ON \( p - C^* \) SUMMING OPERATORS

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**Abstract.** We prove that every bounded linear operator \( T : \mathcal{A} \rightarrow C_p(H) \) such that \( i \circ T : \mathcal{A} \rightarrow B(H) \) is positive (where \( \mathcal{A} \) is a unital \( C^* \)-algebra, \( C_p(H) \) a Schatten class, \( i \) the identity map from \( C_p(H) \) into \( B(H) \)) is \( p - C^* \) summing. This permits us to characterize \( p - C^* \) summing operators in some classes of multipliers.

**Introduction**

Gilles Pisier introduced the notion of \( p - C^* \) summing operator in order to prove Grothendieck’s inequality for noncommutative \( C^* \)-algebras (see [5]). In fact he used, in his proof, only 4 and 2 - \( C^* \) summing operators. In this paper we prove that every bounded operator \( T : \mathcal{A} \rightarrow C_p(H) \) such that \( i \circ T \) is positive (where \( \mathcal{A} \) is a unital \( C^* \)-algebra, \( C_p(H) \) a Schatten class, \( i \) the canonical embedding of \( C_p(H) \) in \( B(H) \)) is \( p - C^* \) summing. We remark that, for \( 1 \leq p < 2 \), the assumption “\( i \circ T \) is positive” cannot be omitted. Using this result, we give the characterisation of \( p - C^* \) summing operators in the class of multiplier operators on \( B(H) \) and positive Herz-Schur multipliers.

**1. On positive \( p - C^* \) summing operators**

A linear map \( T \) from a \( C^* \)-algebra \( \mathcal{A} \) into a Banach space \( X \) is \( p - C^* \) summing (we assume \( p \geq 1 \)) if there is a constant \( c \) such that, for any finite sequence

\[
\{x_i\}_{i=1}^N \subseteq \mathcal{A}^h = \{ x \in \mathcal{A} : x^* = x \},
\]

the following condition holds:

\[
\left( \sum_{i=1}^N \|Tx_i\|^p \right)^{1/p} \leq c \left( \sum_{i=1}^N |x_i|^p \right)^{1/p},
\]

where

\[
|x| = (x^* x)^{1/2}.
\]
The least constant $c$ for which this condition is satisfied is denoted by $c_p(T)$. It is shown in [5] that $T$ is $p - C^*$ summing if and only if there is a constant $c$ and a state $\varphi$ on $\mathcal{A}$ such that, for all $x$ in $\mathcal{A}$,

$$\|Tx\| \leq c\varphi(|x|^p)^{1/p}.$$ 

The least of those constants is equal to $c_p(T)$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two $C^*$-algebras and $T : \mathcal{A} \to \mathcal{B}$ a linear map. $T$ is called positive if $Tx$ is positive in $\mathcal{B}$ for all positive $x$ in $\mathcal{A}$ and completely positive if $T \otimes i_n : \mathcal{A} \otimes M_n \to \mathcal{B} \otimes M_n$ is positive for all natural $n$.

We will use the following notation:

- $B(H)$—algebra of all bounded linear operators on the Hilbert space $H$ equipped with operator norm,
- $C(H)$—the ideal of compact operators on $H$,
- $C_p(H)$—Schatten class, i.e., operators in $C(H)$ of the form

$$\sum_i \lambda_i \varphi_i \otimes \psi_i = \sum_i \lambda_i \langle \varphi_i, \psi_i \rangle \varphi_i,$$

where $\{\varphi_i\}, \{\psi_i\}$ are orthonormal sets in $H$ and $\sum_i |\lambda_i|^p < \infty$ with the norm $\|\sum_i \lambda_i \varphi_i \otimes \psi_i\|_p = (\sum_i |\lambda_i|^p)^{1/p}$.

We will need

**Lemma 1.1.** [7, p. 95]. Let $H$ be a Hilbert space, and let $A, B$ be positive operators in $B(H)$. If $p \geq 2$ and $B \in C_p(H)$ then $\|AB\|_p \leq \|A^{p/2}B^{p/2}\|_2^{2/p}$.

