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# SINGULAR DEGENERATIONS OF CALABI-YAU MANIFOLDS AND THE WEIL-PETERSSON METRIC

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ABSTRACT. Degenerations of Calabi-Yau manifolds with isolated singularities somewhat more general than nodes are shown to be at finite distance from nonsingular manifolds in the Weil-Petersson metric on the appropriate moduli spaces.

If $\mathscr{M}(z)$ is a nonsingular complex threefold with trivial canonical bundle and holomorphic three-form $\Omega_z$, depending holomorphically on $z \in \mathscr{Z}$, for some complex manifold $\mathscr{Z}$, then $\log(\int_{\mathscr{M}(z)} \Omega_z \wedge \bar{\Omega}_z)$ is the Kähler potential for a pseudo-metric on $\mathscr{Z}$, which is a metric provided $\mathscr{Z}$ is an effective deformation space in the sense of [2]. This metric has come to be known in the physics literature as the Weil-Petersson metric, although the strict analogy with moduli spaces of curves holds only for curves of genus one.

This paper is a step in attempting to understand the completion of this Weil-Petersson metric by studying its behaviour with respect to singular degenerations. In particular, we study the case where $\mathscr{Z}$ is a punctured disk, and the holomorphic family $\{\mathscr{M}(z)\}$ can be completed by the inclusion of a variety $\mathscr{M}(0)$ with isolated singularities. We give some sufficient conditions for curves through the deleted point to have finite length. The case of nodal degenerations was discussed in [1].

We proceed mainly by studying the periods of $\Omega_z$. We shall denote by $C(z)$ a locally constant homology class in $H_3(\mathscr{M}(z))$.[1]

$$\int_{C(z)} \Omega_z$$

is a holomorphic, although possibly multiple valued, function of $z$. We shall usually assume that $z$ ranges over a punctured disk, $D - \{0\}$, and write $\mu$ for the monodromy operator on $H_3(\mathscr{M}(z))$. Our principal results, Theorems

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[1]We think of the parameter $z$ somewhat ambiguously as an element either of $\mathscr{Z}$ or of its universal covering space.

A and B below, generalize the case of a nodal degeneration, for which $\mu$ is a transvection and $(1-\mu)^2 = 0$.

**Theorem A.** *Let* $\{(\mathscr{M}(z), \Omega_z)\}_{0<|z|<r}$ *be a family of nonsingular threefolds with nowhere vanishing holomorphic three-form, parametrized holomorphically by a punctured disk about the origin in the complex plane. Suppose that*

(i) $(1-\mu^k)^2 = 0$ *for some positive integer* $k$.

(ii) *For any locally constant homology class,* $C(z) \in H_3(\mathscr{M}(z))$, $\int_{C(z)} \Omega_z$ *is bounded along any ray,* $\arg(z) = $ *constant.*

(iii) *For any locally constant class* $C(z) \in \operatorname{im}(1-\mu)$, *and any fixed value of* $\arg(z)$,

$$\lim_{|z|\to 0} |z|^{-\epsilon} \int_{C(z)} \Omega_z = 0$$

*for some* $\epsilon > 0$.

(iv) $\lim_{z\to 0} \int_{\mathscr{M}(z)} \Omega \wedge \bar{\Omega} \neq 0$ (*it will follow from* (i)–(iii) *that the limit exists*).

