

## FUNCTORIAL FINITE SUBCATEGORIES OVER TRIANGULAR MATRIX RINGS

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**ABSTRACT.** Let  $\Lambda$  and  $\Gamma$  be Artin algebras,  $M$  a  $\Gamma$ - $\Lambda$ -bimodule, and  $R$  the triangular matrix ring of  $\Lambda$ ,  $\Gamma$ , and  $M$ ; assume that  $R$  is also an Artin algebra. The  $R$ -modules are triples  $(U, V, f)$  where  $U$  is a  $\Lambda$ -module,  $V$  is a  $\Gamma$ -module, and  $f$  is a  $\Gamma$ -homomorphism from  $M \otimes U$  to  $V$ . For an Artin algebra  $S$ , let  $\text{mod } S$  denote the category of finitely generated  $S$ -modules. For full subcategories  $\mathcal{S}$  of  $\text{mod } \Lambda$  and  $\mathcal{T}$  of  $\text{mod } \Gamma$ , let  $\text{mod } R_{\mathcal{G}}^{\mathcal{S}}$  denote the full subcategory consisting of the modules  $(U, V, f)$ , where  $U$  is in  $\mathcal{S}$  and  $V$  is in  $\mathcal{T}$ . In this paper it is proved that  $\text{mod } R_{\mathcal{G}}^{\mathcal{S}}$  is functorially finite in  $\text{mod } R$  if and only if  $\mathcal{S}$  is functorially finite in  $\text{mod } \Lambda$  and  $\mathcal{T}$  is functorially finite in  $\text{mod } \Gamma$ . Using this result, we increase the known examples of functorially finite subcategories considerably, hence also the classes of subcategories having relative almost split sequences.

### INTRODUCTION

In studying the category of representations of a partially ordered set, the category of lattices over an order, or other similar structures, one often gets involved in studying full subcategories of the category of modules over an Artin algebra. Usually these subcategories are additive, with splitting idempotents satisfying some additional hypothesis, such as being closed with respect to extensions. Therefore the questions of existence of almost split sequences arise naturally in connection with the presentation of both the simple contravariant and covariant functors. In this connection the notion of functorial finiteness from [AS1] is of importance.

The aim of this note is to show that the functorial finiteness of subcategories is preserved under some triangular matrix ring constructions. It can also be generalized to the setting of comma categories. Using this result for Artin algebras, the existence of almost split sequences for the category of preinjective modules studied by Grecht [Gr] and de la Peña and Simson [PS] follows immediately. This result will increase considerably the number of known examples of other types of subcategories having almost split sequences.

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## 1. PRELIMINARIES

In this section we recall the definitions and notions needed in the rest of the paper and fix the notation.

For a ring  $S$ , we let  $\text{mod } S$  denote the category of finitely presented left  $S$ -modules. Let  $\Lambda$  and  $\Gamma$  be Artin  $T$ -algebras, where  $T$  is a commutative Artin ring, and let  ${}_{\Gamma}M_{\Lambda}$  be a  $\Gamma$ - $\Lambda$ -bimodule, where the induced actions of  $T$  through  $\Lambda$  and  $\Gamma$  commutes and  $M$  is finitely generated as a  $T$ -module. Then form the triangular matrix ring  $R$  given by

$$R = \left\{ \begin{pmatrix} \lambda & 0 \\ m & \gamma \end{pmatrix} \mid \lambda \in \Lambda, m \in {}_{\Gamma}M_{\Lambda}, \text{ and } \gamma \in \Gamma \right\},$$

where the operations are the usual matrix operations. It is clear that  $R$  is also an Artin  $T$ -algebra. A left  $R$ -module may be viewed as a triple  $(U, V, f)$ , where  $U$  is a  $\Lambda$ -module,  $V$  is a  $\Gamma$ -module, and  $f$  is a  $\Gamma$ -homomorphism from  ${}_{\Gamma}M_{\Lambda} \otimes U$  to  $V$ . As an Abelian group,  $(U, V, f)$  is just  $U \times V$ , and the multiplication with an element from  $R$  is given by

$$\begin{pmatrix} \lambda & 0 \\ m & \gamma \end{pmatrix} (u, v) = (\lambda u, mf(u) + \gamma v),$$

for  $\begin{pmatrix} \lambda & 0 \\ m & \gamma \end{pmatrix} \in R$  and  $(u, v) \in U \times V$ . An  $R$ -homomorphism  $\phi: (U, V, f) \rightarrow (U', V', f')$  is a pair  $(\alpha, \beta)$ , where  $\alpha: U \rightarrow U'$  is a  $\Lambda$ -homomorphism and  $\beta: V \rightarrow V'$  is a  $\Gamma$ -homomorphism such that  $f'(M \otimes \alpha) = \beta f$ . An alternative description can be obtained by using the adjoint isomorphism

$$\text{Hom}_{\Gamma}({}_{\Gamma}M_{\Lambda} \otimes U, V) \simeq \text{Hom}_{\Lambda}(U, \text{Hom}_{\Gamma}({}_{\Gamma}M_{\Lambda}, V)).$$

