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# CONVOLUTION EQUATIONS IN CERTAIN BANACH SPACES

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ABSTRACT. For a Banach space $E$ and $p > 0$, the following problem is considered: how to identify a finite Borel measure $\mu$ on $E$ by means of the potential $g(a) = \int_E \|x - a\|^p \, d\mu(x)$, $a \in E$. The solution for infinite-dimensional Hilbert spaces is based on limit correlations between the Fourier transforms of finite-dimensional restrictions of $g$ and $\|x\|^p$. For finite-dimensional subspaces of $L_p$, the Levy representation of norms is used.

## 1. INTRODUCTION

Let $(E, \|\cdot\|)$ be a Banach space, $M_p$ be the set of Borel measures $\mu$ on $E$ satisfying $\int_E (1 + \|x\|)^p \, d\mu(x) < \infty$. For $\mu \in M_p$, let $g$ be the potential of $\mu$:

$$g(a) = \int_E \|x - a\|^p \, d\mu(x), \; a \in E.$$

The problem is to identify $\mu$ by means of $g$.

Thus we are going to study a sort of convolution equation. The uniqueness problem for this equation has been investigated by several authors. For some special Banach spaces $E$ the solution of this equation is unique for all $p > 0$, except for a countable set of exponents $p$, which will be called exceptional for the space $E$. For instance, in the one-dimensional case the exceptional exponents are the even numbers, [19–21], and the same is true for separable Hilbert spaces [see [22, 1, 9] for the finite-dimensional case and [10], [15] for the infinite-dimensional case]. Exceptional for $L_q$-spaces are the numbers $p$, for which $p/q \in \mathbb{N}$ and, besides that, in the case of $n$-dimensional space $l_q^n$ one of the following three conditions must be fulfilled: (a) $p/q < n$; (b) $q$ is an even integer; (c) $q$ and $(p/q) - n$ are odd integers. For the complex space $l_\infty^n$, even integers are exceptional; and for the real space, $p$ is exceptional iff $n + p$ is odd. The set of exceptional exponents $p > 0$ is empty for spaces $C(K)$,

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where $K$ is an infinite metric compact space without isolated points, and for spaces $C_0(\Omega)$, where $\Omega$ is a noncompact, locally compact Hausdorff space. If $K$ contains isolated points, then positive $p \notin \mathbb{N}$ are not exceptional, and the problem is still open for $p \in \mathbb{N}$ [for all these results, see [5, 6, 11, 16].

If $E$ is a finite-dimensional space, the potential $g$ can be considered as the convolution of distributions $\|x\|^p$ and $\mu$ over the space $S = S(\mathbb{R}^n)$ of rapidly decreasing functions. This convolution is, as a rule, naturally connected with the Fourier transform [6], so to solve the convolution equation one can compute $(\|x\|^p)^\wedge$, verify that $(\|x\|^p)^\wedge \neq 0$ on open sets and put $\hat{\mu} = \hat{g}/(\|x\|^p)^\wedge$ (throughout we denote by $\hat{f}$ the Fourier transform of a distribution $f$). However, the complete solution of the inverse problem has been obtained only in the case $E = \mathbb{R}^n$, when $(\|x\|^p)^\wedge$ can easily be computed. For other finite-dimensional spaces only uniqueness theorems are available, because attempts at straightforward computation of $(\|x\|^p)^\wedge$ have been unsuccessful. In the following, an analytic continuation of $\|x\|^p$ was used to check that $(\|x\|^p)^\wedge \neq 0$ on open sets.

In this paper the Fourier transform of norms in some finite-dimensional spaces is computed with the help of isometric embedding of these spaces into $L_p$. In §4 this is done for an arbitrary $n$-dimensional subspace $E = \operatorname{span}(f_1, \dots, f_n)$ of $L_p$. If $p \neq 2, 4, 6, \dots$ then $p$ is exceptional for $E$ iff the joint distribution of $f_1, \dots, f_n$ vanishes on some open cone in $\mathbb{R}^n$. (We recall that this joint distribution is a measure on $\mathbb{R}^n$.) In the same section, any easy proof is obtained for the well-known equimeasurability theorem for $L_p$-isometries [19–21, 17, 7, 12].

In §5 we give some concrete consequences of results of §4. The construction of isometric embedding of $l_q^n$ into $L_p (0 < p < q < 2)$ from [2] is used to obtain the Levy representation of norms and, as a consequence, to express the Fourier transform of $l_q^n$-norm in terms of $q$-stable measures. In §6 we investigate the inverse problem for two-dimensional spaces. An exponent $p > 0$ is exceptional for $E = \operatorname{span}(e_1, e_2)$ iff the $(p+1)$th fractional derivative of the function $\|e_1 + te_2\|^p$ vanishes on some open subset of $\mathbb{R}$. In particular, the exponent $p = 1$ is exceptional for a two-dimensional space iff this space is not strictly convex.

