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A RESULT BY KULIKOV THAT DOES NOT EXTEND TO MODULES OVER GENERAL VALUATION DOMAINS

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ABSTRACT. Let R be a valuation domain. It is proved that every nonzero R-module contains a nonzero pure uniserial submodule if and only if R is rank-one discrete.

It is a classical result in the theory of Abelian groups, due to Kulikov [5], that every Abelian p-group contains a pure cocyclic subgroup (which is necessarily a summand, cocyclic groups being pure-injective). Note that cocyclic p-groups are exactly the torsion uniserial $\mathbb{Z}_p$-modules, where $\mathbb{Z}_p$ is the rank-one discrete valuation domain obtained by localizing the ring of the integers $\mathbb{Z}$ at the prime p.

The goal of this paper is to show that this result is not extendable to valuation domains which are not rank-one discrete. Our main result is the following:

Theorem. *Let R be a valuation domain. Then every nonzero R-module contains a nonzero pure uniserial submodule if and only if R is rank-one discrete.*

The first contribution to the proof of this theorem was given in [2] (see also [3, X.4]), where an example of a *cohesive* R-module (i.e., a module without elements of limit height) with no nonzero uniserial pure submodules was given, under the assumption that the maximal ideal P of R is not principal.

The second author showed in [6] that a cohesive R-module with the above property cannot exist if and only if $J \cdot R_J$ is a principal ideal of R_J, for each prime ideal J of R (equivalently, R is discrete and $\operatorname{Spec}(R)$ is well ordered by the opposite inclusion); note that the proof of the sufficiency was just an adaptation of the example given in [2].

The consequence of the above results is that, in order to complete the proof of the theorem, we must find a noncohesive R-module with no nonzero pure uniserial submodules, where R is any discrete valuation domain with a well-ordered prime spectrum consisting of at least two nonzero ideals.

In fact, we shall construct an R-module S, where R is as in the preceding

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paragraph, which satisfies some particular properties: S has essential socle, all of whose nonzero elements have limit height, and S has a suitable quotient which is a direct sum of cyclics. These properties ensure that our module S has no nonzero pure uniserial submodules.

We will use the following notation. If R is a valuation domain, Q denotes its field of quotients, P its maximal ideal, and $U(R)$ its group of units. For an R-module M and an ideal I of R, we set $M[I] = \{x \in M: rx = 0$ for each $r \in I\}$. For the convenience of the reader, we recall the definition of the height of an element $x \in M$ (see [3, VIII]). The *divisibility set* of x in M is the subset of R: $D_M(x) = \{r \in R: x \in rM\}$; the *height ideal* of x in M is the submodule of Q: $H_M(x) = \cup\{Rr^{-1}: r \in D_M(x)\}$. Then, setting $H_M(x) = J$, the *height* of x in M is defined as follows:

$$h_M(x) = \begin{cases} J/R & \text{if there exists a morphism } \varphi: J \to M \text{ such that } \varphi 1 = x, \\ (J/R)^- & \text{otherwise.} \end{cases}$$

If $h_M(x) = J/R$, then we say that x has *nonlimit* height; otherwise, we say that x has *limit* height.

An R-module U is *uniserial* if its submodules are linearly ordered by inclusion. A submodule N of an R-module M is *pure* if $rM \cap N = rN$ for all $r \in R$; this is actually the definition of RD-purity, which coincides with Cohn's purity if R is a valuation domain. For all these definitions and for general facts on modules over valuation domains we refer to [3]. Finally, we denote by $\mathbb{N}$ the set of natural numbers, zero included, and by $\mathbb{N}^+ = \{n \in \mathbb{N}: n \geq 1\}$.

The module S

Now let R be a valuation domain with principal maximal ideal $P = Rp$, such that the prime ideal $P_1 = \bigcap_{n\in\mathbb{N}} Rp^n$ is nonzero and principal as R_{P_1}-module; thus there is an $r \in P_1$ such that $P_1R_{P_1} = rR_{P_1}$. Note that P_1 is generated by the elements $rp^{-n} (n \in \mathbb{N})$.

