_k u_k(\varphi)$ converges in $L_1$ to some integrable function $u(\varphi)$ and

$$G(z) = \exp \left\{ \sum_k u_k(z) + i\tilde{u_k}(\varphi) \right\} = \exp\{u(z) + i\tilde{u}(z)\}$$

is an outer function. In view of (2.5) we have $\|G\|_\infty < 1$. Q.E.D.

The utility of $G$ is demonstrated by

**Theorem 2.** The following factorization

$$B(\xi) = \frac{G(\xi)}{|G(\xi)|}$$

holds a.e. on $T$, i.e.

$$\arg B(\varphi) = \nu(\varphi) = \tilde{u}(\varphi),$$

$$u(\varphi) = -\sum_k \log \left\{ \frac{1}{|z_k|} \left( 1 + \frac{(1 - |z_k|^2)^2}{4|z_k|^2 \sin^2 \frac{\varphi \cdot \varphi_k}{2}} \right) \right\}.$$  

**Proof.** It follows from

$$\log g_k^2(\varphi) = \log |g_k(\varphi)|^2 + 2i \arg g_k(\varphi) = u_k(\varphi) + i\tilde{u_k}(\varphi),$$

that

$$\tilde{u}_k(\varphi) = 2 \arg g_k(\varphi) = 2 \arg \frac{1 - e^{i(\varphi \cdot \varphi_k)}}{1 - |z_k|e^{i(\varphi \cdot \varphi_k)}} = 2 \arctan \left\{ \frac{1 - |z_k|}{1 + |z_k|} \cot \frac{\varphi \cdot \varphi_k}{2} \right\} = v_k(\varphi).$$
Hence, by M. Riesz's theorem about conjugate functions,

\[ \left\| \tilde{u} - \sum_{k=1}^{n} u_k \right\|_p \leq C_p \left\| \sum_{k=n+1}^{\infty} u_k \right\|_1, \quad 0 < p < 1. \]

Thus, the series (2.2) converges in \( L_p \), \( 0 < p < 1 \), to the function \( \tilde{u} \) and (2.8) is satisfied a.e. Next, according to a theorem by J. Walsh (see [2]), we have \( \ell^2 \cdot \prod_{k=1}^{n} b_k(\varphi) = B(\varphi) \) in the sense of convergence in \( L_2 \) where \( B(\varphi) = \lim_{r \to 1^-} B(re^{i\varphi}) \), from which (2.7) easily follows. Q.E.D.

Note that the possibility of factorization (2.7) with some \( G \in H^1 \) follows from the Adamjan-Arov-Krein theorem [3, 5].

Formula (2.8) plays an important role in the analysis of series (2.2).

3. Conditions for convergence of the argument

**Theorem 3.** (i) The series (2.2) converges in measure to a function \( \nu \), \( \nu \in \bigcap_{p<1} L_p \), and

\[ \|\nu\|_p \leq C_p \sum_k \log \frac{1}{|z_k|}, \quad 0 < p < 1. \]

(ii) The series (2.2) converges a.e. to \( \nu \) and \( \nu \in L_1 \) if

\[ \sum_k (1 - |z_k|) \log \frac{1}{1 - |z_k|} < \infty. \]

In this case

\[ \|\nu\|_1 \leq C \sum_k (1 - |z_k|) + 2 \sum_k (1 - |z_k|) \log \frac{1}{1 - |z_k|}. \]

**Proof.** The statement (i) is already proved. The statement (ii) can be obtained by the direct calculation of \( \|\nu_k\|_1 \):

\[ \|\nu_k\|_1 \leq (2 + \log 2\pi^2)(1 - |z_k|) + 2(1 - |z_k|) \log \frac{1}{1 - |z_k|} \]

Q.E.D.

Note that if condition (3.2) fails [6], it is possible to select \( \{\varphi_k\} \) so that the resulting series (2.2) diverges at every point of \( \mathbb{T} \).

The following statement can be obtained by the direct calculation of \( \|\nu_k\|_p \).

**Theorem 4.** The series (2.2) converges in \( L_p \), \( p > 1 \), if \( \sum_k (1 - |z_k|)^{1/p} < \infty \). In this case,

\[ \|\nu\|_p \leq C_p \sum_k (1 - |z_k|)^{1/p}, \quad p > 1. \]

**Theorem 5.** Let \( \xi \in \mathbb{T} \) and let \( \Gamma_\theta(\xi) = \{z \in \mathbb{D} : |\xi - z| \cos \theta \leq 1 - |z|\} \) be the Stolz angle. If

\[ \sup_{\xi \in \mathbb{T}} \text{card}\{z_k \in \Gamma_\theta(\xi)\} = N < \infty \]

for some \( 0 < \theta < \pi/2 \), then \( \nu \in \bigcap_{p>0} L_p \).
Proof. Condition (3.3) means that the set \( \sigma \) can be decomposed into \( N \) subsequences such that each of them has at most one point in any angle \( \Gamma_\theta(\xi) \). These subsequences are interpolating [4]. Therefore it is sufficient to prove Theorem 5 in the case \( N = 1 \). Let us complete our set \( \sigma \) with points \( \{z'_k\} \) so that \( \text{card}\{z_k, z'_k \in \Gamma_\theta(\xi)\} = 1 \) (see Figure 1). Obviously it is sufficient to prove the statement for \( \sigma' = \sigma \cup \{z'_k\} \), which we simply denote again as \( \sigma(= \{z_k\}) \).

