A GENERIC TORELLI-TYPE THEOREM FOR SINGULAR ALGEBRAIC CURVES WITH AN INVOLUTION

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ABSTRACT. We prove a generic Torelli-type theorem for a special class of singular algebraic curves with an involution. In order to obtain this result we introduce an appropriate mixed Hodge structure on the anti-invariant part of the first homology group, and study its properties.

Let \( X \) be an irreducible projective algebraic curve with an involution \( \sigma \). Suppose \( X \) has only ordinary singularities and let \( \Sigma \) be its singular locus. Let \( \pi: N \to X \) be the normalization of \( X \) and let \( \tau: N \to N \) be the involution induced by \( \sigma \). We suppose that the following condition is satisfied:

\[
(*) \quad \text{The set of fixed points of } \sigma \text{ coincides with } \Sigma \text{ and the involution } \tau \text{ is without fixed points.}
\]

Following J. Carlson [2] one can introduce a polarized mixed Hodge structure (PMHS) on the anti-invariant part of \( H_1(X, \mathbb{Z}) \) with respect to \( \sigma \), denoted by \( H_1^-(X, \mathbb{Z}) \):

\[
0 \to H_1^-(N, \mathbb{Z}) \to H_1^-(X, \mathbb{Z}) \to A \to 0,
\]

where \( H_1^-(N, \mathbb{Z}) \) is the anti-invariant part of \( H_1(N, \mathbb{Z}) \) with respect to \( \tau \) and has a polarized Hodge structure (PHS) of weight \(-1\); \( A \) has PHS of weight \(0\).

It turns out that the latter is isomorphic to the lattice generated by a root-system of the type \( D_n \) when \( \#(\pi^{-1}(\Sigma)) > 2 \).

Using the generic Torelli theorem for the Prym map as proven by Friedman-Smith [4] and Kanev [5], the pair \((N, \tau)\) is uniquely determined by its Prym variety (equivalently by the PHS of \( H_1^-(N, \mathbb{Z}) \)) if the following condition is satisfied.

\( N/\tau \) is a sufficiently general curve of genus \( g \geq 7 \).

We prove the following result:

**Theorem 7.** Let \( X \) be the curve which satisfies (\(*\)). Suppose that \( N/\tau \) is a general curve of genus \( g \geq 15 \). Then \( N, \tau, \) and the set \( \pi^{-1}(\Sigma) \) are uniquely determined by the PMHS of \( H_1^-(X, \mathbb{Z}) \).

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As a consequence we obtain

**Theorem 8.** If \( X \) satisfies the conditions of Theorem 7 and has only one singular point then \( X \) and \( \sigma \) are uniquely determined by the PMHS of \( H_1^-(X, \mathbb{Z}) \).

1. **Preliminaries**

Let \( X \) be an irreducible projective curve with ordinary singularities, and \( n: N \to X \) be its normalization for which

(i) there exist involutions \( \tau: N \to N \) and \( \sigma: X \to X \), such that \( \tau \) has no fixed points and the fixed points of \( \sigma \) are the singularities of \( X \);

(ii) \( \pi \circ \tau = \sigma \circ \tau \).

Such curves can be constructed as follows:

(i) we fix \( (N, \tau) \) to be a smooth irreducible projective algebraic curve with an involution without fixed points;

(ii) we choose a finite subset \( \Omega \subset N \) such that \( \Omega = \bigcup_{i=1}^{k} \Omega_i \), and for each \( i \), \( \tau(\Omega_i) = \Omega_i \) and \( \Omega_i \cap \Omega_j = \emptyset \) if \( i \neq j \);

(iii) we define \( X = N/\rho \), provided with the factor-topology and the induced involution \( \sigma \), where by definition for each \( a, b \in N \)

\[ \{a \rho b\} \text{ iff } \{\text{either } a = b \text{ or } a, b \in \Omega_i \text{ for some } i\}. \]

The condition that \( X \) has only ordinary singularities uniquely determines \( X \) as an algebraic curve.