Let Σ denote the set consisting of two symbols e_0 and e_1 and the finite sequences of 0 and 1:

$$\sum = \{e_0, e_1, (\sigma_1, \sigma_2, \dots, \sigma_n): n \in \mathbb{N}^+, \sigma_i \in \{0, 1\}\}.$$

Given any $s \in \Sigma$, we define its length $l(s)$ by setting

$$l(s) = \begin{cases} -1 & \text{if } s = e_0, \\ 0 & \text{if } s = e_1, \\ n & \text{if } s = (\sigma_1, \sigma_2, \dots, \sigma_n). \end{cases}$$

For each $s = (\sigma_1, \sigma_2, \dots, \sigma_n) \in \Sigma\backslash\{e_0, e_1\}$ and different from $(0, 0, \dots, 0)$, let $k(s) = \max\{i \leq n: \sigma_i = 1\}$. We define a map $*: \Sigma\backslash\{e_0\} \to \Sigma$, sending s

into s^*, by setting

$$s^* = \begin{cases} e_0 & \text{if } s = e_1 \text{ or } s = (0, 0, \dots, 0), \\ e_1 & \text{if } s = (1, 0, \dots, 0); \text{ i.e., } k(s) = 1, \\ (\sigma_1, \sigma_2, \dots, \sigma_{k-1}) & \text{if } s = (\sigma_1, \sigma_2, \dots, \sigma_n), \\ & \text{and } k = k(s) > 1. \end{cases}$$

We remark that $l(s) > l(s^*)$ for all $s \in \Sigma \backslash \{e_0\}$.

Finally, we define the map $t\colon \Sigma\backslash\{e_0\} \to \mathbb{N}$ in the following way:

$$t(s) = \begin{cases} l(s) & \text{if } s = e_1 \text{ or } s = (0, 0, \dots, 0), \\ l(s) - k(s) + 1 & \text{otherwise.} \end{cases}$$

Now let $F = \oplus_{s\in\Sigma} Rs$ be the free R-module with basis Σ and F_0 the submodule of F generated by the elements

$$rpe_0;\ rs - rp^{-t(s)}s^*,\ s \in \Sigma\backslash\{e_0\}.$$

We shall need the following technical result:

Lemma 1. *Each $0 \neq y \in F_0$ can be written in the form*

$$y = \sum_1^n \lambda_i s_i + rcs \tag{1}$$

with $0 \neq c \in R$, $\lambda_i \in R$, $s_i \neq s$, and $l(s) \geq l(s_i)$ for each i.

Proof. If $y \in \mathrm{Re}_0$, then $y = \lambda rpe_0$ for $0 \neq \lambda \in R$, so the claim is true. If $y \notin \mathrm{Re}_0$, then

$$y = ae_0 + \sum_1^m a_i(rs_i - rp^{-t_i}s_i^*)$$

where $a \in R$, $0 \neq a_i \in R$, s_i are distinct elements of $\Sigma\backslash\{e_0\}$, $t_i = t(s_i)$ $(i = 1, \dots, m)$. Choose $j \leq m$ such that $l(s_j)$ is maximal. Then obviously $l(s_j) > l(s_i^*)$ for all $i \leq m$. Set $s = s_j$ and $c = a_j$, and the claim follows. □

Let $S = F/F_0$; for each $s \in \Sigma$, we denote $s + F_0$ by $\bar{s}$; then S is generated by the elements $\{\bar{s}\colon s \in \Sigma\}$, subject to the following conditions:

(2)

$$\begin{cases} pr\bar{e}_0 = 0;\ r\bar{e}_0 = r\bar{e}_1;\ r(\overline{0, 0, \dots, 0}) = rp^{-n}\bar{e}_0 & \text{if } l(0, \dots, 0) = n; \\ r(\overline{1, 0, \dots, 0}) = rp^{-n}\bar{e}_1 & \text{if } l(1, 0, \dots, 0) = n; \\ r\bar{s} = rp^{-t(s)}(\overline{\sigma_1, \dots, \sigma_{k-1}}) = rp^{-t(s)}\overline{s^*} & \text{if } s = (\sigma_1, \dots, \sigma_n) \\ \text{and } k = k(s) \geq 2. \end{cases}$$

From Lemma 1 it follows that $s \neq s'$ in Σ implies $\bar{s} \neq \overline{s'}$ in S.