*Then a smooth path in the disk has finite length in the Weil-Petersson metric, even if it passes through the origin.*

**Theorem B.** *Let* $\mathscr{W}$ *be a fourfold with nowhere vanishing holomorphic four-form* $\Omega_{\mathscr{W}}$. *Let* $D$ *be a disk around the origin and let* $f\colon \mathscr{W} \to D$ *be a proper holomorphic complex valued function on* $\mathscr{W}$ *whose fibres are compact and, except for* $f^{-1}(0)$, *nonsingular. Suppose further that the holomorphic three-form* $\Omega_z$, *defined along* $\mathscr{M}(z) = f^{-1}(z)$ *at all regular points by the relation*

$$\Omega_z \wedge df = \Omega_{\mathscr{W}},$$

*is nowhere vanishing. Finally, suppose that the singularities of* $\mathscr{M}(0)$ *are all isolated and that each of them has a neighborhood in* $\mathscr{W}$ *which admits a coordinate system in which the coordinates may be weighted so that* $f$ *is weighted homogeneous with weight strictly lower than the sum of the weights of the coordinates. Then the holomorphic family* $\{(\mathscr{M}(z), \Omega_z)\}_{z\in D-\{0\}}$ *satisfies the hypotheses of Theorem A.*

*Proof of Theorem* A. By passing to a $k$-fold cyclic cover of the punctured disk we may, without loss of generality, confine our discussion to the case $(1-\mu)^2 = 0$. We observe next that we can regard $\Omega_z$ as an element of $H^3(\mathscr{M}(z), C)$, on which $\mu$ also operates by duality.

We write $\alpha = 1-\mu$ and

$$\hat{\Omega}_z = \Omega_z + \frac{\log(z)}{2\pi i}\alpha(\Omega_z).$$

Although multiple valued, $\hat{\Omega}_z$ depends holomorphically on $z$ and satisfies the identity

$$\Omega_z = \hat{\Omega}_z - \frac{1}{2\pi i}\log(z)\alpha(\hat{\Omega}_z).$$

It follows by a straightforward computation that the periods of $\hat{\Omega}_z$ are single-valued and therefore, by (ii), holomorphic in the unpunctured disk. Also by (iii), the periods of $\alpha(\hat{\Omega}_z)$ vanish at the origin. Now, for $z \neq 0$, we choose a locally constant symplectic basis $\{A_i(z), B_i(z)\}$ for $H_3(\mathscr{M}(z))$, and write

$$a_i(z) = \int_{A_i(z)} \hat{\Omega}_z, \qquad b_i(z) = \int_{B_i(z)} \hat{\Omega}_z,$$

$$c_i(z) = \int_{A_i(z)} \alpha(\hat{\Omega}_z) = \int_{\alpha(A_i(z))} \hat{\Omega}_z, \quad \text{and} \quad d_i(z) = \int_{B_i(z)} \alpha(\hat{\Omega}_z) = \int_{\alpha(B_i(z))} \hat{\Omega}_z.$$

We note that all these functions are single-valued and can be extended to be holomorphic in the unpunctured disk with $c_i(0) = d_i(0) = 0$.

Next we avail ourselves of the identity,

$$\int_{\mathscr{M}(z)} \Omega_z \wedge \overline{\Omega}_z = 2\mathfrak{I}\left(\sum_i \left(\int_{A_i(z)} \Omega_z\right)\left(\int_{B_i(z)} \bar{\Omega}_z\right)\right),$$

of which the right-hand side can be rewritten,

$$(1) \quad 2\mathfrak{I}\left(\sum_i a_i\bar{b}_i + \frac{\log(z)}{2\pi i}\sum_i c_i\bar{b}_i - \frac{\log(\bar{z})}{2\pi i}\sum_i a_i\bar{d}_i + \frac{\log(z)\log(\bar{z})}{4\pi^2}\sum_i c_i\bar{d}_i\right).$$

Because (1) is single-valued, we may infer that the coefficients of $\arg(z)$ and $\arg(z)^2$ vanish identically. From the vanishing of the coefficient of $\arg(z)^2$ it follows that the last sum makes no contribution. The coefficient of $\arg(z)$ in (1) is

$$\frac{1}{\pi}\mathfrak{I}\left(\sum_i c_i\bar{b}_i + \sum_i a_i\bar{d}_i\right).$$

