A NOTE ON COBORDISM OF SURFACE LINKS IN $S^4$

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Abstract. Sato's idea of the asymmetric linking number is used in cyclic branched coverings to give an invariant of the cobordism of embedded surfaces in the 4-sphere.

In this article we consider the link cobordism of surface links in $S^4$. We work in the smooth category.

Let $L = J \cup K$ be a link in $S^4$, where $J$, $K$ are embedded, oriented connected surfaces. $L$ is called semi-boundary [S] if each component bounds an embedded, orientable 3-manifold in $S^4$ which misses the other component. Sato [S] defined the asymmetric linking number, denoted by $\text{alk}(J, K)$, to be the nonnegative generator of the image of $H_1(K : \mathbb{Z}) \to H_1(S^4 \setminus J : \mathbb{Z}) \cong \mathbb{Z}$. He proved that a link is semi-boundary iff

$$\text{alk}(J, K) = 0 = \text{alk}(K, J),$$

and being semi-boundary is preserved under link cobordism. We call two surface links $L_0 = J_0 \cup K_0$ and $L_1 = J_1 \cup K_1$ cobordant if there are disjointly embedded, orientable 3-manifolds $C$ and $E$ in $S^4 \times I$ such that $\partial C = J_0 \cup (-J_1)$, $\partial E = K_0 \cup (-K_1)$, and $C$, $E$ are homeomorphic to $J_0 \times I$, $K_0 \times I$, respectively, where we regard $L_i$ as lying in $S^4 \times \{i\}$. A link is called null-cobordant if it is cobordant to the standardly embedded surfaces (which bound disjoint handlebodies) in $S^4$. Thus alk can be regarded as the first obstruction to links being null-cobordant, and we focus on semi-boundary links from now on. The Sato-Levine invariant was defined [S] for semi-boundary links, and Cochran [C] defined the derived series of this invariant. In this paper we observe that the covering asymmetric linking number can be used as a link cobordism invariant and give examples of links with vanishing Sato-Levine invariant and trivial derivatives which belong to different cobordism classes.

Let $L = J \cup K$ be a 2-component, oriented semi-boundary link. Consider the $n$-fold cyclic branched covering $M$ of $S^4$ along $J$, where $n = p^r$ is a prime power. Then there are $n$ lifts $k_0, \ldots, k_{n-1}$ of $K$ to $M$ since $L$ is
semi-boundary. We can assume that $k_{i+1} = \tau k_j$ where $\tau$ is the generator of covering translations. We regard $n$ as 0 and $n + 1$ as 1, so that the indices of lifts are regarded as lying in $\mathbb{Z}_n$.

**Lemma 1.** $H_1(M; k_j : \mathbb{Q})$ is isomorphic to either 0 or $\mathbb{Q}$. Furthermore, this is a link cobordism invariant. More precisely, let $L_i = J_i \cup K_i$ ($i = 0, 1$) be cobordant links and $M_i, k^i_j$ be their cyclic branched coverings and the lifts of $K_i$ respectively ($i = 0, 1, j = 0, \ldots, n - 1$). Then

$$H_1(M_0; k^0_j : \mathbb{Q}) \cong H_1(M_1; k^1_j : \mathbb{Q}).$$

In particular, this is $\mathbb{Q}$ if $L$ is null-cobordant.

The proof is given later. If $H_1(M; k_j : \mathbb{Q}) = 0$, then $L$ is not null-cobordant. Thus we focus on links with $H_1(M; k_j : \mathbb{Q}) = \mathbb{Q}$. Consider the following homomorphism

$$H_1(k_j; \mathbb{Q}) \rightarrow H_1(M; k_0 : \mathbb{Q}) \cong \mathbb{Q}.$$

**Definition 2.** Define $\xi^j_n = 0$ if this homomorphism is zero, $\xi^j_n = 1$ otherwise ($j = 1, \ldots, n - 1$).

**Theorem 3.** $\xi^j_n \in \mathbb{Z}_2, j \in \mathbb{Z}_n \setminus \{0\}$, are link cobordism invariants.

**Proof of Lemma 1.** Let $X_n$ be the $n$-fold cyclic (unbranched) covering ($X_{\infty}$ denotes the infinite cyclic covering) of $X = S^3 \setminus N(J)$. Then we have an exact sequence ([S-S]) with integral coefficient

$$\cdots \rightarrow H_q(X_{\infty}) \rightarrow H_q(X_n) \rightarrow H_{q-1}(X_{\infty}) \rightarrow \cdots,$$

where $t$ is the homomorphism induced from the generator of covering transformations. Thus we have

$$H_1(X) \rightarrow H_0(X_{\infty}) \xrightarrow{t - 1} H_0(X_{\infty}) \rightarrow \mathbb{Z} \rightarrow 0.$$