We start by verifying some properties of the elements in S.

(P1) For each $s \in \Sigma$ there is an $n(s) \in \mathbb{N}$ such that $rp^{n(s)}\bar{s} = r\bar{e}_0$.

We induct on $l(s)$, the claim being trivial for $l(s) = -1$, i.e., if $s = e_0$. If $l(s) > -1$, then $rp^{t(s)}\bar{s} = r\overline{s^*}$ and $l(s^*) < l(s)$ gives, by induction, $rp^{t(s)+n(s^*)}\bar{s} = rp^{n(s^*)}\overline{s^*} = r\bar{e}_0$.

(P2) For each $x \in S$ there is an $n \in \mathbb{N}$ such that $rp^n x = 0$; hence $P_1^2 S = 0$. It is an immediate consequence of (P1) and relations in (2).

(P3) Every $x \in S$ can be written in the form

$$x = a\overline{e}_0 + \sum_1^n v_i p^{n_i} \overline{s}_i + \sum_1^m \varepsilon_j r p^{-m_j} \overline{s}_j \tag{3}$$

where $a \in R$, the v_i's and ε_j's are either units or zero, $n_i \in \mathbb{N}$ for all i, and $m_j \in \mathbb{N}^+$ for all j.

We can assume $x \notin R\overline{e}_0$. First we show that x is of the form

$$x = a\overline{e}_0 + \sum_1^m a_i \overline{s}_i \tag{4}$$

with $a \in R$, $a_i \in R$ for all i, and $r \in \bigcap_i Pa_i$. We can obviously write $x = b\overline{e}_0 + \sum_1^m b_i \overline{s}_i$ $(b, b_i \in R)$; if $r \in \bigcap_i Pb_i$, then we are done. Let us assume that $b_j = cr$ for a $j \leq m (c \in R)$. If $c \in Rp^{n(s_j)}$, then $cr\overline{s}_j \in R\overline{e}_0$ by (P1), so $b_j \overline{s}_j$ can be replaced by $rd\overline{e}_0$ for a suitable $d \in R$. If $c = up^h$, with $u \in U(R)$ and $h < n(s_j)$ (this implies that $s_j \notin e_1$, since $n(e_1) = 0$), then $rup^h \overline{s}_j = rup^h p^{-t(s_j)} \overline{s_j^*}$. If $h < t(s_j)$, we can replace $rc\overline{s}_j = rup^h \overline{s}_j$ by $rup^{h-t(s_j)} \overline{s_j^*}$, since $r \in Prup^{h-t(s_j)}$; if $h \geq t(s_j)$, we induct on h, since the claim is true for $h = 0$. To conclude, it is enough to note that $r \in \bigcap_i Pa_i$ implies that, for each $i \leq m$, either $a_i = v_i p^{n_i}$ for some $v_i \in U(R)$ and $n_i \in \mathbb{N}$, or $a_i = \varepsilon_i r p^{-m_i}$ for some $\varepsilon_i \in U(R)$ and $m_i \in \mathbb{N}^+$.

(P4) Let $0 \neq x \in S$; then $x \in \bigcup_{k \in N} S[p^k]$ if and only if x can be written in the form

$$x = rup^{-h} \overline{e}_0 + \sum_1^m r\varepsilon_j p^{-m_j} \overline{s}_j \tag{5}$$

where u and the ε_j's are either units or zero, $h \in \mathbb{N}$, and $m_j \in \mathbb{N}^+$ for all j.