Hence

$$(2) \qquad \sum_i c_i\bar{b}_i + \sum_i a_i\bar{d}_i$$

is identically real. Considering (2) as a power series in $z$ and $\bar{z}$ and recalling that $c_i(0) = d_i(0) = 0$, we observe that there is no constant term, and that all terms of degree 0 in $z$ occur only in the right-hand sum while all terms of degree 0 in $\bar{z}$ occur only in the left-hand sum. Necessarily these terms are complex conjugates of one another and hence make no contribution to the coefficient of $\log|z|$ in (1), which is

$$(3) \qquad \frac{-1}{\pi}\mathfrak{R}\left(\sum_i c_i\bar{b}_i - \sum_i a_i\bar{d}_i\right).$$

It follows that (3) admits a factor of $z\bar{z}$, so that

$$\int_{\mathscr{M}(z)} \Omega_z \wedge \bar{\Omega}_z = f(z, \bar{z}) + z\bar{z}\log|z|g(z, \bar{z}),$$

where $f$ and $g$ are real analytic in the disk, and $f(0) \neq 0$ by (iv). The finiteness of the Weil-Petersson distance to the origin follows by a standard computation.

*Proof of Theorem* B. Let $\{\nu_i\}_{1\le i\le n} \subset \mathscr{M}_0$ be the set of singular points of $f$. For each $\nu_i$ we choose coordinates $(x_i, y_i, u_i, v_i)$ with respective weights $(a_i, b_i, c_i, d_i)$, with respect to which $\nu_i$ is at the origin and $f$ is weighted homogeneous with weight $e_i < a_i + b_i + c_i + d_i$. $\Omega_W$ is given near $\nu_i$ by

$$g_i(w)dx_i \wedge dy_i \wedge du_i \wedge dv_i, \tag{4}$$

where $|g_i(w)|$ is bounded both above and below. For $w \in \mathscr{W}$ near $\nu_i$, we write

$$\rho_i(w) = ((x_i\bar{x}_i)^{b_ic_id_i} + (y_i\bar{y}_i)^{a_ic_id_i} + (u_i\bar{u}_i)^{a_ib_id_i} + (v_i\bar{v}_i)^{a_ib_ic_i})^{1/2a_ib_ic_id_i}.$$

For $r$ sufficiently small, we may define $B_i^r$ to be the locus of the inequality $\rho_i(w) \le r$. $B_i^r$ is diffeomorphic to a ball, and its boundary, which we shall denote $S_i^r$, is the locus of the equation $\rho_i(w) = r$ and is diffeomorphic to a sphere. We note that for $|\lambda| \le 1$, the map

$$h_i^\lambda : (x_i, y_i, u_i, v_i) \to (\lambda^{a_i}x_i, \lambda^{b_i}y_i, \lambda^{c_i}u_i, \lambda^{d_i}v_i)$$

maps $\mathscr{M}_z \cap B_i^r$ holomorphically onto $\mathscr{M}_{\lambda^{e_i}z} \cap B_i^{|\lambda|r}$ and satisfies

$$(h_i^\lambda)^*(\Omega_{\lambda_i^e z}) = \lambda^{a_i+b_i+c_i+d_i-e_i}\frac{g_i \circ h_i^\lambda}{g_i}\Omega_z. \tag{5}$$

We fix "radii" $\{\rho_i^0\}$ so that the corresponding $B_i^{\rho_i^0} = B_i$ are defined in the appropriate coordinate neighborhoods and are disjoint. There is no loss of generality in assuming that $f(B_i) = D$ for each $i$, since we can simply restrict the remainder of the discussion to a smaller disk.

We write $\mathscr{W}_i$ for $\mathscr{W} \cap B_i$, and $\mathscr{W}_{\text{comp}}$ for the closure of $\mathscr{W} - \bigcup_i B_i$.

$$\partial\mathscr{W}_{\text{comp}} = \bigcup_i \partial\mathscr{W}_i = \mathscr{W} \cap \bigcup_i S_i,$$

where $S_i$ is the boundary of $B_i$. We write $\mathscr{M}_i(z)$ and $\mathscr{M}_{\text{comp}}(z)$ respectively for $\mathscr{M}(z) \cap \mathscr{W}_i$ and $\mathscr{M}(z) \cap \mathscr{W}_{\text{comp}}$.