A description of the left  $R$ -modules is then obtained as a triple  $(U, V, f)$ , where  $U$  is a  $\Lambda$ -module,  $V$  is a  $\Gamma$ -module, and  $f$  is a  $\Lambda$ -homomorphism from  $U$  to  $\text{Hom}_{\Gamma}({}_{\Gamma}M_{\Lambda}, V)$ . As an Abelian group,  $(U, V, f)$  is as before, but the multiplication with an element from  $R$  is now given by

$$\begin{pmatrix} \lambda & 0 \\ m & \gamma \end{pmatrix} (u, v) = (\lambda u, f(u)(m) + \gamma v).$$

It is left to the reader to describe the  $R$ -homomorphisms in this case.

We now recall the notion of covariantly, contravariantly, and functorially finite subcategories from [AS1].

Recall that a covariant functor  $F$  from a category  $\mathcal{A}$  to the category of sets is said to be finitely generated if there is an object  $A$  in  $\mathcal{A}$  and a natural transformation from  $\mathcal{A}(A, -)$  onto  $F$ . The definition of a finitely generated contravariant functor is analogous.

**Definition.** A full subcategory  $\mathcal{A}$  of a category  $\mathcal{B}$  is said to be

- (i) contravariantly finite in  $\mathcal{B}$  if, for each object  $X$  in  $\mathcal{B}$ , the representable functor  $\mathcal{B}(-, X)$  restricted to  $\mathcal{A}$  is finitely generated as a functor on  $\mathcal{A}$ ,

- (ii) covariantly finite in  $\mathcal{B}$  if, for each object  $X$  in  $\mathcal{B}$ , the representable functor  $\mathcal{B}(X, \cdot)$  restricted to  $\mathcal{A}$  is finitely generated, and
- (iii) functorially finite in  $\mathcal{B}$  if it is both contravariantly and covariantly finite in  $\mathcal{B}$ .

## 2. THE RESULT

Let  $\Gamma$ ,  $\Lambda$ ,  ${}_R M_\Lambda$ , and  $R$  be as before, and let  $\mathcal{S}$  be a full subcategory of  $\text{mod } \Lambda$  and  $\mathcal{T}$  be a full subcategory of  $\text{mod } \Gamma$ . Let  $\text{mod } R_{\mathcal{T}}^{\mathcal{S}}$  denote the full subcategory of  $\text{mod } R$  consisting of the modules of the form  $(U, V, f)$  where  $U$  is in  $\mathcal{S}$  and  $V$  is in  $\mathcal{T}$ . With this notation we have the following result:

**Theorem 2.1.** (i) *The subcategory  $\text{mod } R_{\mathcal{T}}^{\mathcal{S}}$  of  $\text{mod } R$  is contravariantly finite in  $\text{mod } R$  if and only if  $\mathcal{S}$  is contravariantly finite in  $\text{mod } \Lambda$  and  $\mathcal{T}$  is contravariantly finite in  $\text{mod } \Gamma$ .*

(ii) *The subcategory  $\text{mod } R_{\mathcal{T}}^{\mathcal{S}}$  of  $\text{mod } R$  is covariantly finite in  $\text{mod } R$  if and only if  $\mathcal{S}$  is covariantly finite in  $\text{mod } \Lambda$  and  $\mathcal{T}$  is covariantly finite in  $\text{mod } \Gamma$ .*

(iii) *The subcategory  $\text{mod } R_{\mathcal{T}}^{\mathcal{S}}$  of  $\text{mod } R$  is functorially finite in  $\text{mod } R$  if and only if  $\mathcal{S}$  is functorially finite in  $\text{mod } \Lambda$  and  $\mathcal{T}$  is functorially finite in  $\text{mod } \Gamma$ .*

*Proof.* We prove (i) and leave the rest to the reader since the proofs of (i) and (ii) are dual of each other and (iii) follows from (i) and (ii).

