SINGULAR DEGENERATIONS OF CALABI-YAU MANIFOLDS
AND THE WEIL-PETERSSON METRIC

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ABSTRACT. Degenerations of Calabi-Yau manifolds with isolated singularities somewhat more general than nodes are shown to be at finite distance from nonsingular manifolds in the Weil-Petersson metric on the appropriate moduli spaces.

If $M(z)$ is a nonsingular complex threefold with trivial canonical bundle and holomorphic three-form $\Omega_z$, depending holomorphically on $z \in \mathcal{Z}$, for some complex manifold $\mathcal{Z}$, then $\log(\int_{M(z)} \Omega_z \wedge \Omega_z)$ is the Kähler potential for a pseudo-metric on $\mathcal{Z}$, which is a metric provided $\mathcal{Z}$ is an effective deformation space in the sense of [2]. This metric has come to be known in the physics literature as the Weil-Petersson metric, although the strict analogy with moduli spaces of curves holds only for curves of genus one.

This paper is a step in attempting to understand the completion of this Weil-Petersson metric by studying its behaviour with respect to singular degenerations. In particular, we study the case where $\mathcal{Z}$ is a punctured disk, and the holomorphic family $\{M(z)\}$ can be completed by the inclusion of a variety $M(0)$ with isolated singularities. We give some sufficient conditions for curves through the deleted point to have finite length. The case of nodal degenerations was discussed in [1].

We proceed mainly by studying the periods of $\Omega_z$. We shall denote by $C(z)$ a locally constant homology class in $H_3(M(z))$. Let

$$\int_{C(z)} \Omega_z$$

be a holomorphic, although possibly multiple valued, function of $z$. We shall usually assume that $z$ ranges over a punctured disk, $D - \{0\}$, and write $\mu$ for the monodromy operator on $H_3(M(z))$. Our principal results, Theorems

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1We think of the parameter $z$ somewhat ambiguously as an element either of $\mathcal{Z}$ or of its universal covering space.
A and B below, generalize the case of a nodal degeneration, for which \( \mu \) is a transvection and \((1 - \mu)^2 = 0 \).

**Theorem A.** Let \( \{ (\mathcal{M}(z), \Omega_z) \}_{0 < |z| < r} \) be a family of nonsingular threefolds with nowhere vanishing holomorphic three-form, parametrized holomorphically by a punctured disk about the origin in the complex plane. Suppose that

(i) \((1 - \mu^k)^2 = 0 \) for some positive integer \( k \).

(ii) For any locally constant homology class, \( C(z) \in H_3(\mathcal{M}(z)) \), \( \int_{C(z)} \Omega_z \) is bounded along any ray, \( \arg(z) = \text{constant} \).

(iii) For any locally constant class \( C(z) \in \text{im}(1 - \mu) \), and any fixed value of \( \arg(z) \),

\[
\lim_{|z| \to 0} |z|^{-\epsilon} \int_{C(z)} \Omega_z = 0
\]

for some \( \epsilon > 0 \).

(iv) \( \lim_{z \to 0} \int_{C(z)} \Omega_z \wedge \Omega \neq 0 \) (it will follow from (i)-(iii) that the limit exists).

Then a smooth path in the disk has finite length in the Weil-Petersson metric, even if it passes through the origin.

**Theorem B.** Let \( \mathcal{W} \) be a fourfold with nowhere vanishing holomorphic four-form \( \Omega_{\mathcal{W}} \). Let \( D \) be a disk around the origin and let \( f: \mathcal{W} \to D \) be a proper holomorphic complex valued function on \( \mathcal{W} \) whose fibres are compact and, except for \( f^{-1}(0) \), nonsingular. Suppose further that the holomorphic three-form \( \Omega_z \), defined along \( \mathcal{M}(z) = f^{-1}(z) \) at all regular points by the relation

\[
\Omega_z \wedge df = \Omega_{\mathcal{W}},
\]

is nowhere vanishing. Finally, suppose that the singularities of \( \mathcal{M}(0) \) are all isolated and that each of them has a neighborhood in \( \mathcal{W} \) which admits a coordinate system in which the coordinates may be weighted so that \( f \) is weighted homogeneous with weight strictly lower than the sum of the weights of the coordinates. Then the holomorphic family \( \{ (\mathcal{M}(z), \Omega_z) \}_{z \in D - \{0\}} \) satisfies the hypotheses of Theorem A.