A connection between the Fourier transform and isometric embedding into $L_p$ is discussed in §3. In particular, it is proved in this section that, for an arbitrary even function $h$ on $\mathbb{R}$ satisfying some mild additional assumptions, if $\int_{S^{n-1}} h(\langle x, \xi\rangle)\, d\nu(\xi) = 0$ for each $x \in \mathbb{R}^n$, then either $\nu = 0$ or $h$ is a polynomial containing even powers only (here $\langle x, \xi\rangle$ stands for the scalar product in $\mathbb{R}^n$ and $\nu$ is a symmetric charge on the unit sphere $S^{n-1}$ in $\mathbb{R}^n$). This statement for $h(t) = |t|^p$ has been proved by different methods in [8, 14, 18].

In §2 the convolution equation is solved for infinite-dimensional Hilbert spaces. This solution is based on limit correlations between the Fourier transforms of functions $g$ and $\|x\|^p$.

All definitions and facts about distributions over $S(\mathbb{R}^n)$ used in this paper can be found in [3].

## 2. Infinite-dimensional Hilbert spaces

Let $E$ be the separable Hilbert space $l_2$ and $e_i$, $i \in \mathbb{N}$, be the standard basis in $l_2$, $E_n = \operatorname{span}(e_i, i = 1, \dots, n)$. For $x = \sum_1^\infty x_i e_i \in l_2$, we write $x^{(n)} = \sum_1^n x_i e_i$ (so $x^{(n)}$ is the projection of $x$ to $E_n$).

Let $\mu$ be a Borel measure on $l_2$, $p \in \mathbb{R}$. We shall assume that $\mu \in M_p$ if $p > 0$, and that $\mu(l_2) < \infty$ if $p < 0$. Note that the restriction $g_n$ of the potential $g(a) = \int_E \|x - a\|^p \, d\mu(x)$ to the subspace $E_n$ is locally an $L_1$-function on $E_n$ with respect to Lebesgue measure on $E_n$, if $-n < p < 0$ (see, for instance, [13, p. 84]). So $g_n$ can be considered as a distribution over $S(\mathbb{R}^n)$.

**Theorem 1.** *If $p \in \mathbb{R}$, $p \neq 0, 2, 4, \dots$, then for each $\xi \in l_2, \xi \neq 0$,*

$$\hat{\mu}(\xi) = \lim_{n \to \infty} \frac{\hat{g}_n(\xi^{(n)}) \|\xi^{(n)}\|^{n+p} \Gamma(-p/2)}{2^{n+p} \pi^{n/2} \Gamma((n+p)/2)}$$

*Proof.* Let us fix an element $x \in l_2$ and assume first that $-1 < p < 0$. In order to compute the Fourier transform of the function $\|x - a\|^p$ of the variable $a \in E_n$, we can use the following representation, which is an easy consequence of the definition of the $\Gamma$-function:

$$\|x - a\|^p = \frac{2}{\Gamma(-p/2)} \int_0^\infty t^{-1-p} \exp(-t^2 \|x - a\|^2) \, dt.$$

Now for each fixed $t > 0$, the Fourier transform of the function $a \mapsto \exp(-t^2 \|x - a\|^2)$, $a \in E_n$, can easily be computed:

$$(\exp(-t^2 \|x - a\|^2))\,\hat{}\,(\zeta) = \pi^{n/2} t^{-n} \exp(-i\langle x^{(n)}, \zeta\rangle - t^2 \|x - x^{(n)}\|^2 - \|\zeta\|^2/4t^2),$$

$\zeta \in \mathbb{R}^n$. Consequently, for every $\zeta \in \mathbb{R}^n$, $\zeta \neq 0$,

$$(1)\quad (\|x - a\|^p)\,\hat{}\,(\zeta) = \frac{2\pi^{n/2}}{\Gamma(-p/2)} \exp(-i\langle x^{(n)}, \zeta\rangle) \int_0^\infty t^{-1-p-n} \exp(-t^2 \|x - x^{(n)}\|^2 - \|\zeta\|^2/4t^2) \, dt.$$

If we allow $p$ to assume complex values, then both sides of (1) are analytic functions of $p$ in the domain $\{\operatorname{Re} p > -n, p \neq 0, 2, 4, 6, \dots\}$. So these functions admit a unique analytic continuation from the interval $(-1, 0)$, and (1) remains true for all real $p > -n$, $p \neq 0, 2, 4, \dots$.

