A RESULT BY KULIKOV THAT DOES NOT EXTEND TO MODULES OVER GENERAL VALUATION DOMAINS

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Abstract. Let $R$ be a valuation domain. It is proved that every nonzero $R$-module contains a nonzero pure uniserial submodule if and only if $R$ is rank-one discrete.

It is a classical result in the theory of Abelian groups, due to Kulikov [5], that every Abelian $p$-group contains a pure cocyclic subgroup (which is necessarily a summand, cocyclic groups being pure-injective). Note that cocyclic $p$-groups are exactly the torsion uniserial $\mathbb{Z}_p$-modules, where $\mathbb{Z}_p$ is the rank-one discrete valuation domain obtained by localizing the ring of the integers $\mathbb{Z}$ at the prime $p$.

The goal of this paper is to show that this result is not extendable to valuation domains which are not rank-one discrete. Our main result is the following:

Theorem. Let $R$ be a valuation domain. Then every nonzero $R$-module contains a nonzero pure uniserial submodule if and only if $R$ is rank-one discrete.

The first contribution to the proof of this theorem was given in [2] (see also [3, X.4]), where an example of a cohesive $R$-module (i.e., a module without elements of limit height) with no nonzero uniserial pure submodules was given, under the assumption that the maximal ideal $P$ of $R$ is not principal.

The second author showed in [6] that a cohesive $R$-module with the above property cannot exist if and only if $J \cdot R_J$ is a principal ideal of $R_J$, for each prime ideal $J$ of $R$ (equivalently, $R$ is discrete and $\text{Spec}(R)$ is well ordered by the opposite inclusion); note that the proof of the sufficiency was just an adaptation of the example given in [2].

The consequence of the above results is that, in order to complete the proof of the theorem, we must find a noncohesive $R$-module with no nonzero pure uniserial submodules, where $R$ is any discrete valuation domain with a well-ordered prime spectrum consisting of at least two nonzero ideals.

In fact, we shall construct an $R$-module $S$, where $R$ is as in the preceding

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paragraph, which satisfies some particular properties: $S$ has essential socle, all of whose nonzero elements have limit height, and $S$ has a suitable quotient which is a direct sum of cyclics. These properties ensure that our module $S$ has no nonzero pure uniserial submodules.

We will use the following notation. If $R$ is a valuation domain, $Q$ denotes its field of quotients, $P$ its maximal ideal, and $U(R)$ its group of units. For an $R$-module $M$ and an ideal $I$ of $R$, we set $M[I] = \{x \in M : rx = 0 \text{ for each } r \in I\}$. For the convenience of the reader, we recall the definition of the height of an element $x \in M$ (see [3, VIII]). The divisibility set of $x$ in $M$ is the subset of $R$: $D_M(x) = \{r \in R : x \in rM\}$; the height ideal of $x$ in $M$ is the submodule of $Q$: $H_M(x) = \cup\{Rr^{-1} : r \in D_M(x)\}$.

Then, setting $H_M(x) = J$, the height of $x$ in $M$ is defined as follows:

$$h_M(x) = \begin{cases} J/R & \text{if there exists a morphism } \varphi : J \to M \text{ such that } \varphi 1 = x, \\ (J/R)^{-} & \text{otherwise}. \end{cases}$$

If $h_M(x) = J/R$, then we say that $x$ has nonlimit height; otherwise, we say that $x$ has limit height.

An $R$-module $U$ is uniserial if its submodules are linearly ordered by inclusion. A submodule $N$ of an $R$-module $M$ is pure if $rM \cap N = rN$ for all $r \in R$; this is actually the definition of RD-purity, which coincides with Cohn's purity if $R$ is a valuation domain. For all these definitions and for general facts on modules over valuation domains we refer to [3]. Finally, we denote by $\mathbb{N}$ the set of natural numbers, zero included, and by $\mathbb{N}^+ = \{n \in \mathbb{N} : n \geq 1\}$.

**The module $S$**

Now let $R$ be a valuation domain with principal maximal ideal $P = Rp$, such that the prime ideal $P_1 = \bigcap_{n \in \mathbb{N}} Rp^n$ is nonzero and principal as $R_{P_1}$-module; thus there is an $r \in P_1$ such that $P_1R_{P_1} = rR_{P_1}$. Note that $P_1$ is generated by the elements $rp^{-n}(n \in \mathbb{N})$.