Obviously \( M = N/\tau \) is a smooth projective algebraic curve and the induced morphism \( \psi: M \to Y = X/\sigma \) is a normalization. Furthermore \( Y \) has at worst ordinary singularities.

Let \( \phi: N \to M \) and \( \lambda: X \to Y \) be the corresponding factor-morphisms.

2. **Definition and properties of the main exact sequence**

J. Carlson [2] constructed the following exact sequence for \( \pi: N \to X \).

\[ 0 \to H_1(N, \mathbb{Z}) \xrightarrow{\pi^*} H_1(X, \mathbb{Z}) \xrightarrow{\partial} \zeta_x \to 0, \]

where \( \partial \) is a boundary operator, \( \zeta_x \subset \text{Div}^0 N \) is a finitely-generated group, which by means of the natural polarization on \( \text{Div} N \) (for which the points of \( N \) form an orthonormal base) has a representation as an orthogonal sum:

\[ \zeta_x = \bigoplus_{i=1}^{k} \zeta(y_i), \]

where \( y_i = \pi(\Omega_i) \). Furthermore \( \zeta(y_i) \) is spanned by a root-system of type \( A_{r(i)} \), where \( r(i) = \#(\Omega_i) - 1 \). The morphisms \( \sigma \) and \( \tau \) act on (1). Since \( \pi \circ \tau = \sigma \circ \pi \) for the anti-invariant part of the corresponding members of (1) we have:

\[ H_1^-(N, \mathbb{Z}) \xrightarrow{\pi^*} H_1^-(X, \mathbb{Z}) \xrightarrow{\partial} \zeta_x^- \to 0, \]

Since \( \tau(\Omega_i) = \Omega_i \) and \( \tau \) has no fixed points we have

\[ \zeta_x^- = \bigoplus_{i=1}^{k} \zeta^-(y_i) = \bigoplus_{i=1}^{l} \mathbb{Z}(x_i - \tau(x_i)), \]

which is an orthogonal sum. Here \( 2s = \#(\Omega) \) and \( \Omega = \{x_1, \ldots, x_s, \tau(x_1), \ldots, \tau(x_s)\} \). Hence the nonzero elements of \( \zeta_x^- \) with min-
imal length are
\[ \{ \pm(x_1 - \tau(x_1)|x_1 \in \Omega, \ i = 1, 2, \ldots, s}. \]

Let \( A = \partial(H_1^- (X, \mathbb{Z})). \)

**Lemma 1.** (i) If \( \#(\Omega) = 2, \) then \( A \cong \mathbb{Z}, \) a and \( (a, a) = 8; \)
(ii) If \( \#(\Omega) \geq 4 \) and \( R = \{ a \in \zeta_x| (a, a) = 4 \}, \) then \( R \) is a root-system of type \( D_s \) (\( 2s = \#(\Omega) \)) and \( R \) generates \( A. \)

**Proof.** (a) We claim that there is no element \( b \in \zeta_x^- \) for which \( (b, b) = 2 \) and \( b \in A. \) Indeed if there exists such an element then \( b = x - \tau(x) \in A \) with \( x \in \Omega, \) hence there exists \( c \in C_1(N) \) for which \( \partial(c) = x - \tau(x) \) and \( \pi_*(c) \in H_1^- (X, \mathbb{Z}). \) Furthermore \( c + \tau(c) \) is a cycle in \( H_1(N, \mathbb{Z}), \) and \( \pi_*(c) \in H_1^- (X, \mathbb{Z}) \) implies that \( \pi_*(c + \tau(c)) = 0 \) in \( H_1(X, \mathbb{Z}). \) Since \( H_1(N, \mathbb{Z}) \to H_1(X, \mathbb{Z}) \) is injective, it follows that \( c + \tau(c) \) is homologous to \( 0 \) in \( H_1(N, \mathbb{Z}). \)

