CE-EQUIVALENCE, $UV^k$-EQUIVALENCE
AND DIMENSION OF COMPACTA

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Abstract. It is shown that for each $k > 0$ there exists a finite-dimensional continuum $X$ which is not $UV^k$-equivalent, and therefore not CE-equivalent, to any continuum $Y$ such that the dimension of $Y$ is equal to the shape dimension of $X$.

A map of compacta is cell-like (CE) if all point-inverses have trivial shape. On the class $CM_f$ of finite-dimensional compacta, the CE maps generate an equivalence relation known as CE-equivalence. To be precise, $X, Y \in CM_f$ are said to be CE-equivalent if there exist spaces $X_1 = X, X_2, \ldots, X_{2n}, X_{2n+1} = Y$ in $CM_f$ and CE maps $X_{2i} \to X_{2i+1}$, $i = 1, \ldots, n$. CE-equivalence implies shape equivalence, but the converse fails to be true (see S. Ferry [2] or our paper [5]). This leaves the general problem to decide whether a given invariant of CE-equivalence is a shape invariant or not. This paper is concerned with a very natural dimension invariant. For $X \in CM_f$, the CE-dimension of $X$ is defined as the number

$$CE\text{-dim } X = \min\{\dim Y | Y \text{ is CE-equivalent to } X\}.$$ 

Obviously, $CE\text{-dim } X \geq Sd X$ (shape dimension of $X$; see e.g. [4]). Using our notation, we can restate a question which appears in "A list of open problems in shape theory" by J. Dydak and J. Segal:

Is it true that $CE\text{-dim } X = Sd X$ for each $X \in CM_f$?

In other words, is $CE\text{-dim } X$ a shape invariant?

We shall show that the answer is "no." For this purpose it will be convenient to work with the weaker concept of $UV^k$-equivalence, that is, the equivalence relation on the class of metrizable spaces generated by the proper $UV^k$-maps (cf. [3, 5]). Recall that a map is $UV^k$ if all point-inverses are $UV^k$, that is, have vanishing homotopy pro-groups up to dimension $k$. The definition of the $UV^k$-dimension of a metrizable space $X$, abbreviated by $UV^k\text{-dim } X$, can safely be left to the reader. Since CE-equivalence implies $UV^k$-equivalence, we have $CE\text{-dim } X \geq UV^k\text{-dim } X$ for each $X \in CM_f$. 

Received by the editors July 7, 1989.
1980 Mathematics Subject Classification (1985 Revision). Primary 55P55.
Theorem. Let $X_0$ be a continuum (= nonempty connected compactum) such that pro-$\pi_1(X_0)$ is not pro-finite. For each $k > 0$, there exists a compactum $X_k$ with the following properties:

1. $X_k$ and $X_0$ are shape equivalent;
2. $\dim X_k = \max(\text{Sd} X_0, k + 2)$;
3. $\text{UV}^k$-$\dim X_k = k + 1$.

This shows in particular that the difference $\text{CE}$-$\dim X - \text{Sd} X$ can be arbitrarily large even within a fixed shape equivalence class of finite-dimensional continua.