Let $\Sigma$ denote the set consisting of two symbols $e_0$ and $e_1$ and the finite sequences of 0 and 1:

$$\Sigma = \{e_0, e_1, (\sigma_1, \sigma_2, \ldots, \sigma_n) : n \in \mathbb{N}^+, \sigma_i \in \{0, 1\}\}.$$ 

Given any $s \in \Sigma$, we define its length $l(s)$ by setting

$$l(s) = \begin{cases} -1 & \text{if } s = e_0, \\ 0 & \text{if } s = e_1, \\ n & \text{if } s = (\sigma_1, \sigma_2, \ldots, \sigma_n). \end{cases}$$

For each $s = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \Sigma \setminus \{e_0, e_1\}$ and different from $(0, 0, \ldots, 0)$, let $k(s) = \max\{i \leq n : \sigma_i = 1\}$. We define a map $*: \Sigma \setminus \{e_0\} \to \Sigma$, sending $s$
into \( s^* \), by setting
\[
S^* = \begin{cases} 
  e_0 & \text{if } s = e_1 \text{ or } s = (0, 0, \ldots, 0), \\
  e_1 & \text{if } s = (1, 0, \ldots, 0); \text{ i.e., } k(s) = 1, \\
  (\sigma_1, \sigma_2, \ldots, \sigma_{k-1}) & \text{if } s = (\sigma_1, \sigma_2, \ldots, \sigma_n), \\
  \text{and } k = k(s) > 1.
\end{cases}
\]

We remark that \( l(s) > l(s^*) \) for all \( s \in \Sigma \setminus \{e_0\} \).

Finally, we define the map \( t: \Sigma \setminus \{e_0\} \to \mathbb{N} \) in the following way:
\[
t(s) = \begin{cases} 
  l(s) & \text{if } s = e_1 \text{ or } s = (0, 0, \ldots, 0), \\
  l(s) - k(s) + 1 & \text{otherwise}.
\end{cases}
\]

Now let \( F = \bigoplus_{s \in \Sigma} R s \) be the free \( R \)-module with basis \( \Sigma \) and \( F_0 \) the submodule of \( F \) generated by the elements
\[
rpe_0; rs - r p^{-t(s)} s^*, s \in \Sigma \setminus \{e_0\}.
\]

We shall need the following technical result:

**Lemma 1.** Each \( 0 \neq y \in F_0 \) can be written in the form
\[
y = \sum_{i=1}^{n} \lambda_i s_i + rcs
\]

with \( 0 \neq c \in R, \lambda_i \in R, s_i \neq s, \) and \( l(s) \geq l(s_i) \) for each \( i \).

**Proof.** If \( y \in R e_0 \), then \( y = \lambda rpe_0 \) for \( 0 \neq \lambda \in R \), so the claim is true. If \( y \notin R e_0 \), then
\[
y = ae_0 + \sum_{i=1}^{m} a_i (rs_i - r p^{-t_i} s_i^*)
\]

where \( a \in R, 0 \neq a_i \in R, s_i \) are distinct elements of \( \Sigma \setminus \{e_0\}, t_i = t(s_i) \) \((i = 1, \ldots, m)\). Choose \( j \leq m \) such that \( l(s_j) \) is maximal. Then obviously \( l(s_j) > l(s_j^*) \) for all \( i \leq m \). Set \( s = s_j \) and \( c = a_j \), and the claim follows. \( \Box \)

Let \( S = F/F_0 \); for each \( s \in \Sigma \), we denote \( s + F_0 \) by \( \overline{s} \); then \( S \) is generated by the elements \( \{\overline{s}: s \in \Sigma\} \), subject to the following conditions:
\[
(2) \quad \begin{cases} 
  pr\overline{e}_0 = 0; r\overline{e}_0 = r\overline{1}; r(0, 0, \ldots, 0) = rp^{-n}\overline{e}_0 & \text{if } l(0, \ldots, 0) = n; \\
  r(1, 0, \ldots, 0) = rp^{-n}\overline{e}_1 & \text{if } l(1, 0, \ldots, 0) = n; \\
  r\overline{s} = rp^{-t(s)}(\sigma_1, \ldots, \sigma_{k-1}) = rp^{-t(s)} s^* & \text{if } s = (\sigma_1, \ldots, \sigma_n), \\
  \text{and } k = k(s) \geq 2.
\end{cases}
\]

From Lemma 1 it follows that \( s \neq s' \) in \( \Sigma \) implies \( \overline{s} \neq \overline{s'} \) in \( S \).