Let us represent the function \( u(\phi) = \sum_k u_k(\phi) \) in the form

\[
(3.4) \quad u(\phi) = u_{j_\phi}(\phi) + \sum_{k \neq j_\phi} u_k(\phi),
\]

where \( j_\phi \) is the index of the arc \( J_k \) for which \( \phi \in J_k \). We have

\[
u_{j_\phi}(\phi) = \log \left\{ \frac{1}{|z_k|} \left( 1 + \frac{(1 - |z_{j_\phi}|)^2}{4|z_{j_\phi}| \sin^2 \frac{1}{4}(\phi - \phi_{j_\phi})} \right) \right\} \in L_p, \quad \forall p > 0.
\]

The sum in (3.4) can be estimated as follows:

\[
\sum_{k \neq j_\phi} u_k(\phi) \leq \sum_k \log \frac{1}{|z_k|} + \sum_{k \neq j_\phi} \frac{(1 - |z_k|)^2}{4|z_k| \sin^2 \frac{\phi - \phi_k}{2}}.
\]

Figure 1 shows that for \( \phi \notin J_k \), \( (\phi - \phi_k)^2 \geq \frac{1}{4} J_k^2 = C_\theta(1 - |z_k|)^2 \), where \( C_\theta \simeq \tan^2 \theta \). That is why (for the sufficient choice of the constant \( C \), which depends on \( \theta \) only)

\[
4|z_k| \sin^2 \frac{\phi - \phi_k}{2} \geq \frac{1}{C} \left( 4|z_k| \sin^2 \frac{\phi - \phi_k}{2} + (1 - |z_k|)^2 \right).
\]
Hence

\[(3.5) \quad \sum_{k \neq \phi} u_k(\phi) \leq \sum_k \log \frac{1}{|z_k|} + C \sum_{k \neq \phi} (1 - |z_k|) P_z(\phi),\]

where \(P_z(\phi) = \frac{1 - |z|^2}{1 - \bar{z}e^{\phi}}\) is the Poisson kernel. Now it is sufficient to note that

\[(3.6) \quad \sum_{k \neq \phi} (1 - |z_k|) P_z(\phi) \leq \sum_k (1 - |z_k|) P_z(\phi) \]

\[= \int_{\mathbb{D}} P_z(\phi) d\mu(z),\]

where \(\mu\) is the measure on \(\mathbb{D}\) given by \(\mu(\Omega) = \sum_{z_k \in \Omega} (1 - |z_k|)\). Since \(\{z_k\}\) is an interpolating sequence, the measure \(\mu\) is of Carleson type [3, 5]. So [3] the integral in (3.6) determines a function in \(\text{BMO} \subseteq \text{L}_p\). That is why the right-hand side of (3.5) also belongs to \(\bigcap_{p>0} L_p\). The theorem is proved, because \(\nu = \hat{u}\).

Let us represent one more statement which, in fact, is the Frostman’s theorem [1].

**Theorem 6.** The series (2.2) converges absolutely and uniformly to a function \(\nu \in \text{L}_\infty\) iff the Frostman condition,

\[\sup_{\xi \in \mathbb{T}} \sum_k \frac{1 - |z_k|}{|1 - \bar{z}_k \xi|} < \infty,\]

is satisfied.

**Proof.** Since \(|\arg g_k| \leq \frac{\pi}{2}\), we have

\[|\nu_k(\phi)| = 2|\arg g_k(\phi)| \leq \pi \sin|\arg g_k(\phi)|\]

\[= \frac{1 - |z_k|}{|1 - \bar{z}_k \xi|} \pi \cos \frac{\phi - \phi_k}{2} \leq \frac{1 - |z_k|}{|1 - \bar{z}_k \xi|}.\]

Now for the converse. Obviously, the behavior of the function \(\nu\) near the point \(\phi\) is determined only by those points \(z_k \in \sigma\), for which \(\cos \frac{\phi - \phi_k}{2} \geq C > 0\). Then for such a \(\phi\)

\[|\nu_k(\phi)| \geq 2 \sin|\arg g_k(\phi)| = \frac{2(1 - |z_k|)}{|1 - \bar{z}_k \xi|} \cos \frac{\phi - \phi_k}{2}\]

\[\geq 2C \frac{1 - |z_k|}{|1 - \bar{z}_k \xi|}. \quad \text{Q.E.D.}\]

