CONVOLUTION EQUATIONS IN CERTAIN BANACH SPACES

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Abstract. For a Banach space $E$ and $p > 0$, the following problem is considered: how to identify a finite Borel measure $\mu$ on $E$ by means of the potential $g(a) = \int_E \|x - a\|^p \, d\mu(x), \ a \in E$. The solution for infinite-dimensional Hilbert spaces is based on limit correlations between the Fourier transforms of finite-dimensional restrictions of $g$ and $\|x\|^p$. For finite-dimensional subspaces of $L_p$, the Levy representation of norms is used.

1. Introduction

Let $(E, \| \cdot \|)$ be a Banach space, $M_p$ be the set of Borel measures $\mu$ on $E$ satisfying $\int_E (1 + \|x\|)^p \, d\mu(x) < \infty$. For $\mu \in M_p$, let $g$ be the potential of $\mu$:

$$g(a) = \int_E \|x - a\|^p \, d\mu(x), \ a \in E.$$

The problem is to identify $\mu$ by means of $g$.

Thus we are going to study a sort of convolution equation. The uniqueness problem for this equation has been investigated by several authors. For some special Banach spaces $E$ the solution of this equation is unique for all $p > 0$, except for a countable set of exponents $p$, which will be called exceptional for the space $E$. For instance, in the one-dimensional case the exceptional exponents are the even numbers, [19–21], and the same is true for separable Hilbert spaces [see [22, 1, 9] for the finite-dimensional case and [10], [15] for the infinite-dimensional case]. Exceptional for $L_q$-spaces are the numbers $p$, for which $p/q \in \mathbb{N}$ and, besides that, in the case of $n$-dimensional space $l_q^n$ one of the following three conditions must be fulfilled: (a) $p/q < n$; (b) $q$ is an even integer; (c) $q$ and $(p/q) - n$ are odd integers. For the complex space $l_\infty^n$, even integers are exceptional; and for the real space, $p$ is exceptional iff $n + p$ is odd. The set of exceptional exponents $p > 0$ is empty for spaces $C(K)$.
where $K$ is an infinite metric compact space without isolated points, and for spaces $C_0(\Omega)$, where $\Omega$ is a noncompact, locally compact Hausdorff space. If $K$ contains isolated points, then positive $p \notin \mathbb{N}$ are not exceptional, and the problem is still open for $p \in \mathbb{N}$ [for all these results, see [5, 6, 11, 16].

If $E$ is a finite-dimensional space, the potential $g$ can be considered as the convolution of distributions $\|x\|^p$ and $\mu$ over the space $S = S(\mathbb{R}^n)$ of rapidly decreasing functions. This convolution is, as a rule, naturally connected with the Fourier transform [6], so to solve the convolution equation one can compute $\left(\|x\|^p\right)^\wedge$, verify that $\left(\|x\|^p\right)^\wedge \neq 0$ on open sets and put $\hat{\mu} = \hat{g}/(\|x\|^p)^\wedge$ (throughout we denote by $\hat{f}$ the Fourier transform of a distribution $f$). However, the complete solution of the inverse problem has been obtained only in the case $E = \mathbb{R}^n$, when $\left(\|x\|^p\right)^\wedge$ can easily be computed. For other finite-dimensional spaces only uniqueness theorems are available, because attempts at straightforward computation of $\left(\|x\|^p\right)^\wedge$ have been unsuccessful. In the following, an analytic continuation of $\|x\|^p$ was used to check that $\left(\|x\|^p\right)^\wedge \neq 0$ on open sets.

In this paper the Fourier transform of norms in some finite-dimensional spaces is computed with the help of isometric embedding of these spaces into $L^p$. In §4 this is done for an arbitrary $n$-dimensional subspace $E = \text{span}(f_1, \ldots, f_n)$ of $L^p$. If $p \neq 2, 4, 6, \ldots$ then $p$ is exceptional for $E$ iff the joint distribution of $f_1, \ldots, f_n$ vanishes on some open cone in $\mathbb{R}^n$. (We recall that this joint distribution is a measure on $\mathbb{R}^n$.) In the same section, any easy proof is obtained for the well-known equimeasurability theorem for $L_p$-isometries [19–21, 17, 7, 12].