The proof of the theorem is based on the theory of $\text{UV}^k$-components developed in [5]. Clearly, it is no restriction to assume that $\dim X_0 = \text{Sd} X_0$. Let $P_k$ denote the polyhedron obtained by attaching the $(k + 1)$-cell $D^{k+1}$ to the $k$-sphere $S^k = \partial D^{k+1}$ by a map $f : S^k \to S^k$ of degree 2. By [5, Theorem 6.1], there exists a compactum $X_k \supset X_0$ such that the remainder $X' = X_k \setminus X_0$ is a $\text{UV}^1$-component of $X_k$ homeomorphic to $P_k \times (0, \infty)$. By construction, $X_k$ satisfies (1) and (2) (cf. [5, Proposition 4.1]). Since $X'$ is path-connected, $X'$ clearly is a $\text{UV}^k$-component of $X_k$. Now assume that $\text{UV}^k$-$\dim X_k \leq k$. Then there is a compactum $Y$, $\dim Y \leq k$, such that $X_k$ and $Y$ are $\text{UV}^k$-equivalent. By [5, Theorem 2.15] there exists a $\text{UV}^k$-component $Y'$ of $Y$ such that $X'$ and $Y'$ are $\text{UV}^k$-equivalent. From [5, Theorem 1.4] we infer that there are basepoints $x \in X'$ and $y \in Y'$ such that pro-$\pi_i(X', x) \approx$ pro-$\pi_i(Y', y)$ for $i = 0, \ldots, k$. This yields pro-$\pi_i(Y', y) \approx 0$ for $i = 0, \ldots, k - 1$ and pro-$\pi_k(Y', y) \approx \mathbb{Z}_2$. The Hurewicz isomorphism theorem in shape theory (see e.g. [4, Chapter II §4.1, Theorem 1]) implies then pro-$H_k(Y') \approx \mathbb{Z}_2$. On the other hand, $\text{Sd} Y' \leq \dim Y' \leq \dim Y \leq k$, that is, $Y'$ has an HPol-expansion $Y' \to \{Y_o\}$ where all $Y_o$ are polyhedra of dimension $\leq k$. Therefore pro-$H_k(Y')$ can be represented by an inverse system of free abelian groups, which cannot be isomorphic to $\mathbb{Z}_2$. This contradiction proves $\text{UV}^k$-$\dim X_k \geq k + 1$. The equation (3) follows now from Ferry’s observation that $\text{UV}^k$-$\dim X \leq k + 1$ for any compactum $X$ (see [3, Proposition 1.10]).

Remarks. (1) Let $C_{\text{prof}}$ denote the class of continua $X$ such that pro-$\pi_1(X)$ is pro-finite. Our theorem leaves the question whether $\text{CE}$-$\dim X = \text{Sd} X$ for each finite-dimensional $X \in C_{\text{prof}}$. However, we have a complete picture regarding $\text{UV}^k$-$\dim X$ for $X \in C_{\text{prof}}$. On $C_{\text{prof}}$, shape equivalence implies $\text{UV}^k$-equivalence (see [3, Theorem 2]); hence $\text{UV}^k$-$\dim X \leq \text{Sd} X$. Moreover, for compacta of dimension $\leq k$, $\text{UV}^k$-equivalence implies shape equivalence (see [3, Theorem 1]). We infer that $\text{UV}^k$-$\dim X = \text{Sd} X$ whenever $\text{Sd} X \leq k$: Choose a compactum $Y$ shape equivalent to $X$ with $\dim Y = \text{Sd} X$ and a compactum $Z$ $\text{UV}^k$-equivalent to $X$ with $\dim Z = \text{UV}^k$-$\dim X$; then $Y$
and $Z$ must be $UV^k$-equivalent of dimension $\leq k$, whence $\text{Sd} X = \text{Sd} Z \leq \dim Z = UV^k$-dim $X$. On the other hand, if $\text{Sd} X > k$, $UV^k$-dim $X$ can take any of the values $0, 2, 3, \ldots, k + 1$: Let $r > k$ and define $Y_0 = S^r$, $Y_j = S^r \vee S^j$ for $j = 2, \ldots, k$, $Y_{k+1} = S^r \vee P_k$ ($\vee$ denotes one-point union and $P_k$ is taken from the above proof). Then all $Y_i \in \text{C}_{\text{prof}}$, $\text{Sd} Y_i = r$, but $UV^k$-dim $Y_i = i$. Note that neither $UV^k$-dim $X = 1$ nor $\text{Sd} X = 1$ is possible when $X \in \text{C}_{\text{prof}}$.

(2) The counterexamples given in our theorem are not locally connected. However, for each $k > 0$ R. Daverman and G. Venema have constructed an LC$^{k-1}$ continuum $X_k$ of dimension $k + 1$ which is shape equivalent to $S^1$ but not CE-equivalent to $S^1$ (see [1]). Such an $X_k$ has $\text{Sd} X_k = 1$ but is not CE-equivalent to any locally connected one-dimensional compactum (since shape equivalence and CE-equivalence are the same on locally connected one-dimensional compacta, see [1]).

REFERENCES


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