Now consider \( \phi: N \to M. \) Since \( M = N/\tau, \) it follows that \( \phi_*(c + \tau(c)) = 2\phi_*(c) \) is homologous to \( 0 \) in \( H_1(M, \mathbb{Z}). \) Since \( H_1(M, \mathbb{Z}) \) is torsion-free, \( \phi_*(c) \) is also homologous to \( 0. \) Thus, as a loop in \( \pi_1(M), \) \( \phi_*(c) \) is in the commutator subgroup. Since \( \pi_1(N) \subset \pi_1(M) \) is a normal subgroup of index \( 2, \) \( \pi_1(N) \) contains the commutator subgroup. Thus \( \phi_*(c) \) lies in the image of \( \pi_1(N), \) which means that \( \phi_*(c) \) lifts to a closed loop in \( N. \) This contradicts \( \partial(c) = x - \tau(x). \)

(b) We claim that if \( x, y \in \Omega, \) then \( a = x - \tau(x) \pm (y - \tau(y)) \) is an element of \( A. \) It is sufficient to prove the statement for \( a = x - \tau(x) + y - \tau(y) \) since \( x - \tau(x) - (y - \tau(y)) = x - \tau(x) + (y - \tau(y)) \) \( \partial(c) = x - \tau(y) \). Then \( \partial(c - \tau(c)) = x - \tau(x) + y - \tau(y). \)

We complete the proof of Lemma 1 as follows:

**Case (i).** By (a) and (b) \( A \) is generated by \( a = 2(x - y), \) where \( \{x, y\} = \Omega. \)

**Case (ii).** Put \( e_i = x_i - \tau(x_i), \) where \( \Omega = \{x_1, \ldots, x_s, \tau(x_1), \ldots, \tau(x_s)\}. \) Then \( R = \{e_i \pm e_j|i \neq j\}, \) so \( R \) is a root-system of type \( D_s. \) Since \( R = \{e_i \pm e_j\}, \) it follows that \( R \) spans a sublattice \( R' \) of index \( 2 \) in \( \bigoplus_{i=1}^s \mathbb{Z}e_i. \) Then (b) implies \( R' \subset A, \) so that \( R' = A \) or \( A = \bigoplus_{i=1}^s \mathbb{Z}e_i. \) By (a), the latter cannot happen, which proves that \( R \) generates \( A. \) Q.E.D.

**Lemma 2.** The sequence
\[ 0 \to H_1^-(N, \mathbb{Z}) \xrightarrow{\pi_*} H_1^-(X, \mathbb{Z}) \xrightarrow{\partial} A \to 0 \]
is exact.

**Proof.** The exactness at the first member of (2) is derived from the exactness of the first member of (1); the exactness in the third member of (2) is derived from the definition of \( A. \) It remains to prove that \( \text{Ker} \partial = \text{Im} \pi_* \). Obviously \( \text{Ker} \partial \supset \text{Im} \pi_* \). Let \( a \in H_1^-(X, \mathbb{Z}) \) and \( \partial(a) = 0, \) hence there exists \( b \in H_1(N, \mathbb{Z}) \) such that \( \pi_*(b) = a, \) since (1) is exact. We have
\[ 0 = a\pi_*(a) = \pi_*(b) + \sigma_*\pi_*(b) = \pi_*(b + \tau(b)), \]
hence \( b + \tau(b) = 0 \), since \( \pi_* \) is an injection. Hence \( b \in H^1_{\ast}(N, \mathbb{Z}) \) and \( \pi_*(b) = a \), i.e., \( \text{Ker} \, \partial \subset \text{Im} \, \pi_* \). Q.E.D.

3. Construction of PMHS for \( H^1_{\ast}(X, \mathbb{Z}) \)

From the Poincaré duality we get an exact sequence:

\[ 0 \to \hat{\mathcal{A}} \overset{\delta}{\to} H^1_{\ast}(X, \mathbb{Z}) \overset{\pi_*}{\to} H^1_{\ast}(N, \mathbb{Z}) \to 0, \]

where \( \hat{\mathcal{A}} = \text{Hom}_\mathbb{Z}(\mathcal{A}, \mathbb{Z}) \).