**Proof of Theorem A.** By passing to a \( k \)-fold cyclic cover of the punctured disk we may, without loss of generality, confine our discussion to the case \((1 - \mu)^2 = 0 \). We observe next that we can regard \( \Omega_z \) as an element of \( H^3(\mathcal{M}(z), C) \), on which \( \mu \) also operates by duality.

We write \( \alpha = 1 - \mu \) and

\[
\hat{\Omega}_z = \Omega_z + \frac{\log(z)}{2\pi i} \alpha(\Omega_z).
\]

Although multiple valued, \( \hat{\Omega}_z \) depends holomorphically on \( z \) and satisfies the identity

\[
\Omega_z = \hat{\Omega}_z - \frac{1}{2\pi i} \log(z) \alpha(\hat{\Omega}_z).
\]

It follows by a straightforward computation that the periods of \( \hat{\Omega}_z \) are single-valued and therefore, by (ii), holomorphic in the unpunctured disk. Also by
(iii), the periods of $\alpha(\hat{\Omega}_z)$ vanish at the origin. Now, for $z \neq 0$, we choose a locally constant symplectic basis $\{A_i(z), B_i(z)\}$ for $H_3(\mathcal{M}(z))$, and write

$$a_i(z) = \int_{A_i(z)} \hat{\Omega}_z, \quad b_i(z) = \int_{B_i(z)} \hat{\Omega}_z,$$

$$c_i(z) = \int_{A_i(z)} \alpha(\hat{\Omega}_z) = \int_{\alpha(A_i(z))} \hat{\Omega}_z, \quad \text{and} \quad d_i(z) = \int_{B_i(z)} \alpha(\hat{\Omega}_z) = \int_{\alpha(B_i(z))} \hat{\Omega}_z.$$

We note that all these functions are single-valued and can be extended to be holomorphic in the unpunctured disk with $c_i(0) = d_i(0) = 0$.

Next we avail ourselves of the identity,

$$\int_{\mathcal{M}(z)} \Omega_z \wedge \bar{\Omega}_z = 2\mathfrak{j} \left( \sum_i \left( \int_{A_i(z)} \Omega_z \right) \left( \int_{B_i(z)} \Omega_z \right) \right),$$

of which the right-hand side can be rewritten,

$$(1) \quad 2\mathfrak{j} \left( \sum_i a_i b_i + \frac{\log(z)}{2\pi i} \sum_i c_i \bar{b}_i - \frac{\log(z)}{2\pi i} \sum_i a_i \bar{d}_i + \frac{\log(z) \log(z)}{4\pi^2} \sum_i c_i \bar{d}_i \right).$$

Because (1) is single-valued, we may infer that the coefficients of $\arg(z)$ and $\arg(z)^2$ vanish identically. From the vanishing of the coefficient of $\arg(z)^2$ it follows that the last sum makes no contribution. The coefficient of $\arg(z)$ in (1) is

$$\frac{1}{\pi} \mathfrak{Im} \left( \sum_i c_i \bar{b}_i + \sum_i a_i \bar{d}_i \right).$$

Hence

$$(2) \quad \sum_i c_i \bar{b}_i + \sum_i a_i \bar{d}_i$$

is identically real. Considering (2) as a power series in $z$ and $\bar{z}$ and recalling that $c_i(0) = d_i(0) = 0$, we observe that there is no constant term, and that all terms of degree 0 in $z$ occur only in the right-hand sum while all terms of degree 0 in $\bar{z}$ occur only in the left-hand sum. Necessarily these terms are complex conjugates of one another and hence make no contribution to the coefficient of $\log|z|$ in (1), which is

$$(3) \quad \frac{-1}{\pi} \mathfrak{Re} \left( \sum_i c_i \bar{b}_i - \sum_i a_i \bar{d}_i \right).$$

It follows that (3) admits a factor of $z \bar{z}$, so that

$$\int_{\mathcal{M}(z)} \Omega_z \wedge \bar{\Omega}_z = f(z, \bar{z}) + z \bar{z} \log|z| g(z, \bar{z}),$$

where $f$ and $g$ are real analytic in the disk, and $f(0) \neq 0$ by (iv). The finiteness of the Weil-Petersson distance to the origin follows by a standard computation.
Proof of Theorem B. Let \( \{\nu_i\}_{1 \leq i \leq n} \subset \mathcal{M}_0 \) be the set of singular points of \( f \). For each \( \nu_i \) we choose coordinates \((x_i, y_i, u_i, v_i)\) with respective weights \((a_i, b_i, c_i, d_i)\), with respect to which \( \nu_i \) is at the origin and \( f \) is weighted homogeneous with weight \( e_i < a_i + b_i + c_i + d_i \). \( \Omega_W \) is given near \( \nu_i \) by

\[
g_i(w)dx_i \wedge dy_i \wedge du_i \wedge dv_i,
\]

where \( |g_i(w)| \) is bounded both above and below. For \( w \in W \) near \( \nu_i \), we write

\[
\rho_i(w) = \left( (x_i x_i)^{a_i} (y_i y_i)^{b_i} (u_i u_i)^{c_i} (v_i v_i)^{d_i} \right)^{1/2(a_i + b_i + c_i + d_i)}.
\]

