CUSP FORMS ASSOCIATED TO RANK 2
PARABOLIC SUBGROUPS OF KLEINIAN GROUPS

IRWIN KRA

(Communicated by Clifford J. Earle, Jr.)

ABSTRACT. To each rank 2 parabolic subgroup of a Kleinian group \( \Gamma \), we associate a cusp form and a linear functional on the Eichler cohomology space \( PH^1(\Gamma, \Pi_{2q-2}) \). We explore the relation between these two objects and as a consequence we evaluate certain Poincaré series for rank 2 parabolic groups.

This paper is a continuation of the study of automorphic forms associated to elements of Kleinian groups. We treated the case of cusp forms associated to loxodromic elements in [K3] and the holomorphic forms associated to parabolic elements in [K1]. We combine ideas from [K3] and [K1] to investigate the cusp forms associated to rank 2 parabolic subgroups of Kleinian groups.

Let \( A \) and \( B \) be generators of a rank 2 parabolic subgroup \( G \) of a Kleinian group \( \Gamma \). We associate (in §2) to the pair \( (A, B) \) a linear functional

\[
l_{A, B} : PH^1(\Gamma, \Pi_{2q-2}) \to \mathbb{C}
\]
on the space of parabolic cohomology classes for \( \Gamma \), and (in §3) a cusp form \( \phi_{A, B} \in A_q(\Omega, \Gamma) \). One of the main results is (Theorem 4.3) the connection between these two invariants provided by the Bers embedding

\[
\beta^* : A_q(\Omega, \Gamma) \to PH^1(\Gamma, \Pi_{2q-2})
\]
and the Petersson scalar product

\[
(\cdot, \cdot) : A_q(\Omega, \Gamma) \times A_q(\Omega, \Gamma) \to \mathbb{C}.
\]

A by-product of this work is the evaluation of some naturally occurring elliptic sums. See §5.

1. INVARIANTS FOR PARABOLIC ELEMENTS

1.1. A parabolic Möbius transformation \( A \in PSL(2, \mathbb{C}) \) has a unique fixed point \( \alpha = \alpha(A) \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \) and a well-defined translation length

Received by the editors October 30, 1989.
Research partially supported by National Science Foundation grants DMS 8701774 and DMS 8505550.

1We will use the notation and terminology of [K2], [K1] and [K3].

©1991 American Mathematical Society
0002-9939/91 $1.00 + .25 per page

803

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
\( \tau = \tau(A) \in \mathbb{C}^* = \mathbb{C} - \{0\} \). In normal form:

\[
A(z) = z + \tau, \quad A = \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} \text{ if } \alpha = \infty,
\]

and

\[
\frac{1}{A(z) - \alpha} = \frac{1}{z - \alpha} + \tau, \quad A = \begin{bmatrix} 1 + \alpha \tau & -\alpha^2 \tau \\ \tau & 1 - \alpha \tau \end{bmatrix} \text{ if } \alpha \in \mathbb{C}.
\]

For future use, we observe that for finite \( \alpha \),

\[
A'(\alpha) = 1 \text{ and } A''(\alpha) = -2\tau.
\]

If \( C \) is an arbitrary Möbius transformation, then

\[
\alpha(C \circ A \circ C^{-1}) = C(\alpha(A)) \quad \text{and} \quad \tau(C \circ A \circ C^{-1}) = \tau(A)/C'(\alpha).
\]

The above formulae must be properly interpreted for \( \alpha \) or \( C(\alpha) = \infty \). Assume, for example, that \( \alpha \in \mathbb{C} \) and \( C(\alpha) = \infty \). Write

\[
C = \begin{bmatrix} \beta & \delta \\ 1 & -\alpha \end{bmatrix} \in \text{PGL}(2, \mathbb{C}).
\]

Then \( C'(z) = -(\alpha \beta + \delta)/(z - \alpha)^2 \) and we must interpret \( C'(\alpha) \) as \( 1/(\alpha \beta + \delta) \).

1.2. The holomorphic differential form on \( \hat{\mathbb{C}} - \alpha \) for the cyclic group \( \langle A \rangle \) is defined by

\[
h(z) = h_A(z) = \frac{1}{\tau(A)} \text{ if } \alpha = \infty,
\]

and

\[
h(z) = h_A(z) = \frac{-1}{\tau(A) (z - \alpha)^2} \text{ if } \alpha \in \mathbb{C}.
\]

The function \( h \) is canonically associated to the Möbius transformation \( A \) and is characterized by the properties

\[
\int_{z_0}^{A(z_0)} h(z) dz = 1, \quad \text{all } z_0 \in \hat{\mathbb{C}} - \{\alpha\}
\]

(as long as the path of integration stays in \( \hat{\mathbb{C}} - \{\alpha\} \)), and \( h(z)dz \) has simple poles at the two punctures of \( \langle \hat{\mathbb{C}} - \{\alpha\} \rangle \).