Following J. Carlson [2] we define a mixed Hodge structure on \( H^1_{\ast}(X, \mathbb{Z}) \):

(a) Weight filtration on \( H^1_{\ast}(X, \mathbb{Z}) \):

\[ W_{-1} = 0, \quad W_0 = \text{Im} \, \delta, \quad W_1 = H^1_{\ast}(X, \mathbb{Z}); \]

define polarizations on \( W_0 \) and \( W_1/W_0 \) via the polarizations on \( \mathcal{A} \) and on \( H^1_{\ast}(N, \mathbb{Z}) \) introduced above.

(b) Hodge filtration on \( H^1_{\ast}(X, \mathbb{C}) = H^1_{\ast}(X, \mathbb{Z}) \otimes \mathbb{C} \):

\[ F^0 = H^1_{\ast}(X, \mathbb{C}); \]
\[ F^1 = \left\{ \omega \in H^0(X - \Sigma, \Omega^1(X - \Sigma)) \mid \int_{X - \Sigma} \omega \wedge \overline{\omega} < \infty, \sigma^* \omega + \omega = 0 \right\}, \]

here \( \Sigma \) is the singular locus of \( X \).

\[ F^2 = 0. \]

**Lemma 3.** The map \( \pi^* \) gives an isomorphism

\[ F^1 \cong H^0_{\ast}(N, \Omega^1_N). \]

It follows from Lemma 3 that \( F^1 \cap W_0 = 0, \quad W_1/W_0 \otimes \mathbb{C} \cong F^1 \oplus F^1 \) which means that we have MHS on \( H^1_{\ast}(X, \mathbb{Z}) \), \( W_0 \) has PHS of pure weight 0, and \( W_1/W_0 \) has PHS of pure weight 1 (cf. [2]).

**Proof.** Let \( \alpha \in F^1 \). Then \( \pi^*(\alpha) \in H^0(N - \Omega, \Omega^1(N - \Omega)) \). Since

\[ \int_{N - \Omega} \pi^* \alpha \wedge \overline{\pi^* \alpha} = \int_{X - \Sigma} \alpha \wedge \overline{\alpha} < \infty, \]

then \( \pi^* \alpha \in H^0(N, \Omega^1_N) \). We have \( \pi^* \alpha + \tau^* \circ \pi^* \alpha = \pi^* (\alpha + \sigma^* \alpha) = 0 \) on \( N - \Omega \). The 1-form \( \pi^* \alpha \) is holomorphic, so \( \pi^* \alpha + \tau^* \circ \pi^* \alpha = 0 \) on \( N \). Thus we have a linear map \( \pi^*: F^1 \to H^0_{\ast}(N, \Omega^1_N) \). The inverse of this map is obviously defined since \( N \) and \( X \) are birationally isomorphic. Q.E.D.

**Proposition 4.** The group \( H^1_{\ast}(X, \mathbb{Z}) \) has a polarized mixed Hodge structure for which:

(i) \( W_0 H^1_{\ast}(X) = H^1_{\ast}(X, \mathbb{Z}), \quad W_{-1} H^1_{\ast}(X) = \text{Im} \, \pi_* = H^1_{\ast}(N, \mathbb{Z}), \quad W_{-2} H^1_{\ast}(X) = 0; \)
(ii) $F^1 H^1_-(X) = 0$, $F^0 H^1_-(X) = \text{ann}_R(F^1 H^1_-(X)) \cong \overline{F^1 H^1_-(X)} \oplus (A \otimes \mathbb{C})$, $F^{-1} H^1_-(X) = H^1_-(X, \mathbb{C})$, here $F^1 H^1_-(X)^* = \text{Hom}_R(F^1 H^1_-(X), \mathbb{R})$ and the complex conjugation in $H^1_-(X, \mathbb{C})$ is induced by those in $\mathbb{C}$ through $H^1_-(X, \mathbb{C}) = H^1_-(X, \mathbb{Z}) \otimes \mathbb{C}$;

(iii) $\text{Gr}^w_{-1} = W_{-1} / W_{-2}$ has a pure weight $-1$, the polarization of $H^1_-(N, \mathbb{Z})$ is transferred to $\text{Gr}^w_{-1}$ through $\pi_*$. $\text{Gr}^w_{0} = W_{0} / W_{-1}$ has a pure weight $0$ and the polarization of $A$ is transferred to $\text{Gr}^w_{0}$ through $\partial$.