Hence $(t - 1): H_1(X_{\infty} : \mathbb{Z}) \rightarrow H_1(X_{\infty} : \mathbb{Z})$ is surjective. Therefore $(t^n - 1) = (t - 1)^n: H_1(X_{\infty} : \mathbb{Z}_p) \rightarrow H_1(X_{\infty} : \mathbb{Z}_p)$ is also surjective ($n = p^r$ is a prime power). Again using the exact sequence, we have $H_1(X_n : \mathbb{Z}_p) = \mathbb{Z}_p$. But the lift of the meridian of $J$ represents a nontrivial element of infinite order in $H_1(X_n : \mathbb{Z})$. Hence $H_1(X_n : \mathbb{Q}) = \mathbb{Q}$. Thus we have $H_1(M : \mathbb{Q}) = 0$. A Mayer-Vietoris sequence with $\mathbb{Q}$-coefficients gives

$$H_1(\partial N(k_j)) \rightarrow H_1(N(k_j)) \oplus H_1(M \setminus N(k_j)) \rightarrow H_1(M) \rightarrow \mathbb{Q}^{2g} \oplus \mathbb{Q} \rightarrow 0,$$

where $g$ is the genus of $K$. Hence $H_1(M; k_j : \mathbb{Q}) = 0$ or $\mathbb{Q}$. 

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Let \( L_i = J_i \cup K_i \) \((i = 0, 1)\) be cobordant links via \( C, E \), and let \( W \) be the \( n \)-fold cyclic branched covering of \( S^4 \times I \) along \( C, E_j \) \((j = 0, \ldots, n - 1)\) be the lifts of \( E \) to \( W \). Then \( \partial E_j = k_j^0 \cup (-k_j^1) \), where \( k_j^i \) are the lifts of \( K_i \) to \( M_i \), the \( n \)-fold branched covering of \( S^3 \) along \( J_i \). (Note that we have exactly \( n \) lifts of \( E \) because \( L_i \)'s are semi-boundary and \( E \) is homeomorphic to the product \( K_0 \times I \).) The same argument shows that \( H_1(W : \mathbb{Q}) = 0 \) and \( H_1(W \setminus E_0 : \mathbb{Q}) = 0 \) or \( \mathbb{Q} \).

We need to know the homomorphism \( H_2(M_i) \to H_2(W) \). Let \( Y = S^4 \times I \setminus N(C) \), \( Y_n \) (resp. \( Y_\infty \)) be the \( n \)-fold (resp. infinite) cyclic covering of \( Y \). Let \( X_i = S^4 \setminus J_i \), \( X_i^n \) (resp. \( X_i^\infty \)) be the \( n \)-fold (resp. infinite) cyclic covering of \( X_i \). Since \((t - 1): H_1(X_i^\infty : \mathbb{Z}) \to H_1(X_i^\infty : \mathbb{Z})\) is surjective, it is an isomorphism (because \( H_1(X_i^\infty : \mathbb{Z}) \) is a finitely generated module over a Noetherian ring \( \Lambda = \mathbb{Z}[t, t^{-1}] \), see \([S-S]\)). Also we have \( H_2(Y, X_i) = 0 \) since the inclusion induces an isomorphism \( H_*(X_i) \cong H_*(Y) \). Therefore we have the following commutative diagram with \( \mathbb{Z} \)-coefficients:

\[
\begin{array}{ccc}
0 & \to & H_2(X_i^\infty) \\
\downarrow & & \downarrow \\
0 & \to & H_2(Y_\infty) \\
\downarrow & & \downarrow \\
0 & \to & H_2(Y_n, X_i^n) \\
\end{array}
\]

Note that \( i_*: H_2(X_i) \to H_2(Y) \) (the homomorphism induced from the inclusion map) is an isomorphism, and \( H_2(X_i) \cong \mathbb{Z}^h \), where \( h \) is the genus of \( J_i \).

Consider the following splitting

\[
H_2(X_i^\infty : \mathbb{Q}) \cong F(X_i^\infty : \mathbb{Q}) \oplus T(X_i^\infty : \mathbb{Q})
\]

where \( F(\cdot), T(\cdot) \) denote the free and torsion part of \( H_2(\cdot) \) as a \( \Gamma = \Lambda \otimes \mathbb{Q} \)-module respectively (\( \Gamma \) is a PID). Furthermore, \( T(X_i^\infty : \mathbb{Q}) = T^0(X_i^\infty : \mathbb{Q}) \oplus T^1(X_i^\infty : \mathbb{Q}) \), where

\[
T^0(X_i^\infty : \mathbb{Q}) \cong \mathbb{Q} / (t - 1)^{p_1} \oplus \cdots \oplus \mathbb{Q} / (t - 1)^{p_r}
\]

is the \((t - 1)\)-summand and \( T^1(\cdot) \) is the \((t - 1)\)-free summand.

Comparing the above sequence to the following sequence

\[
0 \to \Gamma / (t - 1) \to \Gamma / (t - 1)^{p_k} \to \Gamma / (t - 1)^{p_k} \to \Gamma / (t - 1) \to 0,
\]

where \( \Gamma / (t - 1) \cong \mathbb{Q} \), we conclude that \( T^0(X_i^\infty) = 0 \). (The same is true for \( H_2(Y_\infty) \) and \( H_2(Y_n, X_i^n) \).)