In §5 we give some concrete consequences of results of §4. The construction of isometric embedding of $l^p_q$ into $L^p_\sigma(0 < p < q < 2)$ from [2] is used to obtain the Levy representation of norms and, as a consequence, to express the Fourier transform of $l^p_q$-norm in terms of $q$-stable measures. In §6 we investigate the inverse problem for two-dimensional spaces. An exponent $p > 0$ is exceptional for $E = \text{span}(e_1, e_2)$ iff the $(p + 1)$ th fractional derivative of the function $\|e_1 + te_2\|^p$ vanishes on some open subset of $\mathbb{R}$. In particular, the exponent $p = 1$ is exceptional for a two-dimensional space iff this space is not strictly convex.

A connection between the Fourier transform and isometric embedding into $L^p$ is discussed in §3. In particular, it is proved in this section that, for an arbitrary even function $h$ on $\mathbb{R}$ satisfying some mild additional assumptions, if $\int_{S^{n-1}} h(\langle x, \xi \rangle) d\nu(\xi) = 0$ for each $x \in \mathbb{R}^n$, then either $\nu = 0$ or $h$ is a polynomial containing even powers only (here $\langle x, \xi \rangle$ stands for the scalar product in $\mathbb{R}^n$ and $\nu$ is a symmetric charge on the unit sphere $S^{n-1}$ in $\mathbb{R}^n$). This statement for $h(t) = |t|^p$ has been proved by different methods in [8, 14, 18].

In §2 the convolution equation is solved for infinite-dimensional Hilbert spaces. This solution is based on limit correlations between the Fourier transforms of functions $g$ and $\|x\|^p$. 

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All definitions and facts about distributions over $S(\mathbb{R}^n)$ used in this paper can be found in [3].

2. INFINITE-DIMENSIONAL HILBERT SPACES

Let $E$ be the separable Hilbert space $l_2$ and $e_i, i \in \mathbb{N}$, be the standard basis in $l_2$, $E_n = \text{span}(e_i, i = 1, \ldots, n)$. For $x = \sum_{i=1}^{\infty} x_i e_i \in l_2$, we write $x^{(n)} = \sum_{i=1}^{n} x_i e_i$ (so $x^{(n)}$ is the projection of $x$ to $E_n$).

Let $\mu$ be a Borel measure on $l_2$, $\mu \in \mathbb{R}$. We shall assume that $\mu \in M_p$ if $p > 0$, and that $\mu(l_2) < \infty$ if $p < 0$. Note that the restriction $g_n$ of the potential $g(a) = \int_E \|x - a\|^p d\mu(x)$ to the subspace $E_n$ is locally an $L_1$-function on $E_n$ with respect to Lebesgue measure on $E_n$, if $-n < p < 0$ (see, for instance, [13, p. 84]). So $g_n$ can be considered as a distribution over $S(\mathbb{R}^n)$.

**Theorem 1.** If $p \in \mathbb{R}$, $p \neq 0, 2, 4, \ldots$, then for each $\xi \in l_2$, $\xi \neq 0$,

$$
\hat{\mu}(\xi) = \lim_{n \rightarrow \infty} \frac{\hat{g}_n(\xi^{(n)})\|\xi^{(n)}\|^{n+p} \Gamma(-p/2)}{2^{n+p} \pi^{n/2} \Gamma((n+p)/2)}
$$

**Proof.** Let us fix an element $x \in l_2$ and assume first that $-1 < p < 0$. In order to compute the Fourier transform of the function $\|x - a\|^p$ of the variable $a \in E_n$, we can use the following representation, which is an easy consequence of the definition of the $\Gamma$-function:

$$
\hat{\mu}(\xi) = \lim_{n \rightarrow \infty} \frac{\hat{g}_n(\xi^{(n)})\|\xi^{(n)}\|^{n+p} \Gamma(-p/2)}{2^{n+p} \pi^{n/2} \Gamma((n+p)/2)}
$$

Now for each fixed $t > 0$, the Fourier transform of the function $a \mapsto \exp(-t^2 \|x - a\|^2)$, $a \in E_n$, can easily be computed:

$$
(\exp(-t^2 \|x - a\|^2)) \hat{(\xi)} = \frac{2\pi^{n/2}}{\Gamma(-p/2)} \int_0^\infty t^{-1-p-n} \exp(-t^2 \|x - x^{(n)}\|^2 - \|\xi\|^2/4t^2) dt.
$$

If we allow $p$ to assume complex values, then both sides of (1) are analytic functions of $p$ in the domain $\{\Re p > -n, p \neq 0, 2, 4, 6, \ldots\}$. So these functions admit a unique analytic continuation from the interval $(-1, 0)$, and (1) remains true for all real $p > -n$, $p \neq 0, 2, 4, \ldots$.