As a consequence of (1.1.1) and the basic identity\(^2\)

\[
C(z) - C(\zeta) = (z - \zeta)C'(z)^{1/2}C'(\zeta)^{1/2},
\]

which is valid for all \( C \in \text{PSL}(2, \mathbb{C}) \) and all \( z \) and \( \zeta \in \hat{\mathbb{C}} \), we have

\[
h_{C_{\alpha}A_{\alpha}C^{-1}}(C(z))C'(z) = h_A(z), \quad \text{all } z \in \hat{\mathbb{C}},
\]

(or equivalently,

\[
C_q^*(h_{C_{\alpha}A_{\alpha}C^{-1}}^q) = h_A^q, \quad \text{all } q \in \mathbb{Z}.
\]

\(^2\)This formula (and similar formulae) must be adjusted if \( z, \zeta, C(z), \) or \( C(\zeta) \) equals \( \infty \). We leave such modifications to the reader (generally).
1.3. The polynomial ring of \( A \) acts on the right on the vector space \( \Pi_{2q-2} \) of polynomials of degree \( \leq 2q-2 \) by the Eichler action (from now on \( q \geq 2 \)). The kernel of the operator \([ A - I ]\) is the one-dimensional space with basis \( h_{A}^{1-q} \). The image of \([ A - I ]\) consists of

\[
\Pi_{2q-2}(\alpha) = \{ v \in \Pi_{2q-2} ; v(\alpha) = 0 \}.
\]

For details see [K2, §8.3].

2. Linear functionals on \( PH^{1}(\Gamma, \Pi_{2q-2}) \)

2.1. Let \(( A, B )\) be an ordered pair of generators for a rank 2 parabolic subgroup \( G \) of a Kleinian group \( \Gamma \). Let \( \chi_{1} \) be a cocycle representing a cohomology class in \( PH^{1}(\Gamma, \Pi_{2q-2}) \). Choose \( v \in \Pi_{2q-2} \) such that

\[
\chi_{1}(A) = v \cdot A - v.
\]

Then the cocycle

\[
\chi(\gamma) = \chi_{1}(\gamma) - v \cdot \gamma + v, \quad \gamma \in \Gamma,
\]

is cohomologous to \( \chi_{1} \) and satisfies

\[
\chi(A) = 0.
\]

The cocycle relation

\[
\chi(A) \cdot B + \chi(B) = \chi(A \circ B) = \chi(B \circ A) = \chi(B) \cdot A + \chi(A)
\]

is a consequence of the fact that \( A \) and \( B \) commute. It implies that \( \chi(B) \in \text{kernel}[A-I] \). Thus there exists a \( l(\chi) \in \mathbb{C} \) such that

\[
\chi(B) = l(\chi) h_{A}^{1-q}.
\]

**Proposition.** (a) The rule

\[
\chi \mapsto l(\chi) = l_{A, B}(\chi)
\]

defines an element of the dual space \( PH^{1}(\Gamma, \Pi_{2q-2})^{*} \) of \( PH^{1}(\Gamma, \Pi_{2q-2}) \).

(b) For all \( C \in \Gamma \),

\[
l_{C_{\alpha} A_{\alpha} C_{-1}, C_{B_{\alpha} C_{-1}}} = l_{A, B}.
\]

**Proof.** Let \( \chi_{1} \) be an arbitrary cocycle with \( \chi_{1}(A) = 0 \) and \( \chi_{1} \) cohomologous to \( \chi \). Then there exists a \( v \in \Pi_{2q-2} \) such that

\[
\chi_{1}(\gamma) = \chi(\gamma) - v \cdot \gamma + v, \quad \gamma \in \Gamma.
\]

It follows that \( v \in \text{kernel}[A-I] = \text{kernel}[B-I] \). Thus \( \chi_{1}(B) = \chi(B) \) and \( l \) is well defined.