**4. Converse statements**

Theorem 5 shows that condition (3.2) does not follow from the summability of \(\arg B\) because the assumptions of Theorem 5 pose no other restriction on \(|z_k|\) except for the Blaschke condition (1.2) (the lemma of Naftalevich [7]). However, the following theorem then holds.
Theorem 1. If $\nu = \arg B \in L_1$, then $\forall \Gamma_\theta(\xi)$

$$\sum_{z_k \in \Gamma_\theta(\xi)} (1 - |z_k|) \log \frac{1}{1 - |z_k|} < \infty.$$  

Proof. From the obvious chain of inequalities,

$$\text{Re} \frac{1 - |z_k|}{1 - z_k e^{i\varphi}} = (1 - |z_k|) \left( \frac{1 - |z_k|}{1 + |z_k|} \right) e^{\nu_k(\varphi)}$$

$$= 1 - |z_k| \left( 1 - \frac{1 - |z_k|}{2} \right) e^{\nu_k(\varphi)}$$

$$\leq 1 - \frac{1}{2} \left( \frac{1 - |z_k|}{2} \right) (1 + \nu_k(\varphi))$$

$$\leq 1 - \frac{1}{2} (1 + |z_k|) \nu_k(\varphi),$$

we can conclude that the sum of rational fractions,

$$F(z) = \sum_k \frac{1 - |z_k|}{1 - \overline{z}_k z},$$

belongs to the space $H^1$ together with $\log G(z)$.

We use the duality of the spaces $H^1$ and BMO (see [3, 5]), which provides that for every function $f \in \text{BMO A} = \text{BMO} \cap H^2$ and $F \in H^1$, there exists a finite limit

$$\lim_{r \to 1} \frac{1}{2\pi i} \int_0^{2\pi} f(e^{i\varphi}) \overline{F(re^{i\varphi})} d\varphi,$$

which determines a linear continuous functional on $H^1$. For our function (4.2) this limit is equal to

$$\lim_{r \to 1} \frac{1}{2\pi} \int_\pi \sum_k \frac{1 - |z_k|}{\xi - rz_k} f(\xi) d\xi = \sum_k (1 - |z_k|) f(z_k).$$

That is, for every $f \in \text{BMO A}$,

$$\left| \sum_k (1 - |z_k|) f(z_k) \right| < \infty.$$  

Because $\nu \in L_1$, $\nu = \delta u, u \in L_1, u < 0$, every term of the series $\nu$ also belongs to $L_1$ (the direct consequence of Zygmund’s theorem about the $L \log L$ class [3, 5]). That is why inequality (3.4) is valid for every subsequence $\sigma \cap \Gamma_\theta(\xi)$. Let us take $f(z) = \log \frac{2}{\xi - z} \in \text{BMO A}$. Then (4.3) leads to the inequality

$$\sum_{z_k \in \Gamma_\theta(\xi)} (1 - |z_k|) \log \left| \frac{2}{\xi - z_k} \right| < \infty,$$

which is equivalent to the original one in the Stolz angle. Q.E.D.
Remark. Duality allows us also to obtain uniformly the necessary condition for \( \arg B \in L_1 \):

\[
\sup_{\zeta \in \Gamma} \sum_k (1 - |z_k|) \log \left| \frac{2}{1 - \overline{z_k} \zeta} \right| < \infty.
\]

Now we obtain the converse statement to Theorem 4. As in the previous theorem, using the duality \( H^p - H^q \), \( \frac{1}{p} + \frac{1}{q} = 1 \) and the test function \( (e > 0) \)

\[
f(z) = (\xi - z)^{-1/q} \left( \log \frac{2}{\xi - z} \right)^{-(1+e)/q} \in H^q \quad (\xi \in H^q', \quad q' > q),
\]

one can prove the following

**Theorem 8.** If \( \arg B \in L_p \), \( p > 1 \), then \( \forall \Gamma_{\theta}(\xi) \),

\[
\sum_{z_k \in \Gamma_{\theta}(\xi)} (1 - |z_k|)^{1/p} \left( \frac{1}{1 - |z_k|} \right)^{-(1+e)/q} < \infty, \quad q = \frac{p}{p - 1}.
\]

**Remark 1.** One can see that in Theorems 7 and 8, the Stolz angle \( \Gamma_{\theta}(\xi) \) can be replaced by the orocycle

\[
C_{\alpha}(\xi) \equiv \{ z \in \mathbb{D} : |\xi - z|^2 < \alpha(1 - |z|^2), \quad \alpha > 0 \}
\]

as well as by any domain of the form

\[
C_{\alpha}^\gamma(\xi) \equiv \{ z \in \mathbb{D} : |\xi - z|^\gamma < \alpha(1 - |z|), \quad \gamma > 1 \}.
\]

**Remark 2.** If \( \exists \theta \) such that \( \sigma \in \bigcup_{k=1}^N \Gamma_{\theta}(\xi_k) \), \( N < \infty \), then Theorems 7 and 8 are valid for all \( \sigma \). However, for \( N = \infty \) this is obviously not true (a counterexample is provided by Theorem 5).

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**References**


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