Making in (1) the change of variables $y = 1/t$ and then integrating against $d\mu(x)$, we obtain

$$
\hat{g}_n(\xi) = \frac{2\pi^{n/2}}{\Gamma(-p/2)} \times \int_{l_2} \exp(-i(x^{(n)}, \xi)) \times \left(\int_0^\infty y^{n+p-1} \exp(-y^2 \|\xi\|^2/4 - \|x - x^{(n)}\|^2/y^2) dy\right) d\mu(x)
$$
for every $\zeta \in \mathbb{R}^n$, $\zeta \neq 0$, and $p > -n$, $p \neq 0, 2, 4, \ldots$

Now let us consider an arbitrary $p \in \mathbb{R}$, $p \neq 0, 2, 4, \ldots$, and $\xi \in l_2$, $\xi \neq 0$.

For each $n \in \mathbb{N}$ with $\xi(n) \neq 0$ and $p > -n$, put $\zeta = \xi(n)$ in (2) and divide both sides of (2) by

$$
\frac{2\pi^{n/2}}{\Gamma(-p/2)} \int_0^\infty y^{n+p-1} \exp(-y^2 \|\xi(n)\|^2/4) \, dy = \frac{2^{n+p} \pi^{n/2} \|\xi(n)\|^{-n-p} \Gamma(n+p)}{\Gamma(-p/2)}.
$$

This leads to

$$
\hat{\mu}_n(\xi(n)) \|\xi(n)\|^{n+p} \Gamma(-p/2) = \int_2 \exp(-i(x(n), \xi(n))) \times \int_0^\infty y^{n+p-1} \exp(-y^2 \|\xi(n)\|^2/4 - \|x - x(n)\|^2/4) \, dy \, d\mu(x).
$$

The absolute value of the integrand on the right is majorated by 1 and for each $x \in l_2$ tends to $\exp(-i(x, \xi))$ as $n \to \infty$. In fact, $1 - \exp(-z) < z$ for $z > 0$, so the difference between the fraction on the right-hand side of (3) and the number 1 is less than

$$
\|x - x(n)\|^2 \int_0^\infty y^{n+p-3} \exp(-y^2 \|\xi(n)\|^2/4) \, dy = \frac{\|x - x(n)\|^2 \|\xi(n)\|^2}{2(n + p - 2)},
$$

which tends to zero as $n \to \infty$. Now we can apply the Lebesgue dominated convergence theorem to obtain that the quantity on the right in (3) tends to $\hat{\mu}(\xi)$ as $n \to \infty$.

3. Uniqueness theorem for measures on $S^{n-1}$

Let $H$ denote the set of even, continuous functions $h$ on $\mathbb{R}$ with tempered growth at infinity (i.e., $\lim_{|x| \to \infty} (h(x)/|x|^\rho) = 0$ for some $\rho > 0$), for which $\hat{h}$ is a continuous function on $\mathbb{R}\{0\}$ with tempered growth at infinity.

Let $\nu$ be a symmetric Borel charge of bounded variation on the unit sphere $S^{n-1}$ in $\mathbb{R}^n$.

The distribution $\hat{h}(t) \, d\nu(\xi)$ is defined by

$$
\langle \hat{h}(t) \, d\nu(\xi), \varphi \rangle = \int_{S^{n-1}} d\nu(\xi) \int_{\mathbb{R}} \hat{h}(t) \varphi(t\xi) \, dt = \int_{S^{n-1}} \langle \hat{h}, \varphi(t\xi) \rangle \, d\nu(\xi)
$$

for each $\varphi \in S(\mathbb{R}^n)$ with $0 \notin \text{supp} \varphi$.

**Lemma 1.** For every $\xi_0 \in S^{n-1}$, the Fourier transform of the function $h(\langle x, \xi_0 \rangle)$ coincides on $\mathbb{R}^n \setminus \{0\}$ with the distribution $\hat{h}(t) \, d\delta_{\xi_0}(\xi)$, where $\delta_{\xi_0}$ is the unit mass at the point $\xi_0$. 
Proof. By the Fubini theorem, for every even function \( \varphi \in S(\mathbb{R}^n) \) with \( 0 \notin \text{supp} \varphi \), we have

\[
\langle (h((x, \xi_0))) \hat{}, \varphi \rangle = \langle h((x, \xi_0)), \hat{\varphi} \rangle
= \int_{\mathbb{R}^n} h((x, \xi_0)) \hat{\varphi}(x) \, dx = \int_{\mathbb{R}} h(t) \left( \int_{\{x, \xi_0\}=t} \hat{\varphi}(x) \, dx \right) \, dt.
\]