We observe that $(\mathscr{W}, \bigcup_i \mathscr{W}_i, \mathscr{W}_{\text{comp}})$, with $\mathscr{M}(0)$ deleted, is a fiberwise triad over $D - \{0\}$ and that the pair $(\mathscr{W}_{\text{comp}}, \partial\mathscr{W}_{\text{comp}})$ is a relative fibration over the full disk $D$. It follows that the Mayer-Vietoris sequences of the triads $(\mathscr{M}(z), \bigcup_i \mathscr{M}_i(z), \mathscr{M}_{\text{comp}}(z))$ for distinct values of $z$ are identified locally, while the homology sequences of the pairs $(\mathscr{M}_{\text{comp}}(z), \partial\mathscr{M}_{\text{comp}}(z))$ are identified globally. Moreover, the monodromy induces an automorphism of the Mayer-Vietoris sequence of the triad and acts trivially on the homology sequence of the pair.

By combining the homology sequence of the pair, $(\mathscr{M}(z), \bigcup_i \mathscr{M}_i(z))$ with the excision isomorphism

$$H_3(\mathscr{M}_{\text{comp}}(z), \partial\mathscr{M}_{\text{comp}}(z)) \to H_3\left(\mathscr{M}(z), \bigcup_i \mathscr{M}_i(z)\right),$$

we can see that

$$(1-\mu)(H_3(\mathscr{M}(z))) \subseteq \iota_*\left(H_3\left(\bigcup_i \mathscr{M}_i(z)\right)\right), \tag{6}$$

where $\iota$ is the inclusion of $\bigcup_i \mathscr{M}_i(z)$ in $\mathscr{M}(z)$.

If $C(z)$ is a locally constant homology class such that $C(z) \in \iota_* H_3(\mathscr{M}_i(z_0))$, then we may choose $z_0 \neq 0$ and represent $C(z_0)$ by an explicit piecewise smooth singular cycle $\mathscr{C}(z_0)$ on $\mathscr{M}_i(z_0)$. Then for $|\lambda| \leq 1$, $h_i^\lambda \circ \mathscr{C}(z_0)$ represents $C(\lambda^{e_i} z_0)$. One immediate consequence of this observation is

$$C \in \iota_*(H_3(\mathscr{M}_i)) \Rightarrow \mu^{e_i}(C) = C. \tag{7}$$

(6) and (7) together verify hypothesis (i) of Theorem A for $k = \mathrm{lcm}(\{e_i\})$. We note next that for $0 < r \leq 1$,

$$\int_{h_i^r \circ \mathscr{C}(z_0)} \Omega_{r^e z_0} = \int_{\mathscr{C}(z_0)} (h_i^r)^* (\Omega_{r^e z_0}) = r^{a_i+b_i+c_i+d_i-e_i} \int_{\mathscr{C}(z_0)} \frac{g_i \circ h_i^r}{g_i} \Omega_{z_0}, \tag{8}$$

by (5). We write $|\Omega_{z_0}|$ for the real three-form on $\mathscr{C}(z_0)$ which differs from $\Omega_{z_0}$ by a unimodular factor and is compatible with the appropriate orientation on $\mathscr{C}(z_0)$. We also denote by $K_i$ a bound on $|g_i(w)/g_i(w')|$ for any $w$, $w' \in B_i$. Such a bound exists since $|g_i(w)|$ is bounded both above and below. With this notation in place, we have

$$\left| r^{a_i+b_i+c_i+d_i-e_i} \int_{\mathscr{C}(z_0)} \frac{g_i \circ h_i^r}{g_i} \Omega_{z_0} \right| \leq K_i r^{a_i+b_i+c_i+d_i-e_i} \int_{\mathscr{C}(z_0)} |\Omega_{z_0}|, \tag{9}$$

which, together with (8), establishes hypothesis iii of Theorem A for

$$\varepsilon < \min\left(\left\{\frac{a_i+b_i+c_i+d_i-e_i}{e_i}\right\}\right).$$