Proof. This is an immediate consequence of Lemma 3 and the definition of MHS on a dual group (cf. [2]). Q.E.D.

It follows that the Hodge structure on $\text{Gr}^w_{-1} = W_{-1} H^1_-(X)$ is:

$$F^{-1} W_{-1} H^1_-(X) = F^{-1} H^1_-(X) \cap [W_{-1} H^1_-(X) \otimes \mathbb{Z} \mathbb{C}] = F^1 H^1_-(X)^* \oplus \overline{F^1 H^1_-(X)^*} = W_{-1} H^1_-(X);$$

$$F^0 W_{-1} H^1_-(X) = F^0 H^1_-(X) \cap [W_{-1} H^1_-(X) \otimes \mathbb{Z} \mathbb{C}] = \overline{F^1 H^1_-(X)^*};$$

$$F^1 W_{-1} H^1_-(X) = 0.$$

4. Geometric description of the 1-motive map

By definition we have

$$L^0 H^1_-(X) = [\text{Gr}^w_{0} H^1_-(X) \otimes \mathbb{Z} \mathbb{C}] \cap [\text{Gr}^w_{0} H^1_-(X)]_Z, \quad J^0 W_{-1} H^1_-(X) = [W_{-1} H^1_-(X) \otimes \mathbb{Z} \mathbb{C}] / [W_{-1} H^1_-(X) + F^0 W_{-1} H^1_-(X)].$$

It is clear from Proposition 4 and from the sequence (2) that $H^1_-(X, \mathbb{Z})$ is an extension of $\text{Gr}^w_{0}$ by $\text{Gr}^w_{-1}$. For extensions of this type J. Carlson [2] has constructed a map 1-motive:

$$u: L^0 H^1_-(X) \to J^0 W_{-1} H^1_-(X),$$

which depends only on the mixed Hodge structure of $H^1_-(X, \mathbb{Z})$ and is given as follows:

(i) Let $\{\omega_i\}$ be a basis of $\text{Gr}^w_{-1}$ and $\{\omega^i\}$ be the dual basis in $\left(\text{Gr}^w_{-1}\right)^*$;

(ii) Let $\{\Omega^i\} \subset H^1_-(X, \mathbb{Z})$, for which $\pi^*(\Omega^i) = \omega^i$ for each $i$;

(iii) If $\gamma \in \text{Gr}^w_{0}$ and $\Gamma \in H^1_-(X, \mathbb{Z})$ is such that $\partial(\Gamma) = \gamma$, then $\gamma \leadsto [\Sigma_i \langle \Omega^i, \Gamma \rangle \omega_i]$, where $\langle , \rangle$ is the canonical pairing between $H^1_-(X, \mathbb{Z})$ and $H^1_-(X, \mathbb{Z})$; and $[\alpha]$ is the class of $\alpha$ in $J^0 W_{-1} H^1_-(X)$. 

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Let us recall that for \((N, \tau)\) a Prym variety \(P(N, \tau) = H^{1,0}_{-1}(N)^*/H^1_{-1}(N, \mathbb{Z})\) is defined and a map \(PA: \text{Div}^0(N) \to P(N, \tau)\), where

\[
P A(P - Q) = \left( \int_Q \phi_1, \ldots, \int_Q \phi_{g-1} \right) \pmod{H^1_{-1}(N, \mathbb{Z})}.
\]

Here \(\{\phi_1, \ldots, \phi_{g-1}\}\) is a basis of \(H^{1,0}_{-1}(N)\) and \(H^1_{-1}(N, \mathbb{Z})\) is injected into \(H^{1,0}_{-1}(N)^*\) by integration. \(PA\) is called the Abel–Prym’s map.