The even function \( \varphi(t \xi_0) \) is the Fourier transform of the even function \( \int_{\{x, \xi_0\}=t} \hat{\varphi}(x) \, dx \) of the variable \( t \in \mathbb{R} \). (It is a simple property of a Radon transform [4, p. 19].) So we have

\[
\langle (h((x, \xi_0))) \hat{}, \varphi \rangle = \left( \int_{\mathbb{R}} h(t) \, dt \right) \langle \hat{\varphi}(t \xi_0) \rangle = \langle h(t) \delta_{\xi_0}(\xi), \varphi \rangle
\]

and even distributions \( (h((x, \xi_0))) \hat{} \) and \( h(t) \delta_{\xi_0}(\xi) \) coincide on \( \mathbb{R}^n \setminus \{0\} \).

The next result is a consequence of Lemma 1 and the Fubini theorem:

**Lemma 2.** Let \( h \in H \) and \( \nu \) be a symmetric Borel charge of bounded variation on \( S^{n-1} \). Then the Fourier transform of the function

\[
f(x) = \int_{S^{n-1}} h((x, \xi)) \, d\nu(\xi)
\]

coincides on \( \mathbb{R}^n \setminus \{0\} \) with the distribution \( \hat{h}(t) \, d\nu(\xi) \).

**Proof.** For an arbitrary even function \( \varphi \in S(\mathbb{R}^n) \) with \( 0 \notin \text{supp} \varphi \)

\[
\langle \hat{f}, \varphi \rangle = \langle f, \hat{\varphi} \rangle = \int_{S^{n-1}} d\nu(\xi) \int_{\mathbb{R}^n} h((x, \xi)) \hat{\varphi}(x) \, dx
= \int_{S^{n-1}} d\nu(\xi) \int_{\mathbb{R}} \hat{h}(t) \varphi(t \xi) \, dt = \langle \hat{h}(t) \, d\nu(\xi), \varphi \rangle.
\]

**Theorem 2.** If \( \int_{S^{n-1}} h((x, \xi)) \, d\nu(\xi) = 0 \) for all \( x \in \mathbb{R}^n \), then either \( \nu = 0 \) or \( h \) is a polynomial containing even powers only.

**Proof.** By Lemma 2, \( \hat{h}(t) \, d\nu(\xi) = 0 \) everywhere on \( \mathbb{R}^n \setminus \{0\} \). This is possible only if either \( \nu = 0 \) or \( \hat{h} \) is a distribution with support in \( \{0\} \). Since \( h \) is even, in the last case \( h \) must be a polynomial containing even powers only.

### 4. Finite-dimensional subspaces of \( L_p \)

Let \( (E, \| \cdot \|) \) be an \( n \)-dimensional subspace of \( L_p(\Omega, \sigma) \), where \( p > 0 \) and \( (\Omega, \sigma) \) is a measure space, \( \sigma(\Omega) < \infty \).

Consider an arbitrary basis \( f_1, \ldots, f_n \) in \( E \). Let \( \mu \) be the joint distribution of functions \( f_1, \ldots, f_n \) with respect to \( \sigma \); that is, \( \mu(B) = \sigma\{\omega \in \Omega: (f_1(\omega), \ldots, f_n(\omega)) \in B\} \) for every Borel subset \( B \) of \( \mathbb{R}^n \). Note that \( \mu \) is a finite Borel measure on \( \mathbb{R}^n \).
A measure $\nu$ on the unit sphere $S^{n-1}$ in $\mathbb{R}^n$, defined by

$$\nu(B) = \frac{1}{2} \int_{B \times \mathbb{R}} \|x\|_2^2 d\mu(x)$$

for all Borel subsets $B$ of $S^{n-1}$, will be called the $p$-projection of $\mu$ to $S^{n-1}$. (Here $\|x\|_2$ is the Euclidean norm, $B \times \mathbb{R} = \{y \in \mathbb{R}^n, y \neq 0: y/\|y\|_2 \in B \cup (-B)\}$.)

The norm in $E$ can be represented in the following way:

$$\|x\|^p = \left( \sum_{i=1}^{n} x_i f_i \right)^p = \int_{\Omega} \left( \sum_{i=1}^{n} x_i f_i(\omega) \right)^p d\sigma(\omega) = \int_{\mathbb{R}^n} |(x, \xi)|^p d\mu(\xi)$$

(In such situations we shall say that the norm in $E$ admits the Levy representation with the measure $\mu$.)