If x is as in (5), then (P2) ensures that $x \in S[p^k]$ for a suitable $k \in \mathbb{N}$. Conversely, assume that $0 \neq x \in S[p^k]$ for a $k \in \mathbb{N}^+$, and write x as in (3). Then, for a suitable $t \geq k$, $p^t \varepsilon_j r p^{-m_j} \overline{s}_j = 0$ for all $j \leq m$; so we get

$$0 = p^t x = ap^t \overline{e}_0 + \sum_1^n v_i p^{n_i + t} \overline{s}_i;$$

thus the element of $F: y = ap^t e_0 + \sum_1^n v_i p^{n_i + t} s_i$ belongs to F_0. From Lemma 1 we deduce that $v_i = 0$ for all i and $a = rup^{-h}$ for a suitable $h \in \mathbb{N}$, and u is a unit of R or zero. Hence x is of the desired form (5).

We consider now some structural properties of S. Obviously $S[P_1] = \bigcap_{k \in \mathbb{N}} S[rp^{-k}]$, and $S[P_1] \supseteq \bigcup_{k \in \mathbb{N}} S[p^k]$. The converse inclusion also holds.

Lemma 2. $S[P_1] = \bigcup_{k \in \mathbb{N}} S[p^k]$.

Proof. It is enough to prove that, if $x \notin \bigcup_{k\in\mathbb{N}} S[p^k]$, then there exists $t \in \mathbb{N}$ such that $rp^{-t}x \neq 0$. If $x \in R\overline{e}_0$, then the claim is obvious, since $R\overline{e}_0 \cong R/Rrp$ [or use Lemma 1 and (P4)]. Assume that $x \notin R\overline{e}_0$ and write x as in (3). Then, as in the proof of (P4), one can show that there exists $k \in \mathbb{N}$ such that

$$p^k x = ap^k\overline{e}_0 + \sum_1^n v_i p^{t_i}\overline{s}_i \qquad (t_i \in \mathbb{N})$$

where $p^k x \neq 0$, since $x \notin S[p^k]$. We show that $x' = p^k x \notin S[P_1]$, so $x \notin S[P_1]$. Let $t = \max\{k, t_i : i = 1, \dots, n\} + 1$; assume that $rp^{-t}x' = 0$. Then

$$y = rap^{k-t}e_0 + \sum_1^n v_i p^{t_i - t} r s_i \in F_0;$$

since $r \in \bigcap_i Prp^{t_i - t}$, we deduce from Lemma 1 that $v_i = 0$ for all i, so $x' \in R\overline{e}_0$; since $x' \notin \bigcup_k S[p^k]$, $rp^{-h}x' \neq 0$ for some $h \in \mathbb{N}$; therefore $x' \notin S[P_1]$. □

The following lemmas are crucial in proving that S has no nonzero pure uniserial submodules.

Lemma 3. *The socle $S[P]$ is essential in S.*

Proof. Let $0 \neq x \in S$; we must show that $0 \neq ax \in S[P]$ for some $a \in R$. This is obvious if $x \in \bigcup_{k\in\mathbb{N}} S[p^k]$; otherwise, by Lemma 2, there exists $t \in \mathbb{N}$ such that $rp^{-t}x \neq 0$; but $rp^{-t}x \in \bigcup_{k\in\mathbb{N}} S[p^k]$ by (P2), so the claim follows. □

Now let $\overline{S} = S/S[P_1]$ and $z_s = \overline{s} + S[P_1]$ $(s \in \Sigma)$. In this notation we have the following:

Lemma 4. $\overline{S} = \oplus_{s\in\Sigma} Rz_s$, *where* $Rz_s \cong R/P_1$ *for each* $s \in \Sigma$.

Proof. Since $\overline{S}$ is generated by the elements $z_s (s \in \Sigma)$, it is enough to prove that $\sum_0^n \lambda_i z_{s_i} = 0$ $(\lambda_i \in R)$ implies $\lambda_i z_{s_i} = 0$ for each i, or, equivalently, that $\sum_0^n \lambda_i \overline{s}_i \in S[P_1]$ implies $\lambda_i \overline{s}_i \in S[P_1]$ for each i. We can assume, without loss of generality, that $s_0 = e_0$. We shall prove that $\lambda_i \in P_1$ for all i; hence the claim will follow from Lemma 2 and (P4). Assume, by way of contradiction, that $\lambda_1, \lambda_2, \cdots, \lambda_k \notin P_1$ and $\lambda_i \in P_1$ for $i \geq k+1$ $(k \leq n)$. Then $\sum_{k+1}^n \lambda_i \overline{s}_i \in S[P_1]$ and, for $1 \leq i \leq k$, $\lambda_i = u_i p^{n_i}$ where $u_i \in U(R)$, $n_i \in \mathbb{N}$. So we get:

$$\lambda_0 \overline{e}_0 + \sum_1^k u_i p^{n_i} s_i = x \in S[P_1].$$

If we write x in the form (5) and apply Lemma 1, then we get a contradiction. Thus we have proved that $\lambda_i \in P_1$ for $i = 1, \dots, n$, from which we derive $\lambda_0 \overline{e}_0 \in S[P_1]$. Another application of (P4) and Lemma 1 gives $\lambda_0 \overline{e}_0 = rup^{-h}\overline{e}_0$, so also $\lambda_0 \in P_1$. □

We consider now the heights of the elements in the socle of S.

Lemma 5. *If* $x \in S[P]$, *then* $h_S(x) \geq (P_1^{-2}/R)^-$.

Proof. Since P_1^{-2} is generated by the elements $r^{-1}p^{-n}(n \in \mathbb{N})$, it is enough to prove, looking at (5) in (P4), that for each $s \in \Sigma$ and $h \in \mathbb{N}$: $rp^{-h}\overline{s} \in \bigcap_{n\in\mathbb{N}} rp^n S$. But for each $s \in \Sigma$ there is $s_0 \in \Sigma$ such that $s = s_0^*$ and $r\overline{s}_0 = rp^{-n-h}\overline{s_0^*}$; therefore $rp^{-h}\overline{s} = rp^{-h}\overline{s_0^*} = rp^n\overline{s}_0 \in rp^n S$. Since the choice of n was arbitrary, we are done. □

We can now prove the main property of S.

Proposition 6. *S has no nonzero pure uniserial submodules.*

Proof. Assume, by way of contradiction, that $0 \neq U$ is a pure uniserial submodule of S. By Lemma 3, there exists $0 \neq x \in U \cap S[P]$; by Lemma 5, since U is uniserial pure in S and $P_1^2 S = 0$, it follows that $h_U(x) = h_S(x) = P_1^{-2}/R$. Take $y \in U$ such that $ry = x$. Then, by [3, VIII 1.4], $h_U(y) = rP_1^{-2}/R = P_1^{-1}/R$; note that $y \notin S[P_1]$, because of Lemma 2 and the inequalities $p^k y \neq 0$ for all $k \in \mathbb{N}$. Hence it follows that the element $0 \neq y + S[P_1]$ of $\overline{S}$ has height $\geq P_1^{-1}/R$ in $\overline{S}$, which is obviously absurd, since $\overline{S}$ is a free R/P_1-module, by Lemma 4. □

The proof of our main theorem is now an immediate consequence of Proposition 6. Assume that all the modules over the valuation domain R contain nonzero pure uniserial submodules; then we must prove that R is necessarily rank-one discrete. By Theorem 9 in [6], R is discrete and Spec R is well-ordered by the opposite inclusion; so let $P > P_1 > \cdots$ be the well-ordered sequence of prime ideals of R. We must show that $P_1 = 0$. As a matter of fact, if $P_1 \neq 0$, since P and P_1 satisfy the hypotheses needed to construct the module S, we reach a contradiction by Proposition 6.

Remark. There are already two available examples of noncohesive modules without nonzero pure uniserial submodules over suitable valuation domains. The first is the divisible module ∂, introduced by Fuchs in [1], under the hypothesis that the projective dimension of Q is at least 2. The second is a weakly polyserial module constructed in [4], which can be obtained, with slight modification, over any valuation domain which fails to be almost maximal. From these two examples one obtains the result that a valuation domain, all of whose modules have nonzero pure uniserial submodules, must satisfy the two additional conditions that Spec R is countable and R is almost maximal. However, it is clear from the above discussion that these results are not useful to prove the theorem in this paper.

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