**Proposition 5.** The following diagram is commutative

\[
\begin{array}{ccc}
L^0H^1_{-1}(X) & \xrightarrow{\nu} & J^0W_{-1}H^1_{-1}(X) \\
\partial_1 \cong & A & \xrightarrow{\mu} P(N, \tau)
\end{array}
\]

where \(\nu\) is given by the identification of \(W_{-1}H^1_{-1}(X)\) with \(H^1_{-1}(N, \mathbb{Z})\) and by using the duality given by integration; \(\mu\) is the Abel–Prym’s map, restricted on \(A \subset \text{Div}^0 N\).

**Proof.** Obviously \(L^0H^1_{-1}(X) = \text{Gr}_W H^1_{-1}(X)\) which is isomorphic to \(A\) through \(\partial\) (cf. Proposition 4).

\[
J^0W_{-1}H^1_{-1}(X) = \frac{W_{-1}H^1_{-1}(X) \otimes \mathbb{C}}{W_{-1}H^1_{-1}(X) + F^0W_{-1}H^1_{-1}(X)} = \frac{F^1H^1_{-1}(X)^* \oplus F^1H^1_{-1}(X)^*}{W_{-1}H^1_{-1}(X) + F^0W_{-1}H^1_{-1}(X)}
\]

\[
\cong \frac{F^1H^1_{-1}(X)^*}{W_{-1}H^1_{-1}(X)} \cong \frac{H^{1,0}_{-1}(X)^*}{H^1_{-1}(N, \mathbb{Z})} \cong P(N, \tau).
\]

To calculate \(\mu\) we introduce the well-known symplectic base of \(H_1(N, \mathbb{Z})\):

\[
\{a_1, b_2; a_{g+1}, b_{g+1}; \ldots; a_{g-1}, b_{g-1}; a_{2g-1}, b_{2g-1}; a_g, b_g\}
\]

for which

\[
\tau_*(a_i) = a_{i+g}, \quad \tau_*(b_i) = b_{i+g} \quad \text{for } i = 1, 2, \ldots, g - 1;
\]

\[
\tau_*(a_g) = a_g, \quad \tau_*(b_g) = b_g.
\]

Furthermore, choose a basis \(\{\omega^1, \ldots, \omega^{2g-1}\}\) of \(H^{1,0}_{-1}(N)\) with \(\int a_i \omega^j = \delta^j_i\) for \(i, j = 1, 2, \ldots, 2g - 1\).

Then \(H^{1,0}_{-1}(N) = (\omega^1 - \omega^{g-1}) \cdot \mathbb{C} \oplus \cdots \oplus (\omega^{g-1} - \omega^{2g-1}) \cdot \mathbb{C}\), hence

\[
H^{1,0}_{-1}(N)^* = \left( \int_{a_i-a_{g+1}} \right) \cdot \mathbb{C} \oplus \cdots \oplus \left( \int_{a_{g-1}-a_{-1}} \right) \cdot \mathbb{C}.
\]

In this case \(\mu\) is given as follows:

\[
\gamma \to \left[ \sum_{i=1}^{g-1} \left( \int_{a_i} (\omega^i - \omega^{i+g}) \right) + \left( \int_{a_i-a_{i+g}} \right) \right],
\]

which is exactly the map \(\gamma \to PA(\gamma)\). Q.E.D.
In fact using the same basis of $H^{1,0}(N)$ we have a map $j: P(N, \tau) \to J(N)$, where $J(N)$ is the Jacobi variety of $N$ and
\[
j \circ \mu(\gamma) = \left[ \left( \int_{\Gamma} (\omega^1 - \omega^{g+1}) \right), \ldots, \int_{\Gamma} (\omega^{g-1} - \omega^{2g-1}), 0, -\int_{\Gamma} (\omega^1 - \omega^{g+1}), \ldots, -\int_{\Gamma} (\omega^{g-1} - \omega^{2g-1}) \right].
\]
It is clear that $j \circ \mu = (1 - \tau) \circ Ab$, where $Ab: N \to J(N)$ is the Abel's map for $N$.