Now we apply Lemma 2 with $h(y) = |y|^p$, $y \in \mathbb{R}$, to compute the Fourier transform of $\|x\|^p$. Note $|y|^p \sim (t) = (2^{p+1} \sqrt{\pi} \Gamma((p+1)/2) / \Gamma(-p/2)) |t|^{1-p}$, if $p > 0$, $p \neq 2, 4, 6, \ldots$ [3, p. 217].

**Lemma 3.** If $p > 0$, $p \neq 2, 4, 6, \ldots$ then

$$\left( \|x\|^p \right) \sim \frac{2^{p+1} \sqrt{\pi} \Gamma((p+1)/2)}{\Gamma(-p/2)} |t|^{1-p} d\nu(\xi).$$

It is clear now that $\|x\|^p \sim 0$ on an open subset of $\mathbb{R}^n$ iff $\nu = 0$ on some open subset of $\mathbb{R}^{n-1}$. So we have the following:

**Theorem 3.** If $p > 0$, $p \neq 2, 4, 6, \ldots$ then $p$ is an exceptional exponent for the space $E = \text{span}(f_1, \ldots, f_n) \subset L_p(\Omega, \sigma)$ iff $\mu(B \times \mathbb{R}) = 0$ for some open subset $B$ of $S^{n-1}$, where $\mu$ is the joint distribution of functions $f_1, \ldots, f_n$ with respect to $\sigma$.

Suppose that $\Omega$ is a topological space, $\sigma$ is a finite Borel measure on $\Omega$ which does not vanish on open sets, and $f_1, \ldots, f_n$ are continuous functions on $\Omega$. Let $V$ be the subset of $\mathbb{R}^{n-1}$ consisting of all points of the form $(f_2(\omega)/f_1(\omega), \ldots, f_n(\omega)/f_1(\omega))$ or $(-f_2(\omega)/f_1(\omega), \ldots, -f_n(\omega)/f_1(\omega))$, where $\omega$ runs over the set $\Omega \setminus f_1^{-1}(0)$. In this case $\mu(B \times \mathbb{R}) \neq 0$ for all open subsets $B$ of $S^{n-1}$ iff $V$ is dense in $\mathbb{R}^{n-1}$.

**Example.** Let $\Omega = S^1$ be the unit circle in $\mathbb{R}^2$ with (linear) Lebesgue measure, $p > 0$, $p \neq 2, 4, 6, \ldots$. Then for the space $E_1 = \text{span}(\sin \omega, \sin 2\omega) \subset L_p(S^1)$, we have $V = (-2, 2)$, and $p$ is an exceptional exponent for $E_1$. If $E_2 = \text{span}(\sin 2\omega, \sin 3\omega) \subset L_p(S^1)$, then $V = \mathbb{R}$, and $p$ is not exceptional for $E_2$.

Let $(\Omega, \mathcal{B}, \sigma)$ and $(\Omega', \mathcal{B}', \sigma')$ be measure spaces with finite measures, $p > 0$, $p \neq 2, 4, 6, \ldots$ and $Y$ be an arbitrary (maybe infinite-dimensional) subspace of $L_p(\Omega)$. Suppose that a linear isometry $T$ maps $Y$ into $L_p(\Omega')$. 
The well-known continuation theorem for $L_p$-isometries [see 19, 20, or 7] states that $T$ can be extended to the space $L_p(\Omega, \mathcal{B}_0, \sigma)$ as a linear isometry, where $\mathcal{B}_0$ is a minimal $\sigma$-algebra contained in $\mathcal{B}$, making all functions in the space $Y$ measurable.

This result was obtained as a straightforward consequence of the following equimeasurability theorem for $L_p$-isometries [see 19–21, 17, 7]:

**Theorem 4.** For arbitrary functions $f_1, \ldots, f_n \in Y$ we have $\mu_1 = \mu_2$, where measures $\mu_1$ and $\mu_2$ on $\mathbb{R}^n$ are the joint distributions of the $n$-tuples

$$(1, f_2(\omega)/f_1(\omega), \ldots, f_n(\omega)/f_1(\omega))$$

and

$$(1, T f_2(\omega')/T f_1(\omega'), \ldots, T f_n(\omega')/T f_1(\omega'))$$

with respect to measures $|f_1|^p \, d\sigma$ and $|T f_1|^p \, d\sigma'$ accordingly.