For \( r \) sufficiently small, we may define \( B'_i \) to be the locus of the inequality \( \rho_i(w) \leq r \). \( B'_i \) is diffeomorphic to a ball, and its boundary, which we shall denote \( S'_i \), is the locus of the equation \( \rho_i(w) = r \) and is diffeomorphic to a sphere. We note that for \(|\lambda| \leq 1\), the map

\[
h^i : (x_i, y_i, u_i, v_i) \mapsto (\lambda^a x_i, \lambda^b y_i, \lambda^c u_i, \lambda^d v_i)
\]

maps \( \mathcal{M}_z \cap B'_i \) holomorphically onto \( \mathcal{M}_{\lambda^i} \cap B^{|\lambda|r}_i \) and satisfies

\[
(h^i_\lambda)^* (\Omega_{\lambda^i z}) = \lambda^{a_i + b_i + c_i + d_i - e_i} \frac{g_i}{\rho_i} h^i_\lambda \Omega_z.
\]

We fix “radii” \( \{\rho^0_i\} \) so that the corresponding \( B'_i^0 = B_i \) are defined in the appropriate coordinate neighborhoods and are disjoint. There is no loss of generality in assuming that \( f(B_i) = D \) for each \( i \), since we can simply restrict the remainder of the discussion to a smaller disk.

We write \( \mathcal{W}_i \) for \( \mathcal{W} \cap B'_i \), and \( \mathcal{W}_{\text{comp}} \) for the closure of \( \mathcal{W} \setminus \bigcup_i B_i \).

\[
\partial \mathcal{W}_{\text{comp}} = \bigcup_i \partial \mathcal{W}_i = \mathcal{W} \cap \bigcup_i S_i,
\]

where \( S_i \) is the boundary of \( B_i \). We write \( \mathcal{M}(z) \) and \( \mathcal{M}_{\text{comp}}(z) \) respectively for \( \mathcal{M}(z) \cap \mathcal{W}_i \) and \( \mathcal{M}(z) \cap \mathcal{W}_{\text{comp}} \).

We observe that \( (\mathcal{W}, \bigcup_i \mathcal{W}_i, \mathcal{W}_{\text{comp}}) \), with \( \mathcal{M}(0) \) deleted, is a fiberwise triad over \( D \setminus \{0\} \) and that the pair \( (\mathcal{W}_{\text{comp}}, \partial \mathcal{W}_{\text{comp}}) \) is a relative fibration over the full disk \( D \). It follows that the Mayer-Vietoris sequences of the triads \( (\mathcal{M}(z), \bigcup_i \mathcal{M}(z), \mathcal{M}_{\text{comp}}(z)) \) for distinct values of \( z \) are identified locally, while the homology sequences of the pairs \( (\mathcal{M}_{\text{comp}}(z), \partial \mathcal{M}_{\text{comp}}(z)) \) are identified globally. Moreover, the monodromy induces an automorphism of the Mayer-Vietoris sequence of the triad and acts trivially on the homology sequence of the pair.

By combining the homology sequence of the pair, \( (\mathcal{M}(z), \bigcup_i \mathcal{M}(z)) \) with the excision isomorphism

\[
H_3(\mathcal{M}_{\text{comp}}(z), \partial \mathcal{M}_{\text{comp}}(z)) \to H_3 \left( \mathcal{M}(z), \bigcup_i \mathcal{M}(z) \right),
\]
we can see that

\[(6) \quad (1 - \mu)(H_3(\mathcal{M}(z))) \subseteq \imath_* \left( H_3 \left( \bigcup_i \mathcal{M}_i(z) \right) \right), \]

where \( \imath \) is the inclusion of \( \bigcup_i \mathcal{M}_i(z) \) in \( \mathcal{M}(z) \).