Making in (1) the change of variables $y = 1/t$ and then integrating against $d\mu(x)$, we obtain

$$(2)\quad \hat{g}_n(\zeta) = \frac{2\pi^{n/2}}{\Gamma(-p/2)} \times \int_{l_2} \exp(-i\langle x^{(n)}, \zeta\rangle) \times \left( \int_0^\infty y^{n+p-1} \exp(-y^2 \|\zeta\|^2/4 - \|x - x^{(n)}\|^2/y^2) \, dy \right) d\mu(x)$$

for every $\zeta \in \mathbb{R}^n$, $\zeta \neq 0$, and $p > -n$, $p \neq 0, 2, 4, \dots$.

Now let us consider an arbitrary $p \in \mathbb{R}$, $p \neq 0, 2, 4, \dots$, and $\xi \in l_2$, $\xi \neq 0$. For each $n \in \mathbb{N}$ with $\xi^{(n)} \neq 0$ and $p > -n$, put $\zeta = \xi^{(n)}$ in (2) and divide both sides of (2) by

$$\frac{2\pi^{n/2}}{\Gamma(-p/2)} \int_0^\infty y^{n+p-1} \exp(-y^2\|\xi^{(n)}\|^2/4)\, dy = \frac{2^{n+p}\pi^{n/2}\|\xi^{(n)}\|^{-n-p}\Gamma(\frac{n+p}{2})}{\Gamma(-p/2)}.$$

This leads to

(3)

$$\frac{\hat{g}_n(\xi^{(n)})\|\xi^{(n)}\|^{n+p}\Gamma(-p/2)}{2^{n+p}\pi^{n/2}\Gamma(\frac{n+p}{2})} = \int_{l_2} \exp(-i\langle x^{(n)}, \xi^{(n)}\rangle) \times \frac{\int_0^\infty y^{n+p-1}\exp(-y^2\|\xi^{(n)}\|^2/4 - \|x - x^{(n)}\|^2/y^2)\, dy}{\int_0^\infty y^{n+p-1}\exp(-y^2\|\xi^{(n)}\|^2/4)\, dy}\, d\mu(x).$$

The absolute value of the integrand on the right is majorated by 1 and for each $x \in l_2$ tends to $\exp(-i\langle x, \xi\rangle)$ as $n \to \infty$. In fact, $1 - \exp(-z) < z$ for $z > 0$, so the difference between the fraction on the right-hand side of (3) and the number 1 is less than

$$\|x - x^{(n)}\|^2 \frac{\int_0^\infty y^{n+p-3}\exp(-y^2\|\xi^{(n)}\|^2/4)\, dy}{\int_0^\infty y^{n+p-1}\exp(-y^2\|\xi^{(n)}\|^2/4)\, dy} = \frac{\|x - x^{(n)}\|^2\|\xi^{(n)}\|^2}{2(n+p-2)},$$

which tends to zero as $n \to \infty$. Now we can apply the Lebesgue dominated convergence theorem to obtain that the quantity on the right in (3) tends to $\hat{\mu}(\xi)$ as $n \to \infty$.

## 3. Uniqueness theorem for measures on $S^{n-1}$

Let $H$ denote the set of even, continuous functions $h$ on $\mathbb{R}$ with tempered growth at infinity (i.e., $\lim_{|x|\to\infty}(h(x)/|x|^\rho) = 0$ for some $\rho > 0$), for which $\hat{h}$ is a continuous function on $\mathbb{R}\setminus\{0\}$ with tempered growth at infinity.

Let $\nu$ be a symmetric Borel charge of bounded variation on the unit sphere $S^{n-1}$ in $\mathbb{R}^n$.

The distribution $\hat{h}(t)\, d\nu(\xi)$ is defined by

$$\langle \hat{h}(t)\, d\nu(\xi), \varphi\rangle = \int_{S^{n-1}} d\nu(\xi) \int_{\mathbb{R}} \hat{h}(t)\varphi(t\xi)\, dt = \int_{S^{n-1}} \langle \hat{h}, \varphi(t\xi)\rangle\, d\nu(\xi)$$

for each $\varphi \in S(\mathbb{R}^n)$ with $0 \notin \operatorname{supp}\varphi$.