Lemma 2, above, provides a simple proof of Theorem 4. Indeed, $\|\sum_1^n x_i f_i\|^p = \|\sum_1^n x_i T f_i\|^p$ for each $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, so

$$\begin{align*}
\int_{\Omega} \left| \sum_{i=1}^n x_i f_i(\omega) \right|^p \, d\sigma(\omega) \\
= \int_{f_1(\omega) \neq 0} \left| x_1 + \sum_{i=2}^n x_i f_i(\omega) \right|^p \left| f_i(\omega) \right|^p \, d\sigma(\omega) + \int_{f_1(\omega) = 0} \left| \sum_{i=2}^n x_i f_i(\omega) \right|^p \, d\sigma(\omega) \\
= \int_{\mathbb{R}^n} |(x, \xi)|^p \, d\mu_1(\xi) + \psi(x_2, \ldots, x_n) \\
= \int_{\Omega'} \left| \sum_{i=1}^n x_i T f_i(\omega') \right|^p \, d\sigma'(\omega') = \int_{\mathbb{R}^n} |(x, \xi)|^p \, d\mu_2(\xi) + \psi'(x_2, \ldots, x_n).
\end{align*}$$

Consider the Fourier transforms of these functions of variables $x_1, \ldots, x_n$. The Fourier transforms of functions $\psi$ and $\psi'$ are supported on the hyperplane $\xi_1 = 0$ in $\mathbb{R}^n$. The measures $\mu_1$ and $\mu_2$ are supported on the hyperplane $\xi_1 = 1$ in $\mathbb{R}^n$ so the $p$-projections $\nu_1$ and $\nu_2$ of these measures to $S^{n-1}$ are supported out of the hyperplane $\xi_1 = 0$. By Lemma 2, $\nu_1 = \nu_2$, and therefore $\mu_1 = \mu_2$.

5. Levy representations

Let $(E, \| \cdot \|)$ be an $n$-dimensional Banach space. Suppose that there exists an even function $f \in L_1(\mathbb{R})$ with $(f(\|x\|)) \sim u \in L_1(\mathbb{R}^n)$. Then for every $k \in \mathbb{R}$ and $x \in E$, $x \neq 0$

$$(2\pi)^n f(k \|x\|) = \int_{\mathbb{R}^n} \exp(ik \|x\|/(x, \xi)/\|x\|)) u(\xi) \, d\xi = \int_{\mathbb{R}} \exp(ik \|x\| \gamma) \, du_1(\gamma),$$

where $u_1$ is the image of the charge $u(\xi) \, d\xi$ under the mapping $\xi \mapsto (x, \xi)/\|x\|$. Here $k$ is arbitrary; therefore, $u_1 = \hat{f}$, and $u_1$ does not depend on $x \in E$, $x \neq 0$. 

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Now for every \( p \in \mathbb{R} \) we obtain, assuming that the first integral converges:

\[
\int_{\mathbb{R}^n} |\langle x, \xi \rangle|^p u(\xi) \, d\xi = \|x\|^p \int_{\mathbb{R}^n} \left| \frac{\langle x, \xi \rangle}{\|x\|} \right|^p u(\xi) \, d\xi = \|x\|^p \int_{\mathbb{R}} |t|^p \, du_1(t).
\]

So if the \( p \) th moment of the charge \( u_1 \) exists and is not zero then we have the Levy representation with the charge \( u(\xi) \, d\xi \):

\[
\|x\|^p = \frac{1}{\int_{\mathbb{R}} |t|^p \, du_1(t)} \int_{\mathbb{R}^n} |\langle x, \xi \rangle|^p u(\xi) \, d\xi.
\]

In order to compute the moment, we assume that \( p \in (-1, 0) \) and use the Parseval theorem:

\[
\int_{\mathbb{R}} |t|^{-1-p} f(t) \, dt = \int_{\mathbb{R}} |t|^p \, du_1(t).
\]

If both sides of (5) are analytic functions of the variable \( p \) in some domain in \( \mathbb{C} \), we can use analytic continuation and compute moments for other \( p \)'s.