To address hypotheses (ii) of Theorem A, we let $C(z) \in H_3(\mathscr{M}(z)$ be locally constant and observe that we can represent $C(z_0)$ as a sum of singular chains

$$\mathscr{C}_{\mathrm{comp}}(z_0) + \sum_i \mathscr{C}_i(z_0),$$

where $\mathscr{C}_{\mathrm{comp}}(z_0)$ is a chain on $\mathscr{M}_{\mathrm{comp}}(z_0)$, $\mathscr{C}_i(z_0)$ is a chain on $\mathscr{M}_i(z_0)$ for each $i$, and

$$\partial \mathscr{C}_{\mathrm{comp}}(z_0) = -\sum_i \partial \mathscr{C}_i(z_0). \tag{10}$$

Next, we extend $\mathscr{C}_{\mathrm{comp}}(z_0)$ continuously to a chain $\mathscr{C}_{\mathrm{comp}}(rz_0)$ on $\mathscr{M}_{\mathrm{comp}}(rz_0)$ such that $\partial \mathscr{C}_{\mathrm{comp}}(rz_0)$ is a cycle on $\partial \mathscr{M}_{\mathrm{comp}}(rz_0)$ for $0 \leq r \leq 1$, using the triviality of the relative fibration $(\mathscr{W}_{\mathrm{comp}}, \partial \mathscr{W}_{\mathrm{comp}})$ over the disk $D$. We have

$$\int_{C(z_0)} \Omega_{z_0} = \int_{\mathscr{C}_{\mathrm{comp}}(z_0)} \Omega_{z_0} + \sum_i \int_{\mathscr{C}_i(z_0)} Om_{z_0}. \tag{11}$$

Also

$$\int_{\mathscr{C}_{\text{comp}}(rz_0)} \Omega_{rz_0}$$

is continous for $0 \leq r \leq 1$, and therefore bounded. Now we set

$$\mathscr{E}_i = \mathscr{C}_i(z_0) \cup \bigcup_{0 \leq r \leq 1} (\partial \mathscr{C}_{\text{comp}}(rz_0) \cap B_i),$$

and

$$\mathscr{K}_i = \{\nu_i\} \cup \bigcup_{0<r\leq 1} h_i^r(\mathscr{E}_i).$$

For each $i$ and for $0 \leq r \leq 1$ choose $\mathscr{C}_i(rz_0)$ to be a triangulation of $\mathscr{M}_i \cap \mathscr{K}_i$, oriented so that

$$\partial \mathscr{C}_{\text{comp}}(rz_0) = -\sum_i \partial \mathscr{C}_i(rz_0), \tag{12}$$

and hence

$$\int_{C(rz_0)} \Omega_{rz_0} = \int_{\mathscr{C}_{\text{comp}}(rz_0)} \Omega_{rz_0} + \sum_i \int_{\mathscr{C}_i(rz_0)} \Omega_{rz_0}. \tag{13}$$