We first establish the “only if” part of (i). Assume  $\text{mod } R_{\mathcal{T}}^{\mathcal{S}}$  is contravariantly finite in  $\text{mod } R$ , and let  $X$  be a module in  $\text{mod } \Lambda$ . Consider the module  $(X, 0, 0)$  in  $\text{mod } R$ . By assumption, there exists a module  $(U, V, f)$  in  $\text{mod } R_{\mathcal{T}}^{\mathcal{S}}$  and a homomorphism  $\phi = (\alpha, \beta): (U, V, f) \rightarrow (X, 0, 0)$  such that each homomorphism  $\psi$  from an object  $(U', V', f')$  in  $\text{mod } R_{\mathcal{T}}^{\mathcal{S}}$  to  $(X, 0, 0)$ , factors through  $\phi$ ; this will hold especially for each module of the form  $(U', 0, 0)$  in  $\text{mod } R_{\mathcal{T}}^{\mathcal{S}}$ . But then we get that  $\text{Hom}_{\Lambda}(\cdot, \alpha): \text{Hom}_{\Lambda}(\cdot, U) \rightarrow \text{Hom}_{\Lambda}(\cdot, X)$  restricted to  $\mathcal{S}$  is surjective. This shows that  $\mathcal{S}$  is contravariantly finite in  $\text{mod } \Lambda$ . The proof that  $\mathcal{T}$  is contravariantly finite in  $\text{mod } \Gamma$  when  $\text{mod } R_{\mathcal{T}}^{\mathcal{S}}$  is contravariantly finite in  $\text{mod } R$  is as simple and therefore left to the reader.

Assume now that  $\mathcal{S}$  is contravariantly finite in  $\text{mod } \Lambda$  and that  $\mathcal{T}$  is contravariantly finite in  $\text{mod } \Gamma$ . Let  $(X, Y, g)$  be a module in  $\text{mod } R$ . By assumption, there exists a  $Y_{\mathcal{T}}$  in  $\mathcal{T}$  and a  $\Gamma$ -homomorphism  $t_Y: Y_{\mathcal{T}} \rightarrow Y$  such that

$$\text{Hom}_{\Gamma}(\cdot, Y_{\mathcal{T}})|_{\mathcal{T}} \xrightarrow{\text{Hom}_{\Gamma}(\cdot, t_Y)} \text{Hom}_{\Gamma}(\cdot, Y)|_{\mathcal{T}} \rightarrow 0$$

is exact, where  $|_{\mathcal{T}}$  means restriction to  $\mathcal{T}$ . Now consider the pullback diagram

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow g \\ \text{Hom}_{\Gamma}(M, Y_{\mathcal{T}}) & \xrightarrow{\text{Hom}_{\Gamma}(M, t_Y)} & \text{Hom}_{\Gamma}(M, Y), \end{array}$$

in  $\text{mod } \Lambda$ . From the assumption that  $\mathcal{S}$  is contravariantly finite in  $\text{mod } \Lambda$ , it follows that there exists a  $Z_{\mathcal{T}}$  in  $\mathcal{S}$  and a  $\Lambda$ -homomorphism  $s_Z: Z_{\mathcal{T}} \rightarrow Z$  such that

$$\text{Hom}_{\Lambda}(\ , Z_{\mathcal{T}})|_{\mathcal{T}} \xrightarrow{\text{Hom}_{\Lambda}(\ , s_Z)} \text{Hom}_{\Lambda}(\ , Z)|_{\mathcal{T}} \rightarrow 0$$

is exact. We then obtain the module  $(Z_{\mathcal{T}}, Y_{\mathcal{T}}, h)$  in  $\text{mod } R_{\mathcal{T}}^{\mathcal{S}}$ , where  $h$  is the composed homomorphism  $Z_{\mathcal{T}} \xrightarrow{s_Z} Z \rightarrow \text{Hom}_{\Gamma}(M, Y_{\mathcal{T}})$ , and we also have an  $R$ -homomorphism from  $(Z_{\mathcal{T}}, Y_{\mathcal{T}}, h)$  to  $(X, Y, g)$ . The claim is that this  $R$ -homomorphism  $(Z_{\mathcal{T}}, Y_{\mathcal{T}}, h) \rightarrow (X, Y, g)$  has the property that the induced sequence

$$\text{Hom}_R(\ , (Z_{\mathcal{T}}, Y_{\mathcal{T}}, h))|_{R_{\mathcal{T}}^{\mathcal{S}}} \rightarrow \text{Hom}_R(\ , (X, Y, g))|_{R_{\mathcal{T}}^{\mathcal{S}}} \rightarrow 0$$

is exact.