For instance, if \( E = l^n \), \( f(t) = \exp(-|t|^q) \) and \( 0 < p < q < 2 \), then

\[
u(\xi) = \gamma_q(\xi) = \left( \exp(-\|x\|^q) \right)^{-1} (\xi)
\]

is the density of the \( n \)-dimensional \( q \)-stable measure, and \( u_1 \) is the one-dimensional stable measure. The \( p \) th moment of \( u_1 \) is finite and can easily be computed [23, p. 75]. Indeed, for \( p \in (-1, 0) \)

\[
\int_{\mathbb{R}} |t|^{-1-p} \exp(-|t|^q) \, dt = \frac{2^{p+1} \sqrt{\pi} \Gamma((p+1)/2)}{\Gamma(-p/2)}.
\]

and it follows from (5) and the analytic continuation argument that, for every \( p \in (0, q) \),

\[
\int_{\mathbb{R}} |t|^p \, du_1(t) = \frac{\Gamma(-p/q) 2^{p+2} \sqrt{\pi} \Gamma((p+1)/2)}{q \Gamma(-p/2)}.
\]

Now use (4) to get the Levy representation

\[
\|x\|^q = \frac{q \Gamma(-p/2)}{2^{p+2} \sqrt{\pi} \Gamma((p+1)/2) \Gamma(-p/q)} \int_{\mathbb{R}^n} |\langle x, \xi \rangle|^p \, d\gamma_q(\xi).
\]

Now we can use Lemma 2 to compute the Fourier transform of the function \( \|x\|^q \). Let \( \nu_{p,q} \) be the \( p \)-projection of the \( q \)-stable measure \( \gamma_q(\xi) \, d\xi \) onto \( S^{n-1} \).

**Theorem 5.** If \( 0 < p < q < 2 \) then the Fourier transform of the function \( \|x\|^p_q \) coincides on \( \mathbb{R}^n \setminus \{0\} \) with the distribution \( (q/(2 \Gamma(-p/q))) |t|^{-1-p} \, d\nu_{p,q}(\xi) \). (For the definition of this distribution, see §3.)

6. Two-dimensional spaces

In this section we shall investigate the inverse problem for two-dimensional spaces.

Let \( \mathcal{D}(\Omega) \) denote the space of functions from \( S(\mathbb{R}^2) \) with compact support in an open set \( \Omega \subset \mathbb{R}^2 \), and \( G = \mathbb{R}^2 \setminus \{(x, y): y = 0\} \).
We shall say that a two-dimensional Banach space \( E = \text{span}(e_1, e_2) \) admits the Levy representation with an exponent \( p > 0 \), \( p \neq 2, 4, 6, \ldots \) and a distribution \( \gamma \) over \( S(\mathbb{R}) \) if, for every function \( \varphi \in S(\mathbb{R}^2) \) with \( \phi \in \mathcal{D}(G) \), the following equality holds:

\[
(7) \quad \int_{\mathbb{R}^2} \|xe_1 + ye_2\|^p \varphi(x, y) \, dx \, dy = (\gamma(\xi), \int_{\mathbb{R}^2} |\xi - y|^p \varphi(x, y) \, dx \, dy).
\]

If \( \phi \in \mathcal{D}(G) \), then it follows from Lemma 1 that

\[
(8) \quad \psi(\xi) = (\int_{\mathbb{R}^2} |\xi - y|^p \varphi(x, y) \, dx \, dy) = \frac{C_p}{(2\pi)^2} \int_{\mathbb{R}} |t|^{-1-p} \phi(t\xi, -t) \, dt,
\]

and the function \( \psi \) belongs to \( \mathcal{D}(\mathbb{R}) \). (\( C_p = 2^{p+1} \sqrt{\pi \Gamma((p + 1)/2) / \Gamma(-p/2)} \), here.) Moreover, for every function \( \psi \in \mathcal{D}(\mathbb{R}) \), there exists an even function \( \varphi \in S(\mathbb{R}^2) \) with \( \phi \in \mathcal{D}(G) \) for which (8) holds. Therefore, the distribution \( \gamma \) in (7) is unique, if it exists.

If the space admits a Levy representation, then the Fourier transform of the norm can be computed. Indeed, for each function \( \varphi \in \mathcal{D}(G) \), we get from (7) and (8) that

\[
\langle (\|xe_1 + ye_2\|^p), \varphi \rangle = (\gamma(\xi), \int_{\mathbb{R}^2} |\xi - y|^p \varphi(x, y) \, dx \, dy)
\]

\[
= (\gamma(\xi), \int_{\mathbb{R}} |t|^{-1-p} \phi(t\xi, -t) \, dt)
\]

Lemma 4. \( (\|xe_1 + ye_2\|^p)^\wedge = 0 \) on some open set in \( \mathbb{R}^2 \) iff \( \gamma = 0 \) on some open set in \( \mathbb{R} \).

Now we shall prove the existence of a Levy representation for every two-dimensional space and get a suitable expression for \( \gamma \).