**Lemma 1.** *For every $\xi_0 \in S^{n-1}$, the Fourier transform of the function $h(\langle x, \xi_0\rangle)$ coincides on $\mathbb{R}^n\setminus\{0\}$ with the distribution $\hat{h}(t)\, d\delta_{\xi_0}(\xi)$, where $\delta_{\xi_0}$ is the unit mass at the point $\xi_0$.*

*Proof*. By the Fubini theorem, for every even function $\varphi \in S(\mathbb{R}^n)$ with $0 \notin \operatorname{supp}\varphi$, we have

$$\langle (h(\langle x, \xi_0\rangle))^\frown, \varphi\rangle = \langle h(\langle x, \xi_0\rangle), \hat{\varphi}\rangle$$
$$= \int_{\mathbb{R}^n} h(\langle x, \xi_0\rangle)\hat{\varphi}(x)\,dx = \int_{\mathbb{R}} h(t)\left(\int_{\langle x, \xi_0\rangle = t} \hat{\varphi}(x)\,dx\right)dt.$$

The even function $\varphi(t\xi_0)$ is the Fourier transform of the even function $\int_{\langle x, \xi_0\rangle = t}\hat{\varphi}(x)\,dx$ of the variable $t \in \mathbb{R}$. (It is a simple property of a Radon transform [4, p. 19].) So we have

$$\langle (h(\langle x, \xi_0\rangle))^\frown, \varphi\rangle = \left\langle h, \int_{\langle x, \xi_0\rangle = t}\hat{\varphi}(x)\,dx\right\rangle = \langle \hat{h}, \varphi(t\xi_0)\rangle = \langle \hat{h}(t)\,d\delta_{\xi_0}(\xi), \varphi\rangle$$

and even distributions $(h(\langle x, \xi_0\rangle))^\frown$ and $\hat{h}(t)\,d\delta_{\xi_0}(\xi)$ coincide on $\mathbb{R}^n\backslash\{0\}$.

The next result is a consequence of Lemma 1 and the Fubini theorem:

**Lemma 2.** *Let $h \in H$ and $\nu$ be a symmetric Borel charge of bounded variation on $S^{n-1}$. Then the Fourier transform of the function*

$$f(x) = \int_{S^{n-1}} h(\langle x, \xi\rangle)\,d\nu(\xi)$$

*coincides on $\mathbb{R}^n\backslash\{0\}$ with the distribution $\hat{h}(t)\,d\nu(\xi)$.*

*Proof*. For an arbitrary even function $\varphi \in S(\mathbb{R}^n)$ with $0 \notin \operatorname{supp}\varphi$

$$\langle \hat{f}, \varphi\rangle = \langle f, \hat{\varphi}\rangle = \int_{S^{n-1}} d\nu(\xi)\int_{\mathbb{R}^n} h(\langle x, \xi\rangle)\hat{\varphi}(x)\,dx$$
$$= \int_{S^{n-1}} d\nu(\xi)\int_{\mathbb{R}} \hat{h}(t)\varphi(t\xi)\,dt = \langle \hat{h}(t)\,d\nu(\xi), \varphi\rangle.$$

**Theorem 2.** *If $\int_{S^{n-1}} h(\langle x, \xi\rangle)\,d\nu(\xi) = 0$ for all $x \in \mathbb{R}^n$, then either $\nu = 0$ or $h$ is a polynomial containing even powers only.*

*Proof*. By Lemma 2, $\hat{h}(t)\,d\nu(\xi) = 0$ everywhere on $\mathbb{R}^n\backslash\{0\}$. This is possible only if either $\nu = 0$ or $\hat{h}$ is a distribution with support in $\{0\}$. Since $h$ is even, in the last case $h$ must be a polynomial containing even powers only.

## 4. Finite-dimensional subspaces of $L_p$

Let $(E, \|\cdot\|)$ be an $n$-dimensional subspace of $L_p(\Omega, \sigma)$, where $p > 0$ and $(\Omega, \sigma)$ is a measure space, $\sigma(\Omega) < \infty$.

Consider an arbitrary basis $f_1, \dots, f_n$ in $E$. Let $\mu$ be the joint distribution of functions $f_1, \dots, f_n$ with respect to $\sigma$; that is, $\mu(B) = \sigma\{\omega \in \Omega\colon (f_1(\omega), \dots, f_n(\omega)) \in B\}$ for every Borel subset $B$ of $\mathbb{R}^n$. Note that $\mu$ is a finite Borel measure on $\mathbb{R}^n$.

A measure $\nu$ on the unit sphere $S^{n-1}$ in $\mathbb{R}^n$, defined by

$$\nu(B) = \frac{1}{2}\int_{B\times\mathbb{R}} \|x\|_2^p \, d\mu(x)$$

for all Borel subsets $B$ of $S^{n-1}$, will be called the $p$-projection of $\mu$ to $S^{n-1}$. (Here $\|x\|_2$ is the Euclidean norm, $B\times\mathbb{R} = \{y \in \mathbb{R}^n, y \neq 0 \colon y/\|y\|_2 \in B \cup (-B)\}$.)