2.2. For arbitrary \( C \in \text{PSL}(2, \mathbb{C}) \), the linear map

\[
\wedge : PH^{1}(\Gamma, \Pi_{2q-2}) \to PH^{1}(C \Gamma C^{-1}, \Pi_{2q-2})
\]

---

There are a number of misprints in [K2]. For example, the formula for \( \text{Image}(E-1) \) in §8.3 should read \( \text{Image}(E-1) = \{ p \in \Pi_{2q-2} ; p(a) = 0 \} \).
defined on the cocycle level by
\[ \tilde{\chi}(C \circ \gamma \circ C^{-1}) = \chi(\gamma) \cdot C^{-1}, \quad \gamma \in \Gamma, \]
is a (surjective) isomorphism. For \( \chi \in PZ^1(\Gamma, \Pi_{2q-2}) \) with
\[ \chi(A) = 0 \quad \text{and} \quad \chi(B) = l(\chi)h_A^{1-q}, \]
we have
\[ \tilde{\chi}(C \circ A \circ C^{-1}) = 0 \quad \text{and} \quad \tilde{\chi}(C \circ B \circ C^{-1}) = l(\chi)h_{C_0C_0C}^{1-q}; \]
which shows that
\[ (2.2.1) \quad l_{A,B}(\chi) = l_{C_0C_0C_0C_0C}^{1-q}(\tilde{\chi}). \]

2.3. If \( C \in \Gamma \), then for every cocycle \( \chi \) for \( \Gamma \),
\[ \chi(C \circ \gamma \circ C^{-1}) = \chi(\gamma) \cdot C^{-1} + \chi(C) \cdot C^{-1} \cdot [C \circ \gamma \circ C^{-1} - I] \]
for all \( \gamma \in \Gamma \). Thus \( \chi \) is cohomologous to \( \tilde{\chi} \) and we conclude that
\[ l_{C_0C_0C_0C_0C_0C_0C}^{1-q}(\chi) = l_{C_0C_0C_0C_0C}^{1-q}(\tilde{\chi}) = l_{A,B}(\chi). \]
We have completed the proof of Proposition 2.1.

2.4. Let \((A_1, B_1)\) be a second ordered pair of generators for the group \( G \).
Then, as is well known, there exist integers \( a, b, c, d \) so that
\[ A_1 = A^a \circ B^b, \quad B_1 = A^c \circ B^d \quad \text{and} \quad ad - bc = \varepsilon = \pm 1. \]
We are interested in comparing \( l_{A_1,B_1} \) with \( l_{A,B} \). As a consequence of (2.2.1),
we may assume that \( A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \) and \( B = \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} \) with \( \text{Im} \, \tau \neq 0 \). Thus
\[ A_1 = \begin{bmatrix} 1 & a + b\tau \\ 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & c + d\tau \\ 0 & 1 \end{bmatrix}. \]
Let us choose \( \chi \in PZ^1(\Gamma, \Pi_{2q-2}) \) with
\[ \chi(A) = 0 \quad \text{and} \quad \chi(B) = l(\chi). \]
Then
\[ \chi(A_1) = \chi(B^b) = \chi(B) \cdot [B^{b-1} + \cdots + I] = bl(\chi), \quad \chi(B_1) = dl(\chi). \]
We must replace the cocycle \( \chi \) by a cohomologous cocycle that vanishes at \( A_1 \).
Let
\[ \nu(z) = \frac{bl(\chi)}{a + b\tau} z, \quad z \in \mathbb{C}. \]
The cocycle
\[ \chi_1(\gamma) = \chi(\gamma) - \nu \cdot \gamma + \nu, \quad \gamma \in \Gamma, \]
vanishes at \( A_1 \) and has value
\[ \chi_1(B_1) = \frac{\varepsilon}{a + b\tau} l(\chi). \]
It follows that for all \( \chi \in PH^1(\Gamma, \Pi_{2q-2}) \),
\[
l_{A_1, B_1}(\chi) = \frac{\epsilon}{(a + b \tau)^q} l_{A, B}(\chi).
\]

**Corollary.** The projective class of the linear functional \( l_{A, B} \in PH^1(\Gamma, \Pi_{2q-2})^* \) depends only on the conjugacy class of \( G \) in \( \Gamma \).