To verify this claim, let  $(A, B, f)$  be a module in  $\text{mod } R_{\mathcal{T}}^{\mathcal{S}}$  and let  $\phi = (\alpha, \beta): (A, B, f) \rightarrow (X, Y, g)$  be an  $R$ -homomorphism. By the construction of the morphism  $t_Y: Y_{\mathcal{T}} \rightarrow Y$ , there exists a  $\Gamma$ -homomorphism  $\gamma: B \rightarrow Y_{\mathcal{T}}$  such that  $\beta$  is the composition  $B \xrightarrow{\gamma} Y_{\mathcal{T}} \xrightarrow{t_Y} Y$ . We then obtain the commuting diagram

$$\begin{array}{ccccccc} A & = & A & \xrightarrow{\alpha} & X \\ \downarrow f & & \downarrow \text{Hom}_{\Gamma}(M, \gamma)f & & \downarrow g \\ \text{Hom}_{\Gamma}(M, B) & \xrightarrow{\text{Hom}_{\Gamma}(M, \gamma)} & \text{Hom}_{\Gamma}(M, Y_{\mathcal{T}}) & \xrightarrow{\text{Hom}_{\Gamma}(M, t_Y)} & \text{Hom}_{\Gamma}(M, Y). \end{array}$$

From the property of the pullback

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow g \\ \text{Hom}_{\Gamma}(M, Y_{\mathcal{T}}) & \xrightarrow{\text{Hom}_{\Gamma}(M, t_Y)} & \text{Hom}_{\Gamma}(M, Y), \end{array}$$

there exists a  $\Lambda$ -homomorphism  $\delta: A \rightarrow Z$  such that

$$\begin{array}{ccccc} A & \xrightarrow{\delta} & Z & \longrightarrow & X \\ \downarrow f & & \downarrow & & \downarrow g \\ \text{Hom}_{\Gamma}(M, B) & \xrightarrow{\text{Hom}_{\Gamma}(M, \gamma)} & \text{Hom}_{\Gamma}(M, Y_{\mathcal{T}}) & \xrightarrow{\text{Hom}_{\Gamma}(M, t_Y)} & \text{Hom}_{\Gamma}(M, Y) \end{array}$$

commutes and such that  $\alpha$  is equal to the composed homomorphism  $A \xrightarrow{\delta} Z \rightarrow X$ . By construction of  $s_Z: Z_{\mathcal{T}} \rightarrow Z$ , there exists a  $\Lambda$ -homomorphism

$\eta: A \rightarrow Z_{\mathcal{S}}$  such that the following diagram commutes:

$$\begin{array}{ccccc}
 A & \xrightarrow{\quad \eta \quad} & Z_{\mathcal{S}} & \longrightarrow & X \\
 \downarrow f & & \downarrow & & \downarrow g \\
 \text{Hom}_{\Gamma}(M, B) & \xrightarrow{\text{Hom}_{\Gamma}(M, \gamma)} & \text{Hom}_{\Gamma}(M, Y_{\mathcal{T}}) & \xrightarrow{\text{Hom}_{\Gamma}(M, t_Y)} & \text{Hom}_{\Gamma}(M, Y),
 \end{array}$$

where the composition of the  $\Lambda$ -homomorphisms in the top row is  $\alpha$  and the composition of  $\gamma$  and  $t_Y$  is still  $\beta$ . This completes the proof of the claim as well as the proof of the theorem.

As an immediate corollary of this result we obtain the following:

**Corollary 2.2.** *If  $\mathcal{S}$  is functorially finite in  $\text{mod } \Lambda$ ,  $\mathcal{T}$  is functorially finite in  $\text{mod } \Gamma$ , and both  $\mathcal{S}$  and  $\mathcal{T}$  are closed with respect to extensions, then  $\text{mod } R_{\mathcal{T}}^{\mathcal{S}}$  has almost split sequences.*

*Proof.* When both  $\mathcal{T}$  and  $\mathcal{S}$  are closed with respect to extensions, then this also obviously holds for  $\text{mod } R_{\mathcal{T}}^{\mathcal{S}}$ . Theorem 2.4 of [AS2] states that a functorially finite subcategory of  $\text{mod } S$  for an Artin algebra  $S$  which is closed with respect to extensions has almost split sequences; hence the result follows from the Theorem.

