NORM EXPONENTS AND REPRESENTATION GROUPS

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Abstract. This note provides an upper bound for the exponent of the norm residue group $k^*/\text{Norm}_{K/k}(K^*)$ of a finite Galois extension $K/k$ of number fields that depends on the obstruction to the Hasse norm principle for $K/k$ and on a group theoretical constant.

Let $K/k$ be a finite Galois extension of number fields with Galois group $G = \text{Gal}(K/k)$. We call $v = v(K/k) = \text{exponent of } k^*/\text{Norm}_{K/k}(K^*)$

the norm exponent of $K/k$. Obviously $v$ divides the degree $(K : k)$, and from the density theorem and the local reciprocity isomorphism we see that $\exp(G)$ divides $v$. In this note we derive a “good” upper bound for $v$ which depends on the obstruction to the Hasse norm principle for $K/k$, i.e. on the kernel $\mathcal{H} = \mathcal{H}(K/k)$ of the natural map

$$\tilde{H}^0(G, K^*) \rightarrow \tilde{H}^0(G, A_K^*),$$

where $\tilde{H}^0$ denotes the Tate cohomology in dimension 0 and $A_K^*$ the group of units of the adele ring $A_K$ of $K$, and on a group theoretical constant. It implies and improves all previous results in this respect [4; 7; 8, p. 100]. Tate has observed (see [2, p. 198]) that $\mathcal{H}$ is dual to the kernel $\mathcal{H} = \mathcal{H}(K/k)$ of the localization map

$$H^2(G, C^*) \rightarrow \prod_{\overline{v}} H^2(G_{\overline{v}}, C^*);$$

here $G_{\overline{v}}$ denotes the decomposition group of an extension $\overline{v}$ of the place $v$ of $k$ and cohomology is taken with respect to the trivial group action. A finite group extension $\tilde{G}$ of $G$ is said to be defined by a subgroup $\mathcal{A} \leq H^2(G, C^*)$ if $\mathcal{A}$ is contained in the kernel of the inflation map $H^2(G, C^*) \rightarrow H^2(\tilde{G}, C^*)$. Define

$$\lambda = \lambda(K/k) = \text{minimum of all } \exp(\tilde{G}),$$

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where \( \tilde{G} \) runs over all finite group extensions of \( G \) which are defined by \( \mathcal{H} \). (Note that, in contrast to [7, lines 17/18] it is not required that the embedding problem corresponding to \( \tilde{G} \) is solvable.)

For any natural number \( r \) put

\[
X(k, r) := \bigcap_v \left( k^* \cap (k^*_v)^r \right)/k^r,
\]

where \( v \) runs over all places of \( k \); it is well known that \( X(k, r) \) is trivial if \( r \) is odd and that \( |X(k, r)| \leq 2 \) in any case (see, e.g. [1, p. 93ff]). We prove

1. **Theorem.** \( \nu \) divides \( \lambda \cdot |X(k, \lambda)| \).

*Proof.* Represent every cocycle class \( (f) \in \mathcal{H} \) by a cocycle \( f: G \times G \to \mu_m \), \( \mu_m = \text{group of roots of unity in } \mathbb{C}^* \text{ of order } m = \exp(\mathcal{H}) \), such that the central group extension \( G(f) \) defined by \( (f) \) has minimal exponent. Put \( \tilde{m} = \tilde{m}_f = n \cdot |X(k, n)| \) where \( n = n_f = \exp(G(f)) \). Let \( C_r \) be the cyclic group of order \( r \). We assume that the action of \( G \) on \( C_r \) is trivial. \( (f) \in \mathcal{H} \) implies that the class of the induced cocycle

\[
f': G \times G \to C_m \hookrightarrow C_n
\]

(we identify \( C_r \) with \( \mu_r \)) belongs to the kernel of the homomorphism

\[
H^2(G, C_n) \to \prod_v H^2(G_v, C_n).
\]