We start by verifying some properties of the elements in \( S \).

(P1) For each \( s \in \Sigma \) there is an \( n(s) \in \mathbb{N} \) such that \( rp^{n(s)}\overline{s} = r\overline{e}_0 \).

We induct on \( l(s) \), the claim being trivial for \( l(s) = -1 \), i.e., if \( s = e_0 \). If \( l(s) > -1 \), then \( rp^{l(s)}\overline{s} = rs^* \) and \( l(s^*) < l(s) \) gives, by induction, \( rp^{l(s)+n(s^*)}\overline{s} = rp^{n(s^*)} s^* = r\overline{e}_0 \).
(P2) For each \( x \in S \) there is an \( n \in \mathbb{N} \) such that \( r p^n x = 0 \); hence \( P_1^2 S = 0 \). It is an immediate consequence of (P1) and relations in (2).

(P3) Every \( x \in S \) can be written in the form

\[
x = a \bar{e}_0 + \sum_{i=1}^{n} v_i p^n \bar{s}_i + \sum_{j=1}^{m} \varepsilon_j r p^{-m_j} \bar{s}_j
\]

where \( a \in R \), the \( v_i \)'s and \( \varepsilon_j \)'s are either units or zero, \( n_i \in \mathbb{N} \) for all \( i \), and \( m_j \in \mathbb{N}^+ \) for all \( j \).

We can assume \( x \notin R \bar{e}_0 \). First we show that \( x \) is of the form

\[
x = a \bar{e}_0 + \sum_{i=1}^{m} a_i \bar{s}_i
\]

with \( a \in R \), \( a_i \in R \) for all \( i \), and \( r \in \bigcap_i Pa_i \). We can obviously write \( x = b \bar{e}_0 + \sum_{i=1}^{m} b_i \bar{s}_i \) (\( b, b_i \in R \)); if \( r \in \bigcap_i Pb_i \), then we are done. Let us assume that \( b_j = c r \) for a \( j \leq m(c \in R) \). If \( c \in R p^{n(s_j)} \), then \( c \bar{s}_j \in R \bar{e}_0 \) by (P1), so \( b_j \bar{s}_j \) can be replaced by \( rd \bar{e}_0 \) for a suitable \( d \in R \). If \( c = u p^h \), with \( u \in U(R) \) and \( h < n(s_j) \) (this implies that \( s_j \notin e_i \), since \( n(e_i) = 0 \)), then \( r u p^{h-\varepsilon(s_j)} \bar{s}_j = r u p^{h-\varepsilon(s_j)} \bar{s}_j \). If \( h < t(s_j) \), we can replace \( r c \bar{s}_j = r u p^{h-\varepsilon(s_j)} \bar{s}_j \), since \( r \in Pr u p^{h-\varepsilon(s_j)} \); if \( h \geq t(s_j) \), we induct on \( h \), since the claim is true for \( h = 0 \). To conclude, it is enough to note that \( r \in \bigcap_i Pa_i \) implies that, for each \( i \leq m \), either \( a_i = v_i p^n \) for some \( v_i \in U(R) \) and \( n_i \in \mathbb{N} \), or \( a_i = \varepsilon_i r p^{-m_i} \) for some \( \varepsilon_i \in U(R) \) and \( m_i \in \mathbb{N}^+ \).

(P4) Let \( 0 \neq x \in S \); then \( x \in \bigcup_{k \in \mathbb{N}} S[p^k] \) if and only if \( x \) can be written in the form

\[
x = r u p^{-h} \bar{e}_0 + \sum_{j=1}^{m} r \varepsilon_j r p^{-m_j} \bar{s}_j
\]

where \( u \) and the \( \varepsilon_j \)'s are either units or zero, \( h \in \mathbb{N} \), and \( m_j \in \mathbb{N}^+ \) for all \( j \).

If \( x \) is as in (5), then (P2) ensures that \( x \in S[p^k] \) for a suitable \( k \in \mathbb{N} \). Conversely, assume that \( 0 \neq x \in S[p^k] \) for a \( k \in \mathbb{N}^+ \), and write \( x \) as in (3).