Every full subcategory of  $\text{mod } S$  with only a finite number of isomorphism classes of indecomposable modules is functorially finite in  $\text{mod } S$ , when  $S$  is an Artin algebra. Therefore, letting  $\mathcal{P}$  be the category of projective modules in  $\text{mod } \Lambda$  and letting  $\mathcal{I}$  be the category of injective modules in  $\text{mod } \Gamma$ , which are both closed with respect to extensions, we immediately obtain that the category  $\text{mod } R_{\mathcal{T}}^{\mathcal{S}}$  has almost split sequences. This is the category of prinjective modules studied by Grecht [Gr] and de la Peña and Simson [PS].

In a forthcoming joint paper with K. Igusa and G. Todorov [IST], the category of modules of finite projective dimension over an Artin algebra will be considered. More precisely, let  $S$  be an Artin algebra and let  $\text{fpd mod } S$  be the full subcategory of  $\text{mod } S$  consisting of the modules of finite projective dimension. Some classes of algebras where  $\text{fpd mod } S$  is contravariantly finite will be given, and an example will be constructed showing that  $\text{fpd mod } S$  is not always contravariantly finite in  $\text{mod } S$ . Using Theorem 2.1, we can easily extend the classes of algebras where  $\text{fpd mod } S$  are contravariantly finite in  $\text{mod } S$ .

**Proposition 2.3.** *Let  $\Lambda$ ,  $\Gamma$ ,  ${}_{\Gamma}M_{\Lambda}$  and  $R$  be as before and assume in addition that  ${}_{\Gamma}M$  has finite projective dimension. Then  $\text{fpd mod } R$  is contravariantly finite in  $\text{mod } R$  if and only if  $\text{fpd mod } \Lambda$  is contravariantly finite in  $\text{mod } \Lambda$  and  $\text{fpd mod } \Gamma$  is contravariantly finite in  $\text{mod } \Gamma$ .*

*Proof.* From the next lemma we have that  $\text{fpd mod } R$  is the category  $\text{mod } R_{\mathcal{T}}^{\mathcal{S}}$  where  $\mathcal{S}$  is  $\text{fpd mod } \Lambda$  and  $\mathcal{T}$  is  $\text{fpd mod } \Gamma$ . Hence, the result follows directly from the theorem.

The content of the following lemma may be deduced from results in [FGR], but for the convenience of the reader we include a proof:

**Lemma 2.4.** *Let  $\Lambda$ ,  $\Gamma$ ,  ${}_\Gamma M_\Lambda$ , and  $R$  be as before, and assume that  ${}_\Gamma M$  has finite projective dimension. Then  $f\text{pd mod } R$  is  $\text{mod } R_{\mathcal{T}}^{\mathcal{S}}$ , where  $\mathcal{S}$  is  $f\text{pd mod } \Lambda$ , and  $\mathcal{T}$  is  $f\text{pd mod } \Gamma$ .*

*Proof.* The indecomposable projective  $R$ -modules are either of form  $(P, M \otimes P, id)$  where  $P$  is an indecomposable projective  $\Lambda$ -module or of form  $(0, Q, 0)$  where  $Q$  is an indecomposable projective  $\Gamma$ -module. Let  $(A, B, f)$  be an  $R$ -module of finite projective dimension and let

$$0 \rightarrow (X_n, Y_n, g_n) \rightarrow \cdots \rightarrow (X_1, Y_1, g_1) \rightarrow (X_0, Y_0, g_0) \rightarrow (A, B, f) \rightarrow 0$$

be a projective resolution of  $(A, B, f)$ . Then

$$0 \rightarrow X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow A \rightarrow 0$$

is a projective resolution of  $A$  as a  $\Lambda$ -module and, using the assumption that  ${}_\Gamma M$  has finite projective dimension, we obtain that

$$0 \rightarrow Y_n \rightarrow Y_{n-1} \rightarrow \cdots \rightarrow Y_1 \rightarrow Y_0 \rightarrow B \rightarrow 0$$

is a resolution of  $B$  in terms of  $\Gamma$ -modules of finite projective dimension. Hence,  $A$  is in  $f\text{pd mod } \Lambda$  and  $B$  is in  $f\text{pd mod } \Gamma$ .

Conversely, using induction on the projective dimension on  $A$ , we obtain that if  $(A, B, f)$  is in  $\text{mod } R_{\mathcal{T}}^{\mathcal{S}}$  then  $(A, B, f)$  is in  $f\text{pd mod } R$ .

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