Let \( X_r \) be the kernel of the homomorphism

\[
H^2(G_k, C_r) \to \prod_v H^2(G_{k_v}, C_r),
\]

where \( G_k \) resp. \( G_{k_v} \) are the absolute Galois groups of \( k \) and \( k_v \) respectively. \( X_r \) is dual to \( X(k, r) \). It follows that \( \text{Inf}_{G_k} ((f')) \in X_n \). Since the natural map \( X(k, \tilde{m}) \to X(k, n) \) is trivial, it follows that the canonical homomorphism \( X_n \to X_{\tilde{m}} \) is trivial. Hence the embedding problem defined by the cocycle

\[
f'_f: G \times G \to \mu_m \hookrightarrow \mu_{\tilde{m}}
\]

has a surjective solution (see, e.g., [3, especially pp. 88, 96]). Let \( L_{f_i}/K/k \) be a solution of this embedding problem and denote by \( L \) the compositum of all \( L_{f_i} \), \( (f) \in \mathcal{H} \). Then \( \text{Gal}(L/k) \) is defined by \( \mathcal{H} \). As shown in [6, (2.5)], this means that every element in \( k^* \) which is a norm locally everywhere in \( L/k \) is a global norm in \( K/k \). As remarked earlier the density theorem and local class field theory show that \( \exp(\text{Gal}(L/k)) \) is the l.c.m. of all the local norm exponents of \( L/k \). Hence \( \nu(K/k) \) divides \( e = \exp(\text{Gal}(L/k)) \). The equality \( \exp(\text{Gal}(L_{f_i}/k)) = \tilde{m}_f \) shows that \( e \) divides the l.c.m. of all \( \tilde{m}_f \), \( (f) \in \mathcal{H} \).

Clearly \( \tilde{G} := \times_{(f)} G(f) \), \( (f) \in \mathcal{H} \), is defined by \( \mathcal{H} \), and the exponent of \( \tilde{G} \) equals \( \lambda \cdot |X(k, \lambda)| \). This completes the proof.
2. Remark. Clearly \( \nu(K/k) \) divides the l.c.m. of all \( \nu(K/k^p) \), \( p \) a prime, where \( k^p \) is the fixed field of a \( p \)-Sylow subgroup \( G^p \) of \( G \), because the restriction map \( \widehat{H}^0(G, K^*) \to \widehat{H}^0(G^p, K^*) \) is injective on the \( p \)-part of \( \widehat{H}^0(G, K^*) \).

For any finite group \( G \) define

\[
\delta(G) := \begin{cases} 
1 & \text{if } |G| \text{ is odd}, \\
2 & \text{if } |G| \text{ is even}.
\end{cases}
\]

Every finite abelian group \( G \) has a representation group \( \widetilde{G} \) such that \( \exp(G) \) divides \( \delta(G) \cdot \exp(G) \). This comes from the isomorphism \( H^2(G, \mathbb{C}^*) \cong (G \wedge G)^* \) given by \( (f) \mapsto \omega_{(f)} \) where

\[
\omega_{(f)}(x, y) = f(x, y)/f(y, x), \quad x, y \in G.
\]

Since \( \omega_{(f)} \) is a symplectic pairing on \( G \) we may choose the cocycle \( f \) in such a way that \( f \) is bimultiplicative. Then an easy computation shows that the group extension defined by \( f \) has exponent dividing \( \delta(G) \cdot \exp(G) \).

In [9, Corollary (4.7)] it is shown that every \( p \)-group \( G \), \( p \neq 2 \), such that the class of \( G \) is \( \leq p - 2 \) and such that \( \exp(G) = p \) has a representation group of exponent \( p \).

Furthermore, by [5, V, 24.5], the exponent of any representation group of a finite group \( G \) divides the order of \( G \).

Therefore 1 and 2 give the following result

3. Proposition. Assume that every \( p \)-Sylow subgroup of the finite Galois group \( G = \text{Gal}(K/k) \) is abelian or of exponent \( p \neq 2 \) and class \( \leq p - 2 \). Then \( \nu \) divides \( \delta(G) \cdot |X(k, r)| \cdot \exp(G) \) where \( r = |G| = (K : k) \).

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REFERENCES


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