5. Proof of Theorem 8

For the Abel's map $Ab: N \to J(N)$ we have $(1 - \tau) \circ Ab(D) = Ab((1 - \tau)D)$ for each $D \in \text{Div}^0 N$. Since each generator of $A$ with minimal length has type $(1 - \tau)(P - Q)$ then
\[
j \circ \mu((1 - \tau)(P - Q)) = (1 - \tau) \circ Ab \circ (1 - \tau)(P - Q) = Ab \circ (1 - \tau)^2(P - Q) = 2Ab \circ (1 - \tau)(P - Q) = 2PA(P - Q).
\]
It follows that we must consider the map
\[
\Phi: N \times N \to P(N, \tau) \subset J(N), \quad (P, Q) \mapsto 2 \cdot PA(P - Q).
\]

Lemma 6. Let $N$ be a smooth projective curve with an involution $\tau$ without fixed points. If $N$ is neither a 4-, nor 8-sheeted covering of $\mathbb{P}^1$, then $\Phi(P_1, Q_1) = \Phi(P_2, Q_2)$ if and only if: either $(P_1, Q_1) = (P_2, Q_2)$ or $(P_1, Q_1) = (\tau(Q_2), \tau(P_2))$ or $(P_1, P_2) = (Q_1, Q_2)$.

Proof. $\Phi(P_1, Q_1) = \Phi(P_2, Q_2)$ iff $2PA(P_1 + Q_2 - Q_1 - P_2) = 0$ in $J(N)$. We consider $D = 2(1 - \tau)(P_1 + Q_2 - Q_1 - P_2)$ as an element of $\text{Div}^0 N$. If $D \neq 0$ then $D = D_+ - D_-$, $D_+ > 0$, $\deg D_+ = \deg D_- = 4$ or 8 and $\text{supp} D_+ \cap \text{supp} D_- = \emptyset$. Since $Ab(D) = 0$ in $J(N)$, then by Abel's theorem for $N$ we conclude that there exists a map $N \to \mathbb{P}^1$ of degree 4 or 8 which is impossible by hypothesis. Thus $D = 0$, which is possible only in the cases listed in the lemma. Q.E.D.

Theorem 7. Let $X$ be a curve of 1, for which $N/\tau$ is a generic curve of genus $g(N/\tau) \geq 15$. Then using the mixed Hodge structure of $H^1(X, \mathbb{Z})$, constructed in 4, we can get $N, \tau: N \to N$ and $\Omega$.

Proof. The mixed Hodge structure gives us $J^0W_{-1}H^1_-(X) \cong P(N, \tau)$. Using the generic Torelli theorem (V. Kanev [5], Friedman-Smith [4]) we obtain $N$ and $\phi: N \to M$, which gives us $(N, \tau)$. Let $R$ be the finite subset of $L^0H^1_-(X)$ defined as follows. If there exist elements $b$ of $L^0H^1_-(X)$ with $(b, b) = 4$, then $R = \{ a | (a, a) = 4 \}$. Otherwise put $R = \{ a | (a, a) = 8 \}$ (see Lemma 1). Consider the 1-motive: $u: L^0H^1_-(X) \to J^0W_{-1}H^1_-(X)$. Since $g(N/\tau) \geq 15$ then $N/\tau$ is not a covering of $\mathbb{P}^1$ of degree 4 or 8 (see [1, p. 214]) implying $N$. 

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is not a covering of $\mathbb{P}^1$ of degree 4 or 8. Then by Proposition 5 and Lemma 6 the set $u(R)$ uniquely determines the set $\Omega$. Q.E.D.

**Theorem 8.** Let $X$ satisfy the conditions of Theorem 7. Suppose $X$ has only one singular point. Then $X$ and $\sigma$ are uniquely determined by the PMHS defined in 4.

**Proof.** By using Theorem 7 we reconstruct $N$, $\tau$, and $\Omega$; since $X$ has an ordinary singular point, which is obtained by identification of the points of $\Omega$, we recover $X$ and $\sigma$. Q.E.D.

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