The norm in $E$ can be represented in the following way:

$$\|x\|^p = \left\|\sum_1^n x_i f_i\right\|^p = \int_\Omega \left|\sum_1^n x_i f_i(\omega)\right|^p d\sigma(\omega) = \int_{\mathbb{R}^n} |\langle x, \xi\rangle|^p \, d\mu(\xi)$$
$$= \int_{S^{n-1}} |\langle x, \xi\rangle|^p \, d\nu(\xi).$$

(In such situations we shall say that the norm in $E$ admits the Levy representation with the measure $\mu$.)

Now we apply Lemma 2 with $h(y) = |y|^p$, $y \in \mathbb{R}$, to compute the Fourier transform of $\|x\|^p$. Note $(|y|^p)^\wedge(t) = (2^{p+1}\sqrt{\pi}\Gamma((p+1)/2)/\Gamma(-p/2))|t|^{-1-p}$, if $p > 0$, $p \neq 2, 4, 6, \dots$ [3, p. 217].

**Lemma 3.** *If $p > 0$, $p \neq 2, 4, 6, \dots$ then*

$$(\|x\|^p)^\wedge = \frac{2^{p+1}\sqrt{\pi}\Gamma((p+1)/2)}{\Gamma(-p/2)}|t|^{-1-p} \, d\nu(\xi).$$

It is clear now that $(\|x\|^p)^\wedge = 0$ on an open subset of $\mathbb{R}^n$ iff $\nu = 0$ on some open subset of $\mathbb{R}^{n-1}$. So we have the following:

**Theorem 3.** *If $p > 0$, $p \neq 2, 4, 6, \dots$ then $p$ is an exceptional exponent for the space $E = \operatorname{span}(f_1, \dots, f_n) \subset L_p(\Omega, \sigma)$ iff $\mu(B\times\mathbb{R}) = 0$ for some open subset $B$ of $S^{n-1}$, where $\mu$ is the joint distribution of functions $f_1, \dots, f_n$ with respect to $\sigma$.*

Suppose that $\Omega$ is a topological space, $\sigma$ is a finite Borel measure on $\Omega$ which does not vanish on open sets, and $f_1, \dots, f_n$ are continuous functions on $\Omega$. Let $V$ be the subset of $\mathbb{R}^{n-1}$ consisting of all points of the form $(f_2(\omega)/f_1(\omega), \dots, f_n(\omega)/f_1(\omega))$ or $(-f_2(\omega)/f_1(\omega), \dots, -f_n(\omega)/f_1(\omega))$, where $\omega$ runs over the set $\Omega\backslash f_1^{-1}(0)$. In this case $\mu(B\times\mathbb{R}) \neq 0$ for all open subsets $B$ of $S^{n-1}$ iff $V$ is dense in $\mathbb{R}^{n-1}$.

**Example.** Let $\Omega = S^1$ be the unit circle in $\mathbb{R}^2$ with (linear) Lebesgue measure, $p > 0$, $p \neq 2, 4, 6, \dots$. Then for the space $E_1 = \operatorname{span}(\sin\omega, \sin 2\omega) \subset L_p(S^1)$, we have $V = (-2, 2)$, and $p$ is an exceptional exponent for $E_1$. If $E_2 = \operatorname{span}(\sin 2\omega, \sin 3\omega) \subset L_p(S^1)$, then $V = \mathbb{R}$, and $p$ is not exceptional for $E_2$.

Let $(\Omega, \mathscr{B}, \sigma)$ and $(\Omega', \mathscr{B}', \sigma')$ be measure spaces with finite measures, $p > 0$, $p \neq 2, 4, 6, \dots$ and $Y$ be an arbitrary (maybe infinite-dimensional) subspace of $L_p(\Omega)$. Suppose that a linear isometry $T$ maps $Y$ into $L_p(\Omega')$.

The well-known continuation theorem for $L_p$-isometries [see 19, 20, or 7] states that $T$ can be extended to the space $L_p(\Omega, \mathscr{B}_0, \sigma)$ as a linear isometry, where $\mathscr{B}_0$ is a minimal $\sigma$-algebra contained in $\mathscr{B}$, making all functions in the space $Y$ measurable.