**Theorem 1.2.** Let $\mathcal{A}$ be a unital $C^*$-algebra and $H$ be a Hilbert space. If $T : \mathcal{A} \to C_p(H)$ ($p \geq 1$) is a bounded linear map such that $i \circ T : \mathcal{A} \to B(H)$ is positive (where $i : C_p(H) \to B(H)$ denotes the identity map), then $T$ is $p - C^*$ summing and $c_p(T) \leq \|T\|$. 

**Proof.** Let $x$ be a hermitian element of $\mathcal{A}$, and let $\mathcal{B}$ be a unital $C^*$-algebra generated by $x$. Since $\mathcal{B}$ is commutative, $T : \mathcal{B} \to B(H)$ is completely positive and, by Stinespring's theorem, may be represented in the form $Ty = V^*\pi(y)V$ for all $y \in \mathcal{B}$, where $\pi : \mathcal{B} \to B(R)$ is a unital $^*$-representation on a Hilbert space $R$ and $V : H \to R$ is a bounded linear operator (see [1]).

Setting $V_1 = \begin{bmatrix} 0 & V \\ V & 0 \end{bmatrix}$, an element of $B(R \oplus H)$, and $\pi_1(y) = \begin{bmatrix} \pi(y) & 0 \\ 0 & 0 \end{bmatrix}$, a $^*$-representation of $\mathcal{B}$ in $R \oplus H$, we have $\pi_1(|x|) = |\pi_1(x)| = U^*\pi_1(x)U$ for a unitary $U \in B(R \oplus H)$ which commutes with $\pi_1(x)$, and we may write

$$\|Tx\|_p = \|V^*\pi(x)V\|_p = \|V_1^*\pi_1(x)V_1\|_p = \|V_1^*U\pi_1(|x|^{1/2})\pi_1(|x|^{1/2})V_1\|_p \leq \|V_1^*U\pi_1(|x|^{1/2})\|_{2p}\|\pi_1(|x|^{1/2})V_1\|_{2p} = \|V_1^*\pi_1(|x|^{1/2})U\|_{2p}\|\pi_1(|x|^{1/2})V_1\|_{2p} \leq \|\pi_1(|x|^{1/2})V_1\|_{2p}^2 \leq \|\pi_1(|x|^{1/2})|V_1^*|\|_p^2.$$
An application of Lemma 1.1 gives
\[ ||Tx||_p \leq (\text{tr}|V_1^*V_1|^p \pi_1(|x|^p))^{1/p} \]
\[ = (\text{tr}(V_1^*V_1)^p \pi_1(|x|^p))^{1/p} \]
Since \((V_1^*V_1)^{m/n} = V_1(V_1^*V_1)^{m/n-1}V_1^*\) for all natural \(m, n\) such that \(m > n\) (to see this take the \(n\)th power of both sides), we have, by continuity argument,
\[ (V_1^*V_1)^p = V_1(V_1^*V_1)^{p-1}V_1^* , \]
and it follows that
\[ ||Tx||_p \leq (\text{tr}(V_1^*V_1)^{p-1}V_1^* \pi_1(|x|^p))^{1/p} \]
\[ = (\text{tr}(V_1^*V_1)^{p-1}V_1^* \pi_1(|x|^p)V_1^p)^{1/p} \]
\[ = (\text{tr}(V_1^*V_1)^{p-1}T(|x|^p))^{1/p} . \]
It is easily seen that the functional \(x \rightarrow \text{tr}(Te)^{p-1}T(x)\) is positive on \(\mathscr{A}\) and its norm equals \(||Te||_p\), so the proof is complete. \(\Box\)

Remark 1.3. Proposition 2.3 shows that, for \(1 < p < 2\), the assumption \(i \circ T\) is positive is essential.

Corollary 1.4. Let \(\mathscr{A}\) be a \(C^*\)-algebra with the unit \(e\), and let \(H\) be a Hilbert space. If \(T : \mathscr{A} \rightarrow B(H)\) is a positive linear map and \(Te \in C_p(H)\) \((p \geq 1)\),
then \(T\) is \(p-C^*\) summing, \(c_p(T) \leq ||Te||_p\), \(T(\mathscr{A}) \subset C_p(H)\) and \(T\) is bounded as the map from \(\mathscr{A}\) into \(C_p(H)\) with the norm \(||Te||_p\).