If \( C(z) \) is a locally constant homology class such that \( C(z) \in \imath_* H_3(\mathcal{M}(z_0)) \), then we may choose \( z_0 \neq 0 \) and represent \( C(z_0) \) by an explicit piecewise smooth singular cycle \( \mathcal{C}(z_0) \) on \( \mathcal{M}(z_0) \). Then for \( |\lambda| \leq 1, h_i^\lambda \circ \mathcal{C}(z_0) \) represents \( C(\lambda^\epsilon z_0) \). One immediate consequence of this observation is

\[(7) \quad C \in \imath_* (H_3(\mathcal{M}_i)) \Rightarrow \mu^\epsilon(C) = C. \]

(6) and (7) together verify hypothesis (i) of Theorem A for \( k = \text{lcm}(\{e_i\}) \). We note next that for \( 0 < r \leq 1 \),

\[(8) \quad \int_{\mathcal{C}(z_0)} \Omega_{z_0} = \int_{\mathcal{C}(z_0)} (h_i^r)^* \Omega_{z_0} = r^{a_i + b_i + c_i + d_i - e_i} \int_{\mathcal{C}(z_0)} g_i \circ h_i^r \Omega_{z_0}, \]

by (5). We write \( |\Omega_{z_0}| \) for the real three-form on \( \mathcal{C}(z_0) \) which differs from \( \Omega_{z_0} \) by a unimodular factor and is compatible with the appropriate orientation on \( \mathcal{C}(z_0) \). We also denote by \( K_i \) a bound on \( |g_i(w)/g_i(w')| \) for any \( w, w' \in B_i \). Such a bound exists since \( |g_i(w)| \) is bounded both above and below. With this notation in place, we have

\[(9) \quad r^{a_i + b_i + c_i + d_i - e_i} \int_{\mathcal{C}(z_0)} g_i \circ h_i^r \Omega_{z_0} \leq K_i r^{a_i + b_i + c_i + d_i - e_i} \int_{\mathcal{C}(z_0)} |\Omega_{z_0}|, \]

which, together with (8), establishes hypothesis iii of Theorem A for

\[ \varepsilon < \min \left\{ \left\{ \frac{a_i + b_i + c_i + d_i - e_i}{e_i} \right\} \right\}. \]

To address hypotheses (ii) of Theorem A, we let \( C(z) \in H_3(\mathcal{M}(z)) \) be locally constant and observe that we can represent \( C(z_0) \) as a sum of singular chains

\[ \mathcal{C}_{\text{comp}}(z_0) + \sum_i \mathcal{C}_i(z_0), \]

where \( \mathcal{C}_{\text{comp}}(z_0) \) is a chain on \( \mathcal{M}_{\text{comp}}(z_0) \), \( \mathcal{C}_i(z_0) \) is a chain on \( \mathcal{M}_i(z_0) \) for each \( i \), and

\[(10) \quad \partial \mathcal{C}_{\text{comp}}(z_0) = -\sum_i \partial \mathcal{C}_i(z_0). \]

Next, we extend \( \mathcal{C}_{\text{comp}}(z_0) \) continuously to a chain \( \mathcal{C}_{\text{comp}}(rz_0) \) on \( \mathcal{M}_{\text{comp}}(rz_0) \) such that \( \partial \mathcal{C}_{\text{comp}}(rz_0) \) is a cycle on \( \partial \mathcal{M}_{\text{comp}}(rz_0) \) for \( 0 \leq r \leq 1 \), using the triviality of the relative fibration \( (\mathcal{M}_{\text{comp}}, \partial \mathcal{M}_{\text{comp}}) \) over the disk \( D \). We have

\[(11) \quad \int_{C(z_0)} \Omega_{z_0} = \int_{\mathcal{C}_{\text{comp}}(z_0)} \Omega_{z_0} + \sum_i \int_{\mathcal{C}_i(z_0)} Om_{z_0}. \]
Also

$$\int_{\Omega_{rz0}} \partial \mathcal{C}^{\text{comp}}(rz0)$$

is continuous for $0 \leq r \leq 1$, and therefore bounded. Now we set

$$\mathcal{E}_i = \mathcal{E}_i(z_0) \cup \bigcup_{0 \leq r \leq 1} (\partial \mathcal{E}^{\text{comp}}_{\text{comp}}(rz0) \cap B_i),$$

and

$$\mathcal{R}_i = \{\nu_i\} \cup \bigcup_{0 \leq r \leq 1} h_i'(\mathcal{E}_i).$$