This result was obtained as a straightforward consequence of the following equimeasurability theorem for $L_p$-isometries [see 19–21, 17, 7]:

**Theorem 4.** *For arbitrary functions $f_1, \dots, f_n \in Y$ we have $\mu_1 = \mu_2$, where measures $\mu_1$ and $\mu_2$ on $\mathbb{R}^n$ are the joint distributions of the $n$-tuples*

$$(\mathbf{1}, f_2(\omega)/f_1(\omega), \dots, f_n(\omega)/f_1(\omega))$$

*and*

$$(\mathbf{1}, Tf_2(\omega')/Tf_1(\omega'), \dots, Tf_n(\omega')/Tf_1(\omega'))$$

*with respect to measures $|f_1|^p\,d\sigma$ and $|Tf_1|^p\,d\sigma'$ accordingly.*

Lemma 2, above, provides a simple proof of Theorem 4. Indeed, $\|\sum_1^n x_i f_i\|^p = \|\sum_1^n x_i Tf_i\|^p$ for each $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, so

$$\begin{aligned}
&\int_\Omega \left|\sum_1^n x_i f_i(\omega)\right|^p d\sigma(\omega) \\
&= \int_{f_1(\omega)\neq 0} \left|x_1 + \sum_2^n x_i \frac{f_i(\omega)}{f_1(\omega)}\right|^p |f_1(\omega)|^p\,d\sigma(\omega) + \int_{f_1(\omega)=0} \left|\sum_2^n x_i f_i(\omega)\right|^p d\sigma(\omega) \\
&= \int_{\mathbb{R}^n} |\langle x, \xi\rangle|^p\,d\mu_1(\xi) + \psi(x_2, \dots, x_n) \\
&= \int_{\Omega'} \left|\sum_1^n x_i Tf_i(\omega')\right|^p d\sigma'(\omega') = \int_{\mathbb{R}^n} |\langle x, \xi\rangle|^p\,d\mu_2(\xi) + \psi'(x_2, \dots, x_n).
\end{aligned}$$

Consider the Fourier transforms of these functions of variables $x_1, \dots, x_n$. The Fourier transforms of functions $\psi$ and $\psi'$ are supported on the hyperplane $\xi_1 = 0$ in $\mathbb{R}^n$. The measures $\mu_1$ and $\mu_2$ are supported on the hyperplane $\xi_1 = 1$ in $\mathbb{R}^n$ so the $p$-projections $\nu_1$ and $\nu_2$ of these measures to $S^{n-1}$ are supported out of the hyperplane $\xi_1 = 0$. By Lemma 2, $\nu_1 = \nu_2$, and therefore $\mu_1 = \mu_2$.

## 5. Levy representations

Let $(E, \|\cdot\|)$ be an $n$-dimensional Banach space. Suppose that there exists an even function $f \in L_1(\mathbb{R})$ with $(f(\|x\|))^\wedge = u \in L_1(\mathbb{R}^n)$. Then for every $k \in \mathbb{R}$ and $x \in E$, $x \neq 0$

$$(2\pi)^n f(k\|x\|) = \int_{\mathbb{R}^n} \exp(ik\|x\|(\langle x, \xi\rangle/\|x\|))u(\xi)\,d\xi = \int_{\mathbb{R}} \exp(ik\|x\|y)\,du_1(y),$$

where $u_1$ is the image of the charge $u(\xi)\,d\xi$ under the mapping $\xi \mapsto \langle x, \xi\rangle/\|x\|$. Here $k$ is arbitrary; therefore, $u_1 = \hat{f}$, and $u_1$ does not depend on $x \in E$, $x \neq 0$.

Now for every $p \in \mathbb{R}$ we obtain, assuming that the first integral converges:

$$\int_{\mathbb{R}^n} |\langle x, \xi\rangle|^p u(\xi)\, d\xi = \|x\|^p \int_{\mathbb{R}^n} \left|\frac{\langle x, \xi\rangle}{\|x\|}\right|^p u(\xi)\, d\xi = \|x\|^p \int_{\mathbb{R}} |t|^p \, du_1(t).$$

So if the $p$ th moment of the charge $u_1$ exists and is not zero then we have the Levy representation with the charge $u(\xi)\, d\xi$:

$$\|x\|^p = \frac{1}{\int_{\mathbb{R}} |t|^p \, du_1(t)} \int_{\mathbb{R}^n} |\langle x, \xi\rangle|^p u(\xi)\, d\xi. \tag{4}$$

In order to compute the moment, we assume that $p \in (-1, 0)$ and use the Parseval theorem:

$$\frac{2^{p+1}\sqrt{\pi}\Gamma((p+1)/2)}{\Gamma(-p/2)} \int_{\mathbb{R}} |t|^{-1-p} f(t)\, dt = \int_{\mathbb{R}} |t|^p \, du_1(t). \tag{5}$$

If both sides of (5) are analytic functions of the variable $p$ in some domain in $\mathbb{C}$, we can use analytic continuation and compute moments for other $p$'s.