Proof. Taking unitary element \(u\) instead of hermitian \(x\) in the first part of the proof of Theorem 1.2, we get \(||Te||_p = ||V_1^*V_1||_p = ||V_1^*V_1||_p\), so \(V_1 \in C_{2p}(R \otimes H)\) and \(||V_1||_{2p} = ||Te||_p\), \(||Tu||_p = ||V_1^*\pi_1(u)V_1||_p = ||\pi_1(u)|| ||V_1||_{2p} \leq ||Te||_p\). Since \(\sup{||Tx||_p : ||x|| \leq 1} = \sup{||Tu||_p : u\text{-unitary}}\) (see [3]), we know that \(T\) is bounded as the map from \(\mathscr{A}\) into \(C_p(H)\) and its norm equals \(||Te||_p\). The rest immediately follows from Theorem 1.2. \(\Box\)

We will demonstrate that, in some cases, the assumption \(Te \in C_p(H)\) of Corollary 1.4 is not only sufficient but also necessary.

2. APPLICATION TO MULTIPLIER OPERATORS

Let \(B \in B(H)\) and \(T_B\) be a mapping from \(B(H)\) into \(B(H)\) defined by \(T_B(A) = BAB^*\).

Proposition 2.1. Let \(p \geq 1\). Then \(T_B\) is a \(p-C^*\) summing operator if and only if \(B^*B\) belongs to \(C_p(H)\); moreover, \(c_p(T) = \|B^*B\|_p\).
Proof. Let us assume that $T_B$ is a $p - C^*$ summing operator and \{\varphi_i\} an arbitrary orthonormal basis in $H$. We may write
\[
\left\| \sum_{i=1}^{n} (B^* B \varphi_i, \varphi_i)^p \right\|^p = \sum_{i=1}^{n} \left\| (\varphi_i \otimes \varphi_i) B^* B (\varphi_i \otimes \varphi_i) \right\|^p \leq \sum_{i=1}^{n} \left\| B (\varphi_i \otimes \varphi_i) B^* \right\|^p \leq cp(T_B)^p.
\]
Let $T = B^* B = \int_0^{||T||} \lambda^d E(\lambda)$ be a spectral decomposition of $T$. We have
\[
T \geq \int_\epsilon^{||T||} \lambda^d E(\lambda) \geq E(\epsilon, ||T||),
\]
and, since
\[
\sum_{i} (T \varphi_i, \varphi_i)^p \leq cp(T_B),
\]
for any orthonormal basis \{\varphi_i\}, we get that the operator $E(\epsilon, ||T||)$ is of finite rank. Hence $T$ is compact and may be represented in the following form: $T = \sum_i \lambda_i \varphi_i \otimes \varphi_i$, \{\varphi_i\} is an orthonormal basis in $H$. Since
\[
\sum_i \lambda_i^p = \sum_i (T \varphi_i, \varphi_i)^p \leq cp(T_B),
\]
we infer that $B^* B$ belongs to $C_p(H)$, $\|B^* B\|_p \leq cp(T_B)$.

The converse is an immediate consequence of Corollary 1.4. □

Now we consider the left regular representation on $B(H)$. For $B \in B(H)$, we define $L_B : B(H) \to B(H)$ by the formula $L_B(A) = BA$.

Proposition 2.2. Let $p \geq 2$. Then $L_B$ is $p - C^*$ summing if and only if $B$ belongs to $C_p(H)$; moreover, $cp(L_B) = \|B\|_p$.