2.5. Let \( \Delta \) denote the unit disk, and
\[
\Phi : \Delta \to \text{Hom}^\text{par}(\Gamma, PSL(2, \mathbb{C}))
\]
be a normalized holomorphic family of parabolic homomorphisms on \( \Delta \); this means that for each \( t \in \Delta \), \( \Phi(t) \) is a homomorphism of the Kleinian group \( \Gamma \) into \( PSL(2, \mathbb{C}) \),
\[
\gamma(t) = \Phi(t)(\gamma)
\]
defines a holomorphic map from \( \Delta \) to \( PSL(2, \mathbb{C}) \) for all \( \gamma \in \Gamma \) and is parabolic or the identity whenever \( \gamma \in \Gamma \) is parabolic, and \( \Phi(0) \) is the identity isomorphism. This normalized family \( \Phi \) defines a cocycle \( \chi \) for \( q = 2 \) by
\[
\chi(\gamma)(z) = \frac{\gamma(z)}{\gamma'(z)}, \quad z \in \mathbb{C},
\]
where
\[
\frac{\gamma(z)}{\gamma'(z)} = \frac{\partial}{\partial t} \gamma(t)(z) |_{t=0}, \quad \gamma \in \Gamma, \ z \in \mathbb{C}.
\]
For details, see [GK]. If there exists a holomorphic function \( C \) from a non-empty neighborhood \( \{ t \in \mathbb{C}; |t| < \varepsilon \} \) of \( 0 \) in \( \mathbb{C} \) into \( PSL(2, \mathbb{C}) \) with \( C(0) = I \) and
\[
\gamma(t) = C(t) \circ \gamma \circ C(t)^{-1}, \quad \text{all } \gamma \in \Gamma, \text{ all } |t| < \varepsilon,
\]
then \( \chi \) is a coboundary.

If \( A \) and \( B \) generate a rank 2 parabolic subgroup of \( \Gamma \), then for sufficiently small \( |t| \), we can define a holomorphic function by
\[
\tau(t) = \frac{\tau(B(t))}{\tau(A(t))}.
\]

**Proposition.** We have
\[
\tau'(0) = l_{A, B}(\chi).
\]

**Proof.** By conjugation, we may assume that \( A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \), \( B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) with \( \text{Im} \, \tau_0 \neq 0 \). Write
\[
A(t) = \begin{bmatrix} 1 + a(t) & b(t) \\ -a^2(t) & 1 - a(t) \end{bmatrix}
\]
with \( a \) and \( b \) holomorphic functions with \( a(0) = 0 \), \( b(0) = 1 \). It follows that
\[
\alpha(A(t)) = \frac{b(t)}{a(t)}.
\]
Since \( \alpha(B(t)) = \alpha(A(t)) \), it also follows that
\[
B(t) = \begin{bmatrix} 1 + \beta(t) & \beta(t) \\ -a^2(t) \beta(t) & 1 - \beta(t) \end{bmatrix}
\]
\[
\begin{bmatrix} a(t) \\ b(t) \end{bmatrix} = \begin{bmatrix} 1 & \frac{\alpha(t)}{\beta(t)} \\ -\frac{a^2(t) \beta(t)}{b(t)} & 1 - \frac{\alpha(t)}{b(t)} \end{bmatrix} \begin{bmatrix} 1 + \beta(t) & \beta(t) \\ -a^2(t) \beta(t) & 1 - \beta(t) \end{bmatrix} \begin{bmatrix} a(t) \\ b(t) \end{bmatrix}.
\]
with $\beta$ some holomorphic function satisfying $\beta(0) = \tau_0$. Let

$$C(t) = \begin{bmatrix} b(t)^{-1/2} & 0 \\ a(t)b(t)^{-1/2} & b(t)^{1/2} \end{bmatrix}.$$

Then

$$C(t) \circ A(t) \circ C(t)^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and

$$C(t) \circ B(t) \circ C(t)^{-1} = \begin{bmatrix} 1 & \beta(t)/b(t) \\ 0 & 1 \end{bmatrix}.$$

Since conjugation preserves the trace and does not change the cohomology class, it suffices to assume that

$$A(t) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B(t) = \begin{bmatrix} 1 & \tau(t) \\ 0 & 1 \end{bmatrix}$$

with $\tau$ a holomorphic function in some neighborhood of 0 with $\tau(0) = \tau_0$. It follows that $\chi(A) = 0$ and $\chi(B) = \tau'(0) = l_{A,B}(\chi)$.

3. Cusp forms associated to rank 2 parabolic subgroups of Kleinian groups

3.1. Let $A$ and $B$ generate a rank 2 parabolic subgroup $G$ of the Kleinian group $\Gamma$. Then

$$\varphi_{A,B}(z) = \sum_{\gamma \in G/\Gamma} h_A(\gamma z)^q \gamma'(z)^q, \quad z \in \Omega,$$

defines a cusp form for $\Gamma$ (that is, an element of $A_q(\Omega, \Gamma)$; here $\Omega$ is the region of discontinuity of $\Gamma$).

3.2. For arbitrary $C \in \Gamma$, we have

$$\varphi_{CoAc^{-1},CoBc^{-1}} = \varphi_{A,B}$$
as a consequence of (1.2.3).

3.3. The cusp form $\varphi_{A,B}$ is related to a holomorphic Eichler integral. Assume that $\Gamma$ is a finitely generated Kleinian group. Then

$$\psi_{A,B}(z) = \sum_{\gamma \in \Gamma/G} \frac{\gamma'(\alpha)^q}{\gamma(\alpha) - z}, \quad z \in \Omega,$$

(here $\alpha = \alpha(A)$) defines a holomorphic Eichler integral (that is, an element of $E^c_{1-q}(\Omega, \Gamma)$), with

$$\frac{d^{2q-1}}{dz^{2q-1}} \psi_{A,B}(z) = (2q - 1)! \sum_{\gamma \in \Gamma/G} \frac{\gamma'(\alpha)^q}{(\gamma(\alpha) - z)^{2q}}$$

$$= (2q - 1)! \sum_{\gamma \in \Omega/\Gamma} \frac{\gamma'(z)^q}{(\gamma(z) - \alpha)^{2q}}$$

$$= (-1)^q \tau(A)^q (2q - 1)! \varphi_{A,B}.$$
The Eichler integral $\psi$ is not parabolic with respect to $A$. It satisfies
\[(pd\psi)(A)(\alpha) = (q - \frac{1}{2})A''(\alpha) = (1 - 2q)\tau(A),\]
\[(pd\psi)(B)(\alpha) = (1 - 2q)\tau(B).\]

The above formulae assume that $\langle A, B \rangle$ is the stabilizer of $\alpha$ in $\Gamma$. Adjustments need to be made when $\langle A, B \rangle$ is a proper subgroup of the stabilizer. The cohomology class $pd\psi$ is parabolic with respect to every parabolic element that fixes a point not $\Gamma$-equivalent to $\alpha$. For details, see [K1].

3.4. As a consequence of §3.3, we see that $\psi_{A, B} \neq 0$ provided $\alpha$ is accessible (defined in [K1, §2]) and $\langle A, B \rangle$ is the stabilizer of $\alpha$ in $\Gamma$. As a matter of fact, under this hypothesis, whenever $A$ (or $B$) fixes a component $\Delta$ of $\Omega$, then $\varphi_{A, B}\Delta \neq 0$. More generally, let $\alpha$ be an accessible limit point of $\Gamma$ whose stabilizer $\Gamma(\alpha)$ contains a maximal rank 2 parabolic subgroup $P(\alpha)$ generated by $A$ and $B$ as above. Then (see [K1, §0]) $\Gamma(\alpha)/P(\alpha)$ is a cyclic group of order $n = 1, 2, 3, 4$ or 6. The fixed point $\alpha$ is $q$-admissible if $q \equiv 0$ (mod $n$). We showed in [K1, §5] that $\varphi_{A, B} \neq 0$ if and only if $\alpha$ is $q$-admissible.

4. Linear functionals on cusp forms associated to rank 2 parabolic subgroups

4.1. Let $\Gamma$ be a nonelementary Kleinian group. The Bers map
\[\beta^* : A_\Omega(\Omega, \Gamma) \to PH^1(\Gamma, \Pi_{2q-2})\]
is discussed in [K2, §0.2]. For $\psi \in A_\Omega(\Omega, \Gamma)$, we let $F$ be a potential for $\mu = \lambda^{2-2q}\psi\overline{\psi}$ (here $\lambda$ is the Poincaré metric on the region of discontinuity $\Omega$ of $\Gamma$) and $\beta^*(\psi) = pd F$. We need to construct specific potentials for $\mu$.

4.2. Let $A$ be a parabolic element of $\Gamma$. Let $F$ be a potential for $\mu$ that vanishes at $A^j(a), j = 0, 1, \ldots, 2q - 2$, where $a \in \hat{\mathbb{C}}$ is arbitrary with $a \neq a(A)$. We claim that $(pd F)(A) = 0$. The polynomial $(pd F)(A) = F \cdot A - F$ automatically vanishes at $\alpha$ and at $A^j(a), j = 0, \ldots, 2q - 3$. It is thus identically zero. Thus we see that
\[(4.2.1) F(A(z))A^j(z)^{1-q} - F(z) = 0, \text{ all } z \in \hat{\mathbb{C}}.\]

It follows that $F(\alpha) = 0$ since by induction $F(A^j(a)) = 0$ for all $j \in \mathbb{Z}$ and $\lim_{j \to \infty} A^j(a) = \alpha$ (the potential $F$ is a continuous function).

As a consequence, the potential $F_1$ for $\mu$ that vanishes at $\alpha$ and at $A^j(a), j = 0, \ldots, 2q - 3$, agrees with $F$. We have constructed two potentials
\[F(z) = \frac{p(z)}{2\pi i} \int_{\Omega} \frac{\mu(\zeta)}{(\zeta - z)p(\zeta)} \, d\zeta \wedge d\overline{\zeta}, \text{ all } z \in \hat{\mathbb{C}},\]
that satisfy (4.2.1) by choosing
\[p(z) = \prod_{j=0}^{2q-2} (z - A^j(a)) \text{ or } p(z) = (z - \alpha) \prod_{j=0}^{2q-3} (z - A^j(a)).\]
4.3. We can now formulate the following

**Theorem.** Let $A$ and $B$ generate a rank 2 parabolic subgroup of the Kleinian group $\Gamma$. Then for all $\psi \in A_q(\Omega, \Gamma)$,

$$l_{A,B} (\beta^*(\psi)) = -i(\varphi_{A,B}, \psi).$$

**Proof.** Let $a \in \hat{C}$, $a \neq \alpha = \alpha(A)$. Let $F$ be the potential for $\mu = \lambda^{2-2q} \overline{\psi}$ that vanishes at $A^j(a)$, $j = 0, \ldots, 2q - 2$. Then, as observed above, for $\chi = \text{pd} F$, $l = l_{A,B}$, we have $\chi(A) = 0$, and hence

$$\chi(B)(z) = l(\chi)(-\tau(A))^{q-1}(z - \alpha)^{2q-2}, \quad z \in \hat{C}.$$ 

Let $\omega$ be a standard fundamental domain for $\langle A, B \rangle$ such that $\partial \omega$ consists of oriented sides $c_1$, $c_2$, $-A(c_1)$, $-B(c_2)$ and the winding number of $\partial \omega$ around $\alpha$ is $-1$. We compute

$$\left(\frac{-1}{\tau(A)} \right)^q \int_{\partial \omega} \frac{F(z)dz}{(z - \alpha)^{2q}} = \left(\frac{-1}{\tau(A)} \right)^q \left[ \int_{c_1} - \int_{A(c_1)} + \int_{c_2} - \int_{B(c_2)} \right] \frac{F(z)dz}{(z - \alpha)^{2q}}. $$

Now

$$\int_{c_1} \frac{F(z)dz}{(z - \alpha)^{2q}} = \int_{c_1} \frac{F(Az)A'(z)^{1-q}dz}{(z - \alpha)^{2q}} = \int_{A(c_1)} \frac{F(z)dz}{(z - \alpha)^{2q}},$$

and

$$\int_{c_2} \frac{F(z)dz}{(z - \alpha)^{2q}} = \int_{c_2} \frac{F(Bz)B'(z)^{1-q} - \chi(B)(z)dz}{(z - \alpha)^{2q}} = \int_{B(c_2)} \frac{F(z)dz}{(z - \alpha)^{2q}} - l(\chi)(-\tau(A))^{q-1} \int_{c_1} \frac{dz}{(z - \alpha)^2}.$$ 

We conclude that

$$\left(\frac{-1}{\tau(A)} \right)^q \int_{\partial \omega} \frac{F(z)dz}{(z - \alpha)^{2q}} = l(\chi).$$

On the other hand ($G = \langle A, B \rangle$)

$$\left(\frac{-1}{\tau(A)} \right)^q \int_{\partial \omega} \frac{F(z)dz}{(z - \alpha)^{2q}} = - \left(\frac{-1}{\tau(A)} \right)^q \int_{\omega} \frac{\partial F(z)}{\partial \overline{z}} \left[ \frac{F(z)}{(z - \alpha)^{2q}} \right] dz \wedge d\overline{z}$$

$$= - \left(\frac{-1}{\tau(A)} \right)^q \int_{\partial \Omega/G} \frac{\lambda^{2-2q}(z)\psi(z)}{(z - \alpha)^{2q}} dz \wedge d\overline{z}$$

$$= - \int_{\partial \Omega/\Gamma} \varphi_{A,B}(z)\lambda^{2-2q}(z)\overline{\psi(z)} dz \wedge d\overline{z}$$

$$= i(\varphi_{A,B}, \psi).$$

For a justification of the use of Stokes’ theorem, see [K3, §2.4].
4.4. Let us assume that $\Gamma$ is finitely generated and use the notation of §3.4. We have obtained the

**Corollary.** The natural restrictions

$$PH^1(\Gamma, \Pi_{2q-2}) \to PH^1(\Gamma(\alpha), \Pi_{2q-2})$$

and

$$H^1(\Gamma, \Pi_{2q-2}) \to H^1(\Gamma(\alpha), \Pi_{2q-2})$$

are surjective provided $\alpha$ is accessible.

We note that we can replace $\Gamma(\alpha)$ by $P(\alpha)$ if and only if $\alpha$ is $q$-admissible. For $\alpha$ not $q$-admissible, every cohomology class for $\Gamma$ restricts to a coboundary for $\Gamma(\alpha)$ (and thus also for $P(\alpha)$). Example D of [K1, §7] shows that the assumption that $\alpha$ be accessible cannot be dropped.

5. **Some Formulae for Rank 2 Parabolic Subgroups**

5.1. Let $G$ be a subgroup of $\Gamma$. We denote by $\Theta_{G,\Gamma}$ the relative Poincaré series operator mapping $q$-forms for $G$ to $q$-forms for $\Gamma$. We abbreviate $\Theta_{G,\Gamma}$ by $\Theta_G$ and observe that

$$\Theta_{\Gamma} = \Theta_{G,\Gamma} \Theta_{\Gamma}.$$

5.2. We consider the special case of the rank 2 parabolic group $G$ generated by $A(z) = z+1$, $B(z) = z+\tau$, $z \in \mathbb{C}$, $\text{Im} \, \tau \neq 0$.

For $k \in \mathbb{Z}^+$, we define

$$g_\tau(z) = \frac{1}{2\pi i} \frac{\tau(\tau-1)\cdots(\tau-k)}{(z-\tau)(z-1)\cdots(z-k)}, \quad z \in \mathbb{C},$$

and

$$\Theta g_\tau(z) = \sum_{\gamma \in G} g_\tau(\gamma(z)), \quad z \in \mathbb{C}.$$

We observe that $\Theta g_\tau$ is a Poincaré series for the group $G$ with $q = \frac{1}{2}(k+2) \geq 2$ for $k$ even and $q = \frac{1}{2}(k+3) \geq 2$ for $k$ odd. Thus we conclude quite easily that

$$\Theta g_\tau = (\Theta_G)(g_\tau)$$

converges absolutely and uniformly on compact subsets of $\mathbb{C}$ and defines a doubly periodic function

$$\Theta g_\tau(z+1) = \Theta g_\tau(z) = \Theta g_\tau(z+\tau), \quad \forall z \in \mathbb{C},$$

for $G$. It is most convenient to regard $\Theta g_\tau$ as an automorphic form (for appropriate $q$, as above). Since $\Theta g_\tau$ has at most simple poles at the lattice points, it must be a constant $c(\tau)$ that depends, a priori, on $\tau$ and $k$.

**Theorem.** For all $\tau \in \mathbb{C}$ with $\text{Im} \, \tau \neq 0$, $c(\tau) = 1$. 

Proof. There exist groups $\Gamma$ (see [Kl, §7]) with $G \subset \Gamma$ and $\dim A_q(\Omega, \Gamma) > 0$. Let us assume that $k$ is even, $k = 2q - 2$ for some integer $q \geq 2$. We will obtain an alternate formula for $l_{A,B}(\beta^* \psi)$ with $\psi \in A_q(\Omega, \Gamma)$. Let $F$ be the potential for $\lambda^{2q} \psi$ that vanishes at $A_j(0) = j$, $j = 0, 1, \ldots, 2q - 2$. Then 

\[
F(A(z)) - F(z) = 0, \quad \text{all } z \in \mathbb{C},
\]

and 

\[
F(B(z)) - F(z) = l(\chi), \quad \text{all } z \in \mathbb{C}.
\]

It follows that 

\[
F(\tau) = l(\chi).
\]

But 

\[
F(\tau) = \frac{\tau(\tau - 1) \cdots (\tau - 2q + 2)}{2\pi i} \int_{\Omega} \lambda^{2q}(\zeta) \psi(\zeta) \frac{d\zeta \wedge d\overline{\zeta}}{\zeta(\zeta - 1) \cdots (\zeta - 2q + 2)(\zeta - \tau)}
\]

Comparison with (4.3.1) yields the theorem for even $k$. For $k$ odd, we use the potential $F$ that vanishes at $\infty$ and at $0, 1, \ldots, 2q - 3$.

Corollary. We have the equality 

\[
\Theta g_r = h_A^q.
\]

In the next section we generalize the above corollary to motions $A$ other than $[1 1 0]$. 

Remark. Rank 2 parabolic groups figure prominently in the fundamental work of Earle and Marden [EM] on compactifications for moduli spaces. The above theorem for $q = 2$ is connected to quasiconformal mappings and is clearly encountered in [EM]. It also appears in Nag’s work on Teichmüller theory [N]. Nag’s methods (limited to $q = 2$) rely on the study of deformations of complex structures on tori.

5.3. Let $A$ and $B$ generate an arbitrary rank 2 parabolic group $G$. For $a \in \hat{G}$, $a \neq \alpha = \alpha(A) = \alpha(B)$, with $B(a) \in \mathbb{C}$, define 

\[
g_{A,B,a}(z) = \frac{1}{2\pi i} \frac{1}{z - B(a)} \prod_{j=0}^{2q-2} \frac{B(a) - A^j(a)}{z - A^j(a)}, \quad z \in \hat{G}.
\]

A straightforward calculation using (1.2.1) shows that for $C \in \text{PSL}(2, \mathbb{C})$ 

\[
g_{C_{oA}C^{-1}, C_{oBO}C^{-1}, C(a)}(C(z))C'(z)^q = C'(B(a))^{q-1} g_{A,B,a}(z), \quad \text{all } z \in \hat{G},
\]

or (equivalently) 

\[
C_q^*(g_{C_{oA}C^{-1}, C_{oBO}C^{-1}, C(a)}) = C'(B(a))^{q-1} g_{A,B,a}.
\]

For $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix}$ with $\text{Im } \tau \neq 0$, $a = 0$, we have 

\[
g_{A,B,a} = g_{\tau}.
\]
Thus choosing \( C \in PSL(2, \mathbb{C}) \) with
\[
C(\alpha) = \infty, \quad C(\alpha) = 0, \quad C(A(\alpha)) = 1,
\]
we see that
\[
C \circ A \circ C^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad C \circ B \circ C^{-1} = \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix},
\]
for some \( \tau \in \mathbb{C} \) with \( \text{Im} \ \tau \neq 0 \). Hence
\[
C_q^*(g_{C(B(\alpha))}) = C'(B(\alpha))^{q-1} g_{A,B,a}.
\]
Assume now that \( \alpha \in \mathbb{C} \). Then the above formula yields
\[
(5.3.1) \quad C_q^*\left(g_{\tau(B)/\tau(A)}\right) = \left(\frac{1 + \tau(B)(a - \alpha)^2}{-\tau(A)(a - \alpha)^2}\right)^{q-1} g_{A,B,a}.
\]
For \( \alpha = \infty \), the corresponding formula is
\[
(5.3.2) \quad C_q^*\left(g_{\tau(B)/\tau(A)}\right) = \frac{1}{\tau(A)^{q-1}} g_{A,B,a}.
\]
Using the well-known relation
\[
(5.3.3) \quad \Theta_G \circ C_q^* = C_q^* \circ \Theta_{C_G^{-1}},
\]
we obtain the following

**Theorem.** Let \( A \) and \( B \) generate a rank 2 parabolic group \( G \) with fixed point \( \alpha \). Then
\[
\Theta_G(g_{A,B,a}) = \left(\frac{1 + \tau(B)(a - \alpha)^2}{-\tau(A)(a - \alpha)^2}\right)^{1-q} h_A^q \quad \text{for } \alpha \in \mathbb{C},
\]
\[
= \tau(A)^{q-1} h_A^q \quad \text{for } \alpha = \infty.
\]
**Proof.** We apply the Poincaré operator \( \Theta_G \) to both sides of (5.3.1), use (5.3.3) and the Corollary to Theorem 5.2 along with (1.2.3).

**Remark.** Similar formulae hold for the function
\[
\frac{1}{2\pi i} \frac{1}{z - B(a)} \frac{B(a) - \alpha}{z - \alpha} \sum_{j=0}^{2q-3} \frac{B(a) - A^j(a)}{z - A^j(a)}.
\]

**References**


**Department of Mathematics, State University of New York, Stony Brook, New York 11794-3651**