For instance, if $E = l_q^n$, $f(t) = \exp(-|t|^q)$ and $0 < p < q \le 2$, then $u(\xi) = \gamma_q(\xi) = (\exp(-\|x\|_q^q))^\wedge(\xi)$ is the density of the $n$-dimensional $q$-stable measure, and $u_1$ is the one-dimensional stable measure. The $p$ th moment of $u_1$ is finite and can easily be computed [23, p. 75]. Indeed, for $p \in (-1, 0)$

$$\int_{\mathbb{R}} |t|^{-1-p} \exp(-|t|^q)\, dt = \frac{2\Gamma(-p/q)}{q},$$

and it follows from (5) and the analytic continuation argument that, for every $p \in (0, q)$,

$$\int_{\mathbb{R}} |t|^p \, du_1(t) = \frac{\Gamma(-p/q) 2^{p+2}\sqrt{\pi}\,\Gamma((p+1)/2)}{q\Gamma(-p/2)}.$$

Now use (4) to get the Levy representation

$$\|x\|_q^p = \frac{q\Gamma(-p/2)}{2^{p+2}\sqrt{\pi}\Gamma((p+1)/2)\Gamma(-p/q)} \int_{\mathbb{R}^n} |\langle x, \xi\rangle|^p \, d\gamma_q(\xi).$$

Now we can use Lemma 2 to compute the Fourier transform of the function $\|x\|_q^p$. Let $\nu_{p,q}$ be the $p$-projection of the $q$-stable measure $\gamma_q(\xi)\, d\xi$ onto $S^{n-1}$.

**Theorem 5.** *If* $0 < p < q < 2$ *then the Fourier transform of the function* $\|x\|_q^p$ *coincides on* $\mathbb{R}^n \setminus \{0\}$ *with the distribution* $(q/(2\Gamma(-p/q)))|t|^{-1-p}\, d\nu_{p,q}(\xi)$. (*For the definition of this distribution, see* §3.)

## 6. Two-dimensional spaces

In this section we shall investigate the inverse problem for two-dimensional spaces.

Let $\mathscr{D}(\Omega)$ denote the space of functions from $S(\mathbb{R}^2)$ with compact support in an open set $\Omega \subset \mathbb{R}^2$, and $G = \mathbb{R}^2 \setminus \{(x, y) : y = 0\}$.

We shall say that a two-dimensional Banach space $E = \operatorname{span}(e_1, e_2)$ admits the Levy representation with an exponent $p > 0$, $p \neq 2, 4, 6, \ldots$ and a distribution $\gamma$ over $S(\mathbb{R})$ if, for every function $\varphi \in S(\mathbb{R}^2)$ with $\hat{\varphi} \in \mathscr{D}(G)$, the following equality holds:

$$(7) \qquad \int_{\mathbb{R}^2} \|xe_1 + ye_2\|^p \varphi(x, y)\, dx\, dy = \langle \gamma(\xi), \int_{\mathbb{R}^2} |x\xi - y|^p \varphi(x, y)\, dx\, dy \rangle .$$

If $\hat{\varphi} \in \mathscr{D}(G)$, then it follows from Lemma 1 that

$$(8) \qquad \psi(\xi) = \int_{\mathbb{R}^2} |x\xi - y|^p \varphi(x, y)\, dx\, dy = \frac{C_p}{(2\pi)^2} \int_{\mathbb{R}} |t|^{-1-p} \hat{\varphi}(t\xi, -t)\, dt,$$

and the function $\psi$ belongs to $\mathscr{D}(\mathbb{R})$. ( $C_p = 2^{p+1}\sqrt{\pi}\Gamma((p+1)/2)/\Gamma(-p/2)$, here.) Moreover, for every function $\psi \in \mathscr{D}(\mathbb{R})$, there exists an even function $\varphi \in S(\mathbb{R}^2)$ with $\hat{\varphi} \in \mathscr{D}(G)$ for which (8) holds. Therefore, the distribution $\gamma$ in (7) is unique, if it exists.

If the space admits a Levy representation, then the Fourier transform of the norm can be computed. Indeed, for each function $\varphi \in \mathscr{D}(G)$, we get from (7) and (8) that

$$\begin{aligned} \langle (\|xe_1 + ye_2\|^p)^\frown, \varphi \rangle &= \int_{\mathbb{R}^2} \|xe_1 + ye_2\|^p \hat{\varphi}(x, y)\, dx\, dy \\ &= \left\langle \gamma(\xi), \int_{\mathbb{R}^2} |x\xi - y|^p \hat{\varphi}(x, y)\, dx\, dy \right\rangle \\ &= C_p \left\langle \gamma(\xi), \int_{\mathbb{R}} |t|^{-1-p} \varphi(t\xi, -t)\, dt \right\rangle . \end{aligned}$$

**Lemma 4.** *$(\|xe_1 + ye_2\|^p)^\frown = 0$ on some open set in $\mathbb{R}^2$ iff $\gamma = 0$ on some open set in $\mathbb{R}$.*

Now we shall prove the existence of a Levy representation for every two-dimensional space and get a suitable expression for $\gamma$.