For an arbitrary \( \varphi \in S(\mathbb{R}^2) \) let us define the function

\[
\varphi_1(t) = \int_{\mathbb{R}} \left| x \right|^{p+1} \varphi(x, tx) \, dx.
\]

For every integer \( n \geq p + 3 \), there exists a constant \( k_n \) such that \( |\varphi(x, y)| \leq k_n (1 + (x^2 + y^2)^{1/2})^{-n} \) for all \( x, y \in \mathbb{R} \). Therefore, \( \varphi_1 \) is a continuous function on \( \mathbb{R} \) satisfying

\[
|\varphi_1(t)| \leq k_n \int_{\mathbb{R}} \frac{|x|^{p+1} \, dx}{(1 + (x^2 + t^2)^{1/2})^n} \leq k_1 (1 + t^2)^{-1-p/2}
\]

for some \( k > 0 \). In particular, \( \varphi_1 \in L_1(\mathbb{R}) \).

As we mentioned, for each function \( \psi \in \mathcal{D}(\mathbb{R}) \) there exists an even function \( \varphi \in S(\mathbb{R}^2) \) with \( \phi \in \mathcal{D}(G) \) satisfying \( \psi(\xi) = \int_{\mathbb{R}^2} |\xi - y|^p \varphi(x, y) \, dx \, dy = \int_{\mathbb{R}} |\xi - t|^p \varphi_1(t) \, dt = (|t|^p \ast \varphi_1)(\xi) \). We can use Lemma 1 from [10] to verify
that $\dot{\psi}(t) = C_p|t|^{-1-p}\dot{\varphi}_1(t)$ for all $t \in \mathbb{R}, \ t \neq 0$. Hence $C_p\varphi_1 = (|t|^{p+1}\dot{\psi}(t))^\gamma = \psi^{(p+1)}$ is the $(p + 1)$th fractional derivative of the function $\psi$ ($\gamma$ denotes the inverse Fourier transform).

Now we define the $(p + 1)$th fractional derivative of the function $\|e_1 + te_2\|^p$. For every $\psi \in \mathcal{L}(\mathbb{R})$, we put

$$\langle (\|e_1 + te_2\|^p)^{(p+1)}, \psi \rangle = \int_\mathbb{R} \|e_1 + te_2\|^p \psi^{(p+1)}(t) \, dt.$$  

(The integral on the right-hand side exists by (9).) If the Fourier transform $(\|e_1 + te_2\|^p)^\gamma$ is regular, then the fractional derivative can be computed:

$$(\|e_1 + te_2\|^p)^{(p+1)} = (\|e_1 + te_2\|^p)^\gamma(x)^\gamma.$$  

**Theorem 6.** If $p > 0$, $p \neq 2, 4, 6, \ldots$, then (1) an arbitrary two-dimensional space $E$ admits the Levy representation with exponent $p$ and distribution $\gamma = (1/C_p)(\|e_1 + te_2\|^p)^{(p+1)}$ and (2) an exponent $p$ is exceptional for $E$ iff $(\|e_1 + te_2\|^p)^{(p+1)} = 0$ on some open subset of $\mathbb{R}$.

**Proof.** For every function $\varphi \in S(\mathbb{R}^2)$ with $\varphi \in \mathcal{S}(\mathcal{G})$ we have

$$\left\langle (\|e_1 + te_2\|^p)^{(p+1)}(\xi), \int_{\mathbb{R}^2} |x\xi - y|^p \varphi(x, y) \, dx \, dy \right\rangle$$

$$= C_p \int_{\mathbb{R}} \|e_1 + te_2\|^p \varphi_1(t) \, dt$$

$$= C_p \int_{\mathbb{R}} \|e_1 + te_2\|^p \, dt \int_{\mathbb{R}} |x|^{p+1} \varphi(x, tx) \, dx$$

$$= C_p \int_{\mathbb{R}^2} |xe_1 + ye_2|^p \varphi(x, y) \, dx \, dy.$$  

The second statement follows now from Lemma 4.

**Corollary.** The exponent $p = 1$ is exceptional for a two-dimensional space $E$ iff this space is not strictly convex.

Indeed, if $p = 1$, then the $(p + 1)$th fractional derivative coincides with the ordinary second derivative. So the exponent $p = 1$ is exceptional for $E$ iff $\|e_1 + te_2\|'' = 0$ on some open segment in $\mathbb{R}$; i.e., $\|e_1 + te_2\|$ is a linear function on some open segment in $\mathbb{R}$, and so $E$ is not strictly convex.

**Bibliography**


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