Then, for a suitable \( t \geq k \), \( p^t \varepsilon_j r p^{-m_j} \bar{s}_j = 0 \) for all \( j \leq m \); so we get

\[
0 = p^t x = ap^t \bar{e}_0 + \sum_{i=1}^{n} v_i p^n \bar{s}_i;
\]

thus the element of \( F \): \( y = a p^t \bar{e}_0 + \sum_{i=1}^{n} v_i p^n \bar{s}_i \) belongs to \( F_0 \). From Lemma 1 we deduce that \( v_i = 0 \) for all \( i \) and \( a = r u p^{-h} \) for a suitable \( h \in \mathbb{N} \), and \( u \) is a unit of \( R \) or zero. Hence \( x \) is of the desired form (5).

We consider now some structural properties of \( S \). Obviously \( S[P_1] = \bigcap_{k \in \mathbb{N}} S[r p^{-k}] \), and \( S[P_1] \supseteq \bigcup_{k \in \mathbb{N}} S[p^k] \). The converse inclusion also holds.

**Lemma 2.** \( S[P_1] = \bigcup_{k \in \mathbb{N}} S[p^k] \).
Proof. It is enough to prove that, if $x \notin \bigcup_{k \in \mathbb{N}} S[p^k]$, then there exists $t \in \mathbb{N}$ such that $r p^{-t} x \neq 0$. If $x \in R \overline{e}_0$, then the claim is obvious, since $R \overline{e}_0 \cong R / R p$ [or use Lemma 1 and (P4)]. Assume that $x \notin R \overline{e}_0$ and write $x$ as in (3). Then, as in the proof of (P4), one can show that there exists $k \in \mathbb{N}$ such that

$$p^k x = a p^k \overline{e}_0 + \sum_{i=1}^{n} v_i p^{t_i} \overline{s}_i \quad (t_i \in \mathbb{N})$$

where $p^k x \neq 0$, since $x \notin S[p^k]$. We show that $x' = p^k x \notin S[P_1]$, so $x \notin S[P_1]$. Let $t = \max\{k, t_i; i = 1, \ldots, n\} + 1$; assume that $r p^{-t} x' = 0$. Then

$$y = r a p^{k-t} u^0 + \sum_{i=1}^{n} v_i p^{t_i-t} r s_i \in F_0;$$

since $r \in \bigcap_i Pr p^{l_i-t}$, we deduce from Lemma 1 that $v_i = 0$ for all $i$, so $x' \in R \overline{e}_0$; since $x' \notin \bigcup_k S[p^k]$, $r p^{-h} x' \neq 0$ for some $h \in \mathbb{N}$; therefore $x' \notin S[P_1]$. □

The following lemmas are crucial in proving that $S$ has no nonzero pure uniserial submodules.

Proof. Let $0 \neq x \in S$; we must show that $0 \neq ax \in S[P]$ for some $a \in R$. This is obvious if $x \in \bigcup_{k \in \mathbb{N}} S[p^k]$; otherwise, by Lemma 2, there exists $t \in \mathbb{N}$ such that $r p^{-t} x \neq 0$; but $r p^{-t} x \in \bigcup_{k \in \mathbb{N}} S[p^k]$ by (P2), so the claim follows. □

Now let $S = S / S[P_1]$ and $z_s = s + S[P_1]$ ($s \in \Sigma$). In this notation we have the following:

Lemma 4. $S = \bigoplus_{s \in \Sigma} R z_s$, where $R z_s \cong R / P_1$ for each $s \in \Sigma$.
Proof. Since $S$ is generated by the elements $z_s (s \in \Sigma)$, it is enough to prove that $\sum_{k=0}^{n} \lambda_i z_{s_i} = 0$ ($\lambda_i \in R$) implies $\lambda_i z_{s_i} = 0$ for each $i$, or, equivalently, that $\sum_{k=0}^{n} \lambda_i \overline{s}_i \in S[P_1]$ implies $\lambda_i \overline{s}_i \in S[P_1]$ for each $i$. We can assume, without loss of generality, that $s_0 = e_0$. We shall prove that $\lambda_i \in P_1$ for all $i$; hence the claim will follow from Lemma 2 and (P4). Assume, by way of contradiction, that $\lambda_1, \lambda_2, \ldots, \lambda_k \notin P_1$ and $\lambda_i \in P_1$ for $i \geq k + 1$ ($k \leq n$). Then $\sum_{k+1}^{n} \lambda_i \overline{s}_i \in S[P_1]$ and, for $1 \leq i \leq k$, $\lambda_i = u_i p^{n_i}$ where $u_i \in U(R)$, $n_i \in \mathbb{N}$. So we get:

$$\lambda_0 \overline{e}_0 + \sum_{i=1}^{k} u_i p^{n_i} \overline{s}_i = x \in S[P_1].$$

If we write $x$ in the form (5) and apply Lemma 1, then we get a contradiction. Thus we have proved that $\lambda_i \in P_1$ for $i = 1, \ldots, n$, from which we derive $\lambda_0 \overline{e}_0 \in S[P_1]$. Another application of (P4) and Lemma 1 gives $\lambda_0 \overline{e}_0 = r u p^{-h} \overline{e}_0$, so also $\lambda_0 \in P_1$. □
We consider now the heights of the elements in the socle of $S$.

**Lemma 5.** If $x \in S[P]$, then $h_S(x) \geq (P_1^{-2}/R)^{-}$.

**Proof.** Since $P_1^{-2}$ is generated by the elements $r^{-1}p^{-n}(n \in \mathbb{N})$, it is enough to prove, looking at (5) in (P4), that for each $s \in \Sigma$ and $h \in \mathbb{N}$: $r^{p^{-n}s} \in \bigcap_{n \in \mathbb{N}} rp^nS$. But for each $s \in \Sigma$ there is $s_0 \in \Sigma$ such that $s = s_0^*$ and $r^{p^{-n}s_0} = rp^{-n-h}s_0^*$; therefore $rp^{-h}s = rp^{-h}s_0^* = rp^nS$. Since the choice of $n$ was arbitrary, we are done. □

We can now prove the main property of $S$.

**Proposition 6.** $S$ has no nonzero pure uniserial submodules.

**Proof.** Assume, by way of contradiction, that $0 \neq U$ is a pure uniserial submodule of $S$. By Lemma 3, there exists $0 \neq x \in U \cap S[P]$; by Lemma 5, since $U$ is uniserial pure in $S$ and $P_1^2S = 0$, it follows that $h_U(x) = h_S(x) = P_1^{-2}/R$. Take $y \in U$ such that $ry = x$. Then, by [3, VIII 1.4], $h_U(y) = rP_1^{-2}/R = P^{-1}/R$; note that $y \notin S[P_1]$, because of Lemma 2 and the inequalities $p^ky \neq 0$ for all $k \in \mathbb{N}$. Hence it follows that the element $0 \neq y + S[P_1]$ of $\overline{S}$ has height $\geq P_1^{-1}/R$ in $\overline{S}$, which is obviously absurd, since $\overline{S}$ is a free $R/P_1$-module, by Lemma 4. □

The proof of our main theorem is now an immediate consequence of Proposition 6. Assume that all the modules over the valuation domain $R$ contain nonzero pure uniserial submodules; then we must prove that $R$ is necessarily rank-one discrete. By Theorem 9 in [6], $R$ is discrete and Spec $R$ is well-ordered by the opposite inclusion; so let $P > P_1 > \cdots$ be the well-ordered sequence of prime ideals of $R$. We must show that $P_1 = 0$. As a matter of fact, if $P_1 \neq 0$, since $P$ and $P_1$ satisfy the hypotheses needed to construct the module $S$, we reach a contradiction by Proposition 6.

**Remark.** There are already two available examples of noncohesive modules without nonzero pure uniserial submodules over suitable valuation domains. The first is the divisible module $\partial$, introduced by Fuchs in [1], under the hypothesis that the projective dimension of $Q$ is at least 2. The second is a weakly polyserial module constructed in [4], which can be obtained, with slight modification, over any valuation domain which fails to be almost maximal. From these two examples one obtains the result that a valuation domain, all of whose modules have nonzero pure uniserial submodules, must satisfy the two additional conditions that Spec $R$ is countable and $R$ is almost maximal. However, it is clear from the above discussion that these results are not useful to prove the theorem in this paper.
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