For an arbitrary $\varphi \in S(\mathbb{R}^2)$ let us define the function

$$\varphi_1(t) = \int_{\mathbb{R}} |x|^{p+1} \varphi(x, tx)\, dx .$$

For every integer $n \geq p + 3$, there exists a constant $k_n$ such that $|\varphi(x, y)| \leq k_n(1 + (x^2 + y^2)^{1/2})^{-n}$ for all $x, y \in \mathbb{R}$. Therefore, $\varphi_1$ is a continuous function on $\mathbb{R}$ satisfying

$$(9) \qquad |\varphi_1(t)| \leq k_n \int_{\mathbb{R}} \frac{|x|^{p+1}\, dx}{(1 + (x^2 + t^2x^2)^{1/2})^n} \leq k(1 + t^2)^{-1-p/2}$$

for some $k > 0$. In particular, $\varphi_1 \in L_1(\mathbb{R})$.

As we mentioned, for each function $\psi \in \mathscr{D}(\mathbb{R})$ there exists an even function $\varphi \in S(\mathbb{R}^2)$ with $\hat{\varphi} \in \mathscr{D}(G)$ satisfying $\psi(\xi) = \int_{\mathbb{R}^2} |x\xi - y|^p \varphi(x, y)\, dx\, dy = \int_{\mathbb{R}} |\xi - t|^p \varphi_1(t)\, dt = (|t|^p * \varphi_1)(\xi)$. We can use Lemma 1 from [10] to verify that $\hat{\psi}(t) = C_p|t|^{-1-p}\hat{\varphi}_1(t)$ for all $t \in \mathbb{R}$, $t \neq 0$. Hence $C_p\varphi_1 = (|t|^{p+1}\hat{\psi}(t))^{\vee} = \psi^{(p+1)}$ is the $(p+1)$ th fractional derivative of the function $\psi$ ($\vee$ denotes the inverse Fourier transform).

Now we define the $(p+1)$th fractional derivative of the function $\|e_1 + te_2\|^p$. For every $\psi \in \mathscr{D}(\mathbb{R})$, we put

$$\langle(\|e_1 + te_2\|^p)^{(p+1)}, \psi\rangle = \int_{\mathbb{R}} \|e_1 + te_2\|^p \psi^{(p+1)}(t)\, dt.$$

(The integral on the right-hand side exists by (9).) If the Fourier transform $(\|e_1 + te_2\|^p)^{\wedge}$ is regular, then the fractional derivative can be computed:

$$(\|e_1 + te_2\|^p)^{(p+1)} = (|x|^{p+1}(\|e_1 + te_2\|^p)^{\wedge}(x))^{\vee}.$$

**Theorem 6.** *If $p > 0$, $p \neq 2, 4, 6, \dots$, then* (1) *an arbitrary two-dimensional space $E$ admits the Levy representation with exponent $p$ and distribution $\gamma = (1/C_p)(\|e_1 + te_2\|^p)^{(p+1)}$ and* (2) *an exponent $p$ is exceptional for $E$ iff $(\|e_1 + te_2\|^p)^{(p+1)} = 0$ on some open subset of $\mathbb{R}$.*

*Proof.* For every function $\varphi \in S(\mathbb{R}^2)$ with $\hat{\varphi} \in \mathscr{D}(G)$ we have

$$\begin{aligned}
&\left\langle (\|e_1 + te_2\|^p)^{(p+1)}(\xi), \int_{\mathbb{R}^2} |x\xi - y|^p \varphi(x, y)\, dx\, dy \right\rangle \\
&= C_p \int_{\mathbb{R}} \|e_1 + te_2\|^p \varphi_1(t)\, dt \\
&= C_p \int_{\mathbb{R}} \|e_1 + te_2\|^p\, dt \int_{\mathbb{R}} |x|^{p+1} \varphi(x, tx)\, dx \\
&= C_p \int_{\mathbb{R}^2} \|xe_1 + ye_2\|^p \varphi(x, y)\, dx\, dy.
\end{aligned}$$

The second statement follows now from Lemma 4.

**Corollary.** *The exponent $p = 1$ is exceptional for a two-dimensional space $E$ iff this space is not strictly convex.*

Indeed, if $p = 1$, then the $(p+1)$ th fractional derivative coincides with the ordinary second derivative. So the exponent $p = 1$ is exceptional for $E$ iff $\|e_1 + te_2\|'' = 0$ on some open segment in $\mathbb{R}$; i.e., $\|e_1 + te_2\|$ is a linear function on some open segment in $\mathbb{R}$, and so $E$ is not strictly convex.

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