Moreover

$$h_i^r(|\mathscr{C}_i(z_0)|) = |\mathscr{C}_i(r^{e_i} z_0)| \cap h_i^r(B_i). \tag{14}$$

where $|\mathscr{C}|$ denotes the support of a chain. By (14), we have

$$\int_{\mathscr{C}_i(r^{e_i} z)} |\Omega_{r^{e_i} z}| = \int_{\mathscr{C}_i(r^{e_i} z) \cap h_i^r(B_i)} |\Omega_{r^{e_i} z_0}| + \int_{\mathscr{C}_i(r^{e_i} z) - h_i^r(B_i)} |\Omega_{r^{e_i} z}|, \tag{15}$$

where $r > 0$ , $0 < z/z_0 \leq 1$, and $|\Omega|$ is defined as preceding (9). The domain of the right hand integral is bounded away from $\nu_i$ even for $z = 0$. It follows that this integral is proper and continuous at $z = 0$ and so is bounded, say by $I$. Analogous to (9), we also have

$$\int_{\mathscr{C}_i(r^{e_i} z) \cap h_i^r(B_i)} |\Omega_{r^{e_i} z}| \leq K_i r^{a_i+b_i+c_i+d_i-e_i} \int_{\mathscr{C}_i(z)} |\Omega_z|. \tag{16}$$

We fix $\mathbf{r}$ so that $0 < \mathbf{r} < 1$ and $\mathbf{R} = K_i \mathbf{r}^{a_i+b_i+c_i+d_i-e_i} < 1$. It now follows from (15) and (16) that

$$\int_{\mathscr{C}_i(\mathbf{r}^{e_i} z)} |\Omega_{\mathbf{r}^{e_i} z}| \leq I + \mathbf{R} \int_{\mathscr{C}_i(z)} |\Omega_z|, \tag{17}$$

and hence, by induction, that

$$\int_{\mathscr{C}_i(\mathbf{r}^{ke_i} z)} |\Omega_{\mathbf{r}^{ke_i} z}| \leq I \sum_{j=0}^{k-i} \mathbf{R}^j + \mathbf{R}^k \int_{\mathscr{C}_i(z)} |\Omega_z|. \tag{18}$$

Finally, we let $J$ be a bound on $\int_{\mathscr{C}_i(rz_0)} |\Omega_{rz_0}|$ for $\mathbf{r} \leq r \leq 1$. Then it follows from (18) that

$$\int_{\mathscr{C}_i(rz_0)} |\Omega_{rz_0}| \leq \frac{1}{1-\mathbf{R}} \max\{I, J\}, \qquad 0 < r \leq 1. \tag{19}$$

Hypothesis (ii) of Theorem A now follows from (13) and (19). The verification of hypothesis (iv) of Theorem A follows from the continuity throughout $D$ of

$$\int_{\mathscr{M}_{\mathrm{comp}}(z)} \Omega_z \wedge \bar{\Omega}_z ,$$

which does not vanish at $z = 0$. This completes the proof of Theorem B.

Theorem B applies most directly to degenerations of Calabi–Yau threefolds realized as hypersurfaces in an algebraic fourfold. Let $\mathscr{X}$ be a fourfold, $\xi$ be a section of $K^*_{\mathscr{X}}$ with isolated singularities satisfying the homogeneity hypothesis of Theorem B, and $\eta$ be a nonsingular section of $K^*_{\mathscr{X}}$ which is nonvanishing at the singular points of $\xi$. Let $\Omega_{\mathscr{X}}$ be a local section of $K_{\mathscr{X}}$, $h = \langle \xi , \Omega_{\mathscr{X}} \rangle$ and $g = \langle \eta , \Omega_{\mathscr{X}} \rangle$. It is straightforward to verify that if we restrict $z$ to be sufficiently small and set $\mathscr{W} = \{(x , z) | (\xi + z\eta)(x) = 0\}$ with $f(x , z) = z$ and $\Omega_{\mathscr{W}}$ defined by

$$\Omega_{\mathscr{W}} \wedge (dh + zdg) = \Omega_{\mathscr{X}} \wedge dz ,$$

then $\Omega_{\mathscr{W}}$ is independent of the choice of $\Omega_{\mathscr{X}}$ and therefore global, and $\mathscr{W}$, $f$ and $\Omega_{\mathscr{W}}$ satisfy the hypotheses of Theorem B.

## References


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