For each $i$ and for $0 \leq r \leq 1$ choose $\mathcal{E}_i(rz0)$ to be a triangulation of $\mathcal{M}_i \cap \mathcal{R}_i$, oriented so that

$$\partial \mathcal{E}^{\text{comp}}(rz0) = -\sum_i \partial \mathcal{E}_i(rz0),$$

and hence

$$\int_{C(rz0)} \Omega_{rz0} = \int_{\mathcal{E}^{\text{comp}}(rz0)} \Omega_{rz0} + \sum_i \int_{\mathcal{E}_i(rz0)} \Omega_{rz0}.$$  \hspace{1cm} (13)

Moreover

$$h_i'(|\mathcal{E}_i(z_0)|) = |\mathcal{E}_i(r^i, z_0) \cap h_i'(B_i)|,$$

where $|\mathcal{E}|$ denotes the support of a chain. By (14), we have

$$\int_{\mathcal{E}_i(r^i, z)} |\Omega_{rz}| = \int_{\mathcal{E}_i(r^i, z) \cap h_i'(B_i)} |\Omega_{rz}| + \int_{\mathcal{E}_i(r^i, z) \setminus h_i'(B_i)} |\Omega_{rz}|,$$

where $r > 0$, $0 < z/z_0 \leq 1$, and $|\Omega|$ is defined as preceding (9). The domain of the right hand integral is bounded away from $\nu_i$, even for $z = 0$. It follows that this integral is proper and continuous at $z = 0$ and so is bounded, say by $I$. Analogous to (9), we also have

$$\int_{\mathcal{E}_i(r^i, z) \cap h_i'(B_i)} |\Omega_{rz}| \leq K_i r^{a_i+b_i+c_i+d_i-e_i} \int_{\mathcal{E}_i(z)} |\Omega_z|.$$  \hspace{1cm} (16)

We fix $r$ so that $0 < r < 1$ and $R = K_i r^{a_i+b_i+c_i+d_i-e_i} < 1$. It now follows from (15) and (16) that

$$\int_{\mathcal{E}_i(r^i, z)} |\Omega_{rz}| \leq I + \frac{R}{R} \int_{\mathcal{E}_i(z)} |\Omega_z|,$$

and hence, by induction, that

$$\int_{\mathcal{E}_i(r^i, z)} |\Omega_{rz}| \leq \sum_{j=0}^{k-1} R^j + R^k \int_{\mathcal{E}_i(z)} |\Omega_z|.$$  \hspace{1cm} (18)

Finally, we let $J$ be a bound on $\int_{\mathcal{E}_i(rz0)} |\Omega_{rz0}|$ for $r \leq r \leq 1$. Then it follows from (18) that

$$\int_{\mathcal{E}_i(rz0)} |\Omega_{rz0}| \leq \frac{1}{1 - R} \max\{I, J\}, \hspace{1cm} 0 < r \leq 1.$$  \hspace{1cm} (19)
Hypothesis (ii) of Theorem A now follows from (13) and (19). The verification of hypothesis (iv) of Theorem A follows from the continuity throughout \( D \) of

\[
\int_{\mathcal{A}^{\text{comp}}(z)} \Omega_z \wedge \Omega_z,
\]

which does not vanish at \( z = 0 \). This completes the proof of Theorem B.

Theorem B applies most directly to degenerations of Calabi-Yau threefolds realized as hypersurfaces in an algebraic fourfold. Let \( \mathcal{X} \) be a fourfold, \( \xi \) be a section of \( K_{\mathcal{X}}^* \) with isolated singularities satisfying the homogeneity hypothesis of Theorem B, and \( \eta \) be a nonsingular section of \( K_{\mathcal{X}}^* \) which is nonvanishing at the singular points of \( \xi \). Let \( \Omega_{\mathcal{X}} \) be a local section of \( K_{\mathcal{X}}^* \), \( h = \langle \xi, \Omega_{\mathcal{X}} \rangle \) and \( g = \langle \eta, \Omega_{\mathcal{X}} \rangle \). It is straightforward to verify that if we restrict \( z \) to be sufficiently small and set \( \mathcal{W} = \{(x, z)| (\xi + z\eta)(x) = 0\} \) with \( f(x, z) = z \) and \( \Omega_{\mathcal{X}} \) defined by

\[
\Omega_{\mathcal{X}} \wedge (dh + zdg) = \Omega_{\mathcal{X}} \wedge dz,
\]

then \( \Omega_{\mathcal{X}} \) is independent of the choice of \( \Omega_{\mathcal{X}} \) and therefore global, and \( \mathcal{W}, f \) and \( \Omega_{\mathcal{X}} \) satisfy the hypotheses of Theorem B.

REFERENCES


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