Proof. Let us assume that $L_B$ is $p - C^*$ summing and that \{\varphi_i\} is an arbitrary orthonormal basis in $H$. Then
\[
\sum_{i=1}^{n} (B^* B \varphi_i, \varphi_i)^{p/2} = \sum_{i=1}^{n} \left\| (\varphi_i \otimes \varphi_i) B^* B (\varphi_i \otimes \varphi_i) \right\|^{p/2} \leq \sum_{i=1}^{n} \left\| B (\varphi_i \otimes \varphi_i) \right\|^p \leq cp(L_B)^{p/2}.
\]

Following the reasoning from Proposition 2.1, we state that $B^* B \in C_{p/2}(H)$ and $\|B^* B\|_{p/2} \leq cp(L_B)^{p/2}$, hence $\|B\|_p \leq cp(L_B)$. If $B \in C_p(H)$ then $\|BA\| = \|BA^2 B^*\|^{1/2} \leq cp(L_B)^{1/2} = cp(L_B)^{1/2} (\|A\|^p)^{1/2}$, where $\varphi$ is a state on $B(H)$. We have $cp(L_B) \leq cp(L_B)^{1/2} = \|B^* B\|_{p/2} \leq \|B\|_p$. □

Proposition 2.3. If $1 \leq p < 2$, $B \in B(H)$ and $B \neq 0$, then $L_B$ is not $p - C^*$ summing.

Proof. To see this, let us show first that, for any $\varphi \in H$, $\|\varphi\| = 1$, $L_\varphi \otimes \varphi$ is not $p - C^*$ summing. For all natural $n$ we can find orthonormal vectors \{\varphi_i\}_{i=1}^{n}
such that
\[ \varphi = \sum_{i=1}^{n} \frac{1}{\sqrt{n}} \varphi_i \]
and
\[ \sum_{i=1}^{n} \| (\varphi \otimes \varphi)(\varphi_i \otimes \varphi_i) \|^p = \sum_{i=1}^{n} (\varphi_i \varphi_i)^p = n^{(2-p)/2}, \]
so the operator \( L_{\varphi \otimes \varphi} \) is not \( p - C^* \) summing.

Let us assume that \( B \in B(H), B \neq 0 \) and \( L_B \) is \( p - C^* \) summing. We can find \( \xi \in H, \|\xi\| = 1 \), such that
\[ (\xi \otimes \xi)B = \xi \otimes B^*\xi \neq 0. \]
We see that \( L_{\xi \otimes B^*\xi} \) is \( p - C^* \) summing, so
\[ L_{B^*\xi \otimes B^*\xi} = L_{(\xi \otimes B^*\xi)^* (\xi \otimes B^*\xi)} \]
is non-zero and \( p - C^* \) summing, and we are done. \( \Box \)

Let \( H \) be a Hilbert space and \( \{\varphi_i\} \) an orthonormal basis of \( H \). Let \( M = (m_{ij}) \) be a positive Herz-Schur multiplier, i.e., for every \( A \in B(H) \), there is a \( B \) in \( B(H) \) such that
\[ \langle B\varphi_j, \varphi_i \rangle = m_{ij} \langle A\varphi_j, \varphi_i \rangle, \]
for any finite sequence of complex numbers
\[ \{\xi_i\}_{i=1}^{N}, \quad \sum_{ij} m_{ij} \xi_i \xi_j \geq 0. \]
We see that \( M \) defines a bounded positive linear operator from \( B(H) \) into \( B(H) \). It is known that \( M \) is a positive Herz-Schur multiplier if and only if there is a Hilbert space \( R \) and a sequence \( \{x_i\} \in R \) such that, for some constant \( c, \|x_i\| \leq c \) for all \( i \) and \( m_{ij} = \langle x_i, x_j \rangle \) (see [2]), but we will not use this fact.

**Proposition 2.4.** \( M \) is \( p - C^* \) summing if and only if \( \sum_i m_{ii}^p < \infty \); moreover, \( c_p(M) = (\sum_i m_{ii}^p)^{1/p} \).

**Proof.** Assuming that \( M \) is \( p - C^* \) summing, we obtain
\[ \sum_i m_{ii}^p = \sum_i \| M(\varphi_i \otimes \varphi_i) \|^p \leq c_p^p(M). \]
The converse is an immediate consequence of Corollary 1.4. \( \Box \)

**Remark 2.5.** It is easily seen, in view of Corollary 1.4, that a positive Herz-Schur multiplier is \( p - C^* \) summing if and only if its image is contained in \( C_p(H) \).
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