Hence \( T(X_i^\infty : \mathbb{Q}) \cong \Lambda / \lambda_1 \oplus \cdots \oplus \Lambda / \lambda_m \) where \( \lambda_j \) is normalized so that \( \lambda_j \in \Lambda \) and coefficients are relatively prime \((j = 1, \ldots, m)\).
On the other hand, \( \text{Cok}(t - 1 : H^i_2(X^i_\infty : Z) \to H^i_2(X^i_\infty : Z)) \) is isomorphic to \( H^i_2(X^i_\infty) \otimes_\Lambda Z \), where \( Z \) is regarded as \( \Lambda \)-module via the augmentation map \( \Lambda \to Z, t \to 1 \) [S-S]. Since \( H^2_2(X) \) is torsion free, we have \( \lambda_j(1) = \pm 1, j = 1, \ldots, m \) [S-S]. Then the same argument as Theorem 3 in [Sum] shows that

\[
\text{Cok}(t^n - 1 : T(X^i_\infty : Q) \to T(X^i_\infty : Q)) = 0 \quad (n = p^r).
\]

The same is true for \( H^2_2(Y^i_\infty, X^i_\infty) \) and we have \( H^2_2(Y_n, X^i_n : Q) = 0 \) (since \( H^2_2(Y^i_\infty, X^i_\infty) \) is \( \Gamma \)-torsion), and hence \( i_* : H^i_2(X^i_n : Q) \to H^i_2(Y^i_n : Q) \) is an epimorphism.

Since \( t - 1 \) is an isomorphism on \( T(X^i_\infty) \), the sequence

\[
0 \to F(X^i_\infty) \xrightarrow{t-1} F(X^i_\infty) \to H^2_2(X^i) \to 0
\]

shows that \( \text{rank}_F F(X^i_\infty) = 2g \) where \( g \) is the genus of \( J_i \). The same is true for \( Y^i_\infty \) and the similar exact sequences for \( t^n - 1 \) show that \( \dim_Q H^2_2(X^i_n) = 2gn = \dim_Q H^2_2(Y^i_n) \). Since \( i_* : H^i_2(X^i_n : Q) \to H^i_2(Y^i_n) \) is an epimorphism of vector spaces of the same dimension, it is an isomorphism.

A Mayer-Vietoris sequence shows that \( i_* : H^i_2(M_i : Q) \to H^i_2(W : Q) \) is an isomorphism. Also we have the following Mayer-Vietoris sequences with \( Q \)-coefficients:

\[
\begin{array}{c}
H^2_2(M_i) \to H^1_1(\partial N(k^i_j)) \\
\downarrow \cong \downarrow \equiv \\
H^2_2(W) \to H^1_1(\partial N(E_j)) \\
\to H^1_1(N(k^i_j)) \oplus H^1_1(M_i \backslash \text{Int } N(k^i_j)) \to 0 \\
\cong \downarrow \downarrow \\
\to H^1_1(N(E_j)) \oplus H^1_1(W \backslash \text{Int } N(E_j)) \to 0
\end{array}
\]

It follows that

\[
i_* : H^i_1(M_i \backslash k^i_j : Q) \to H^i_1(W \backslash E_j : Q)
\]

is an isomorphism for \( i = 0, 1 \) and the lemma follows. Q.E.D.

**Proof of Theorem 3.** We use the same notation as in the proof of Lemma 1. Consider the following diagram with \( Q \)-coefficients:

\[
\begin{array}{c}
H^1_1(k^0_j) \to H^1_1(M_0 \backslash k^0_0) \cong Q \\
\downarrow \downarrow \\
H^1_1(E_j) \to H^1_1(W \backslash E_0) \cong Q \\
\uparrow \uparrow \\
H^1_1(k^1_j) \to H^1_1(M_1 \backslash k^1_0) \cong Q
\end{array}
\]

Each vertical homomorphism is induced from an inclusion map and an isomorphism in \( Q \)-coefficients. Hence the top homomorphism is zero iff so is the bottom homomorphism. Therefore \( \zeta^n_j \) is well-defined. Q.E.D.

**Example.** Let \( L_m = K_{m,0} \cup K_{m,1} \) be an “untwisted spun link” indicated in Figure 1, where \( m \) is a positive integer. (Regard \( S^4 \) as \( B^3 \times S^1 \cup S^2 \times B^2 \).
The circle in the 3-ball becomes a torus in $S^4$ after spinning, and the spun arc together with two disks in $S^2 \times B^2$ forms an $S^2$ in $S^4$.) Thus $K_{m,0}$ is homeomorphic to $S^2$, and $K_{m,1}$ to a torus. Furthermore, $K_{m,0}$ and $K_{m,1}$ are unknotted. One can calculate

$$\xi_j^n(L_m) = \begin{cases} 1 & \text{if } j \equiv \pm m \pmod{n}, \\ 0 & \text{otherwise,} \end{cases}$$

for any prime power $n$. Hence $L_m$ and $L_{m'}$ are not cobordant to each other unless $m = m'$. Note that the Sato-Levine invariant vanishes and Cochran’s derivative is trivial for any